Solutions to B Problems

CHAPTER 2

B-2-1.

$$f(t) = 0 \qquad t < 0$$
$$= t\,e^{-2t} \qquad t \geq 0$$

Note that

$$\mathcal{L}[t] = \frac{1}{s^2}$$

Referring to Equation (2-2), we obtain

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[t\,e^{-2t}] = \frac{1}{(s+2)^2}$$

B-2-2.

(a)

$$f_1(t) = 0 \qquad t < 0$$
$$= 3\sin(5t + 45^\circ) \qquad t \geq 0$$

Note that

$$3\sin(5t + 45^\circ) = 3\sin 5t \cos 45^\circ + 3\cos 5t \sin 45^\circ$$
$$= \frac{3}{\sqrt{2}}\sin 5t + \frac{3}{\sqrt{2}}\cos 5t$$

So we have

$$F_1(s) = \mathcal{L}[f_1(t)] = \frac{3}{\sqrt{2}}\,\frac{5}{s^2+5^2} + \frac{3}{\sqrt{2}}\,\frac{s}{s^2+5^2}$$
$$= \frac{3}{\sqrt{2}}\,\frac{s+5}{s^2+25}$$

(b)

$$f_2(t) = 0 \qquad t < 0$$
$$= 0.03(1 - \cos 2t) \qquad t \geq 0$$

$$F_2(s) = \mathcal{L}[f_2(t)] = 0.03\,\frac{1}{s} - 0.03\,\frac{s}{s^2+2^2} = \frac{0.12}{s(s^2+4)}$$

B-2-3.

$$f(t) = 0 \qquad t < 0$$
$$= t^2 e^{-at} \qquad t \geq 0$$

Note that

$$\mathcal{L}[t^2] = \frac{2}{s^3}$$

Referring to Equation (2-2), we obtain

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[t^2 e^{-at}] = \frac{2}{(s + a)^3}$$

B-2-4.

$$f(t) = 0 \qquad t < 0$$
$$= \cos 2\omega t \cos 3\omega t \qquad t \geq 0$$

Noting that

$$\cos 2\omega t \cos 3\omega t = \tfrac{1}{2}(\cos 5\omega t + \cos \omega t)$$

we have

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[\tfrac{1}{2}(\cos 5\omega t + \cos \omega t)]$$
$$= \frac{1}{2}\left(\frac{s}{s^2 + 25\omega^2} + \frac{s}{s^2 + \omega^2}\right) = \frac{(s^2 + 13\omega^2)s}{(s^2 + 25\omega^2)(s^2 + \omega^2)}$$

B-2-5. The function f(t) can be written as

$$f(t) = (t - a)\,1(t - a)$$

The Laplace transform of f(t) is

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[(t - a)\,1(t - a)] = \frac{e^{-as}}{s^2}$$

B-2-6.

$$f(t) = c\,1(t - a) - c\,1(t - b)$$

The Laplace transform of f(t) is

$$F(s) = c\,\frac{e^{-as}}{s} - c\,\frac{e^{-bs}}{s} = \frac{c}{s}(e^{-as} - e^{-bs})$$

B-2-7. The function f(t) can be written as

$$f(t) = \frac{10}{a^2} - \frac{12.5}{a^2} 1(t - \frac{a}{5}) + \frac{2.5}{a^2} 1(t - a)$$

So the Laplace transform of f(t) becomes

$$F(s) = \mathcal{L}[f(t)] = \frac{10}{a^2}\frac{1}{s} - \frac{12.5}{a^2}\frac{1}{s} e^{-(a/5)s} + \frac{2.5}{a^2}\frac{1}{s} e^{-as}$$

$$= \frac{1}{a^2 s}(10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as})$$

As a approaches zero, the limiting value of F(s) becomes as follows:

$$\lim_{a \to 0} F(s) = \lim_{a \to 0} \frac{10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as}}{a^2 s}$$

$$= \lim_{a \to 0} \frac{\frac{d}{da}(10 - 12.5\, e^{-(a/5)s} + 2.5\, e^{-as})}{\frac{d}{da} a^2 s}$$

$$= \lim_{a \to 0} \frac{2.5\, s\, e^{-(a/5)s} - 2.5\, s\, e^{-as}}{2as}$$

$$= \lim_{a \to 0} \frac{\frac{d}{da}(2.5\, e^{-(a/5)s} - 2.5\, e^{-as})}{\frac{d}{da} 2a}$$

$$= \lim_{a \to 0} \frac{-0.5\, s\, e^{-(a/5)s} + 2.5\, s\, e^{-as}}{2}$$

$$= \frac{-0.5\, s + 2.5\, s}{2} = \frac{2s}{2} = s$$

B-2-8. The function f(t) can be written as

$$f(t) = \frac{24}{a^3} t - \frac{24}{a^2} 1(t - \frac{a}{2}) - \frac{24}{a^3}(t - a)\, 1(t - a)$$

So the Laplace transform of f(t) becomes

$$F(s) = \frac{24}{a^3}\frac{1}{s^2} - \frac{24}{a^2}\frac{1}{s} e^{-\frac{1}{2}as} - \frac{24}{a^3}\frac{e^{-as}}{s^2}$$

$$= \frac{24}{a^3}\left(\frac{1}{s^2} - \frac{a}{s} e^{-\frac{1}{2}as} - \frac{e^{-as}}{s^2}\right)$$

The limiting value of F(s) as a approaches zero is

$$\lim_{a \to 0} F(s) = \lim_{a \to 0} \frac{24(1 - as\ e^{-\frac{1}{2}as} - e^{-as})}{a^3 s^2}$$

$$= \lim_{a \to 0} \frac{\frac{d}{da}\, 24(1 - as\ e^{-\frac{1}{2}as} - e^{-as})}{\frac{d}{da}\, a^3 s^2}$$

$$= \lim_{a \to 0} \frac{24(-s\ e^{-\frac{1}{2}as} + \frac{as^2}{2}\ e^{-\frac{1}{2}as} + s\ e^{-as})}{3a^2 s^2}$$

$$= \lim_{a \to 0} \frac{\frac{d}{da}\, 8(-\ e^{-\frac{1}{2}as} + \frac{as}{2}\ e^{-\frac{1}{2}as} + e^{-as})}{\frac{d}{da}\, a^2 s}$$

$$= \lim_{a \to 0} \frac{8\left[\frac{s}{2}\ e^{-\frac{1}{2}as} + \frac{s}{2}\ e^{-\frac{1}{2}as} + \frac{as}{2}\left(\frac{-s}{2}\right) e^{-\frac{1}{2}as} - se^{-as}\right]}{2as}$$

$$= \lim_{a \to 0} \frac{\frac{d}{da}\, (4\ e^{-\frac{1}{2}as} - as\ e^{-\frac{1}{2}as} - 4\ e^{-as})}{\frac{d}{da}\, a}$$

$$= \lim_{a \to 0} \frac{-2s\ e^{-\frac{1}{2}as} - s\ e^{-\frac{1}{2}as} + as\ \frac{s}{2}\ e^{-\frac{1}{2}as} + 4s\ e^{-as}}{1}$$

$$= -2s - s + 4s = s$$

B-2-9.

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

$$= \lim_{s \to 0} \frac{s\ 5(s + 2)}{s(s + 1)} = \frac{5 \times 2}{1} = 10$$

B-2-10.

$$f(0+) = \lim_{t \to 0+} f(t) = \lim_{s \to \infty} \frac{s\ 2(s + 2)}{s(s + 1)(s + 3)} = 0$$

B-2-11. Define

$$y = \dot{x}$$

Then

$$y(0+) = \dot{x}(0+)$$

The initial value of y can be obtained by use of the initial value theorem as follows:

$$y(0+) = \lim_{s \to \infty} sY(s)$$

Since

$$Y(s) = \mathcal{L}_+[y(t)] = \mathcal{L}_+[\dot{x}(t)] = sX(s) - x(0+)$$

we obtain

$$y(0+) = \lim_{s \to \infty} sY(s) = \lim_{s \to \infty} s[sX(s) - x(0+)]$$

$$= \lim_{s \to \infty} [s^2X(s) - sx(0+)]$$

B-2-12. Note that

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^2}{dt^2} f(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

Define

$$g(t) = \frac{d^2}{dt^2} f(t)$$

Then

$$\mathcal{L}\left[\frac{d^3}{dt^3} f(t)\right] = \mathcal{L}\left[\frac{d}{dt} g(t)\right] = sG(s) - g(0)$$

$$= s[s^2F(s) - sf(0) - \dot{f}(0)] - \ddot{f}(0)$$

$$= s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0)$$

B-2-13.

$$\int_0^T f(t)\, e^{-st}\, dt = \int_0^b a\, e^{-st}\, dt = a \left. \frac{e^{-st}}{-s} \right|_0^b$$

$$= a\left(\frac{e^{-bs} - 1}{-s}\right) = a\left(\frac{1 - e^{-bs}}{s}\right)$$

Referring to Problem A-2-12, we have

$$F(s) = \frac{\int_0^T f(t)\, e^{-st}\, dt}{1 - e^{-Ts}} = \frac{a(1 - e^{-bs})}{s(1 - e^{-Ts})}$$

B-2-14.

$$\int_{0-}^{T-} f(t)\, e^{-st}\, dt = \int_{0-}^{0+} \delta(t)\, e^{-st}\, dt = 1$$

Referring to Problem A-2-12, we obtain

$$F(s) = \frac{1}{1 - e^{-Ts}}$$

B-2-15.

(a)

$$F_1(s) = \frac{s + 5}{(s + 1)(s + 3)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 3}$$

where

$$a_1 = \frac{s + 5}{s + 3}\bigg|_{s = -1} = \frac{4}{2} = 2$$

$$a_2 = \frac{s + 5}{s + 1}\bigg|_{s = -3} = \frac{2}{-2} = -1$$

$F_1(s)$ can thus be written as

$$F_1(s) = \frac{2}{s + 1} - \frac{1}{s + 3}$$

and the inverse Laplace transform of $F_1(s)$ is

$$f_1(t) = 2\, e^{-t} - e^{-3t}$$

(b)

$$F_2(s) = \frac{3(s + 4)}{s(s + 1)(s + 2)} = \frac{a_1}{s} + \frac{a_2}{s + 1} + \frac{a_3}{s + 2}$$

where

$$a_1 = \frac{3(s + 4)}{(s + 1)(s + 2)}\bigg|_{s = 0} = \frac{3 \times 4}{2} = 6$$

$$a_2 = \frac{3(s + 4)}{s(s + 2)}\bigg|_{s = -1} = \frac{3 \times 3}{(-1) \times 1} = -9$$

$$a_3 = \left.\frac{3(s + 4)}{s(s + 1)}\right|_{s = -2} = \frac{3 \times 2}{(-2)(-1)} = 3$$

$F_2(s)$ can thus be written as

$$F_2(s) = \frac{6}{s} - \frac{9}{s + 1} + \frac{3}{s + 2}$$

and the inverse Laplace transform of $F_2(s)$ is

$$f_2(t) = 6 - 9\, e^{-t} + 3\, e^{-2t}$$

B-2-16.

(a)

$$F_1(s) = \frac{6s + 3}{s^2} = \frac{6}{s} + \frac{3}{s^2}$$

The inverse Laplace transform of $F_1(s)$ is

$$f_1(t) = 6 + 3t$$

(b)

$$F_2(s) = \frac{5s + 2}{(s + 1)(s + 2)^2} = \frac{a}{s + 1} + \frac{b_2}{(s + 2)^2} + \frac{b_1}{s + 2}$$

where

$$a = \left.\frac{5s + 2}{(s + 2)^2}\right|_{s = -1} = \frac{-5 + 2}{1^2} = -3$$

$$b_2 = \left.\frac{5s + 2}{s + 1}\right|_{s = -2} = \frac{-10 + 2}{-2 + 1} = 8$$

$$b_1 = \left.\frac{d}{ds}\left(\frac{5s + 2}{s + 1}\right)\right|_{s = -2} = \left.\frac{5(s + 1) - (5s + 2)}{(s + 1)^2}\right|_{s = -2}$$

$$= \frac{5(-1) - (-10 + 2)}{1^2} = 3$$

$F_2(s)$ can thus be written as

$$F_2(s) = \frac{-3}{s + 1} + \frac{8}{(s + 2)^2} + \frac{3}{s + 2}$$

and the inverse Laplace transform of $F_2(s)$ is

$$f_2(t) = -3\ e^{-t} + 8t\ e^{-2t} + 3\ e^{-2t}$$

B-2-17.

$$F(s) = \frac{2s^2 + 4s + 5}{s(s+1)} = 2 + \frac{2}{s+1} + \frac{5}{s(s+1)}$$

$$= 2 + \frac{2}{s+1} + \frac{5}{s} - \frac{5}{s+1} = 2 - \frac{3}{s+1} + \frac{5}{s}$$

The inverse Laplace transform of F(s) is

$$f(t) = 2\ \delta(t) - 3\ e^{-t} + 5$$

B-2-18.

$$F(s) = \frac{s^2 + 2s + 4}{s^2} = 1 + \frac{2}{s} + \frac{4}{s^2}$$

The inverse Laplace transform of F(s) is

$$f(t) = \delta(t) + 2 + 4t$$

B-2-19.

$$F(s) = \frac{s}{s^2 + 2s + 10} = \frac{s + 1 - 1}{(s+1)^2 + 3^2}$$

$$= \frac{s+1}{(s+1)^2 + 3^2} - \frac{3}{(s+1)^2 + 3^2}\ \frac{1}{3}$$

Hence

$$f(t) = e^{-t} \cos 3t - \frac{1}{3} e^{-t} \sin 3t$$

B-2-20.

$$F(s) = \frac{s^2 + 2s + 5}{s^2(s+1)} = \frac{a}{s^2} + \frac{b}{s} + \frac{c}{s+1}$$

where

$$a = \left. \frac{s^2 + 2s + 5}{s+1} \right|_{s=0} = 5$$

$$b = \left. \frac{(2s + 2)(s + 1) - (s^2 + 2s + 5)}{(s + 1)^2} \right|_{s = 0} = \frac{2 - 5}{1} = -3$$

$$c = \left. \frac{s^2 + 2s + 5}{s^2} \right|_{s = -1} = \frac{1 - 2 + 5}{1} = 4$$

Hence

$$F(s) = \frac{5}{s^2} + \frac{-3}{s} + \frac{4}{s + 1}$$

The inverse Laplace transform of F(s) is

$$f(t) = 5t - 3 + 4\, e^{-t}$$

B-2-21.

$$F(s) = \frac{2s + 10}{(s + 1)^2 (s + 4)} = \frac{a}{(s + 1)^2} + \frac{b}{s + 1} + \frac{c}{s + 4}$$

where

$$a = \left. \frac{2s + 10}{s + 4} \right|_{s = -1} = \frac{-2 + 10}{3} = \frac{8}{3}$$

$$b = \left. \frac{2(s + 4) - (2s + 10)}{(s + 4)^2} \right|_{s = -1} = \frac{6 - 8}{3^2} = \frac{-2}{9}$$

$$c = \left. \frac{2s + 10}{(s + 1)^2} \right|_{s = -4} = \frac{-8 + 10}{9} = \frac{2}{9}$$

Hence

$$F(s) = \frac{8}{3} \frac{1}{(s + 1)^2} - \frac{2}{9} \frac{1}{s + 1} + \frac{2}{9} \frac{1}{s + 4}$$

The inverse Laplace transform of F(s) is

$$f(t) = \frac{8}{3}\, te^{-t} - \frac{2}{9}\, e^{-t} + \frac{2}{9}\, e^{-4t}$$

B-2-22.

$$F(s) = \frac{1}{s^2(s^2 + \omega^2)} = \left(\frac{1}{s^2} - \frac{1}{s^2 + \omega^2}\right)\frac{1}{\omega^2}$$

The inverse Laplace transform of F(s) is

$$f(t) = \frac{1}{\omega^2}\,(t - \frac{1}{\omega}\sin\omega t)$$

B-2-23.

$$F(s) = \frac{c}{s^2}\,(1 - e^{-as}) - \frac{b}{s}\,e^{-as} \qquad a > 0$$

The inverse Laplace transform of F(s) is

$$f(t) = ct - c(t - a)1(t - a) - b\,1(t - a)$$

B-2-24. A MATLAB program to obtain partial-fraction expansions of the given function F(s) is given below.

```
num = [0  0  0  0  1];
den = [1  3  2  0  0];
[r,p,k] = residue(num,den)

r =

  -0.2500
   1.0000
  -0.7500
   0.5000

p =

  -2
  -1
   0
   0

k =

   []
```

From this computer output we obtain

$$F(s) = \frac{1}{s^4 + 3s^3 + 2s^2} = \frac{-0.25}{s+2} + \frac{1}{s+1} + \frac{-0.75}{s} + \frac{0.5}{s^2}$$

The inverse Laplace transform of F(s) is

$$f(t) = -0.25\, e^{-2t} + e^{-t} - 0.75 + 0.5t$$

B-2-25. A possible MATLAB program to obtain partial-fraction expansions of the given function F(s) is given below.

```
num = [0  0  3  4  1];
den = [1  2  5  8  10];
[r,p,k] = residue(num,den)

r =

  0.3661 - 0.4881i
  0.3661 + 0.4881i
 -0.3661 - 0.0006i
 -0.3661 + 0.0006i


p =

  0.2758 + 1.9081i
  0.2758 - 1.9081i
 -1.2758 + 1.0309i
 -1.2758 - 1.0309i


k =

   []
```

From this computer output we obtain

$$F(s) = \frac{3s^2 + 4s + 1}{s^4 + 2s^3 + 5s^2 + 8s + 10}$$

$$= \frac{0.3661 - j0.4881}{s - 0.2758 - j1.9081} + \frac{0.3661 + j0.4881}{s - 0.2758 + j1.9081}$$

$$+ \frac{-0.3661 - j0.0006}{s + 1.2758 - j1.0309} + \frac{-0.3661 + j0.0006}{s + 1.2758 + j1.0309}$$

Since the poles are complex quantities, we may rewrite F(s) as follows:

$$F(s) = \frac{0.7322s + 1.6607}{(s - 0.2758)^2 + 1.9081^2} + \frac{-0.7322s - 0.9329}{(s + 1.2758)^2 + 1.0309^2}$$

$$= \frac{0.7322(s - 0.2758) + 1.9081 \times 0.9762}{(s - 0.2758)^2 + 1.9081^2}$$

$$+ \frac{-0.7322(s + 1.2758) + 1.0309 \times 0.001204}{(s + 1.2758)^2 + 1.0309^2}$$

Then, the inverse Laplace transform of F(s) is obtained as

$$f(t) = 0.7322\, e^{0.2758t} \cos 1.9081t$$
$$+ 0.9762\, e^{0.2758t} \sin 1.9081t$$
$$- 0.7322\, e^{-1.2758t} \cos 1.0309t$$
$$+ 0.001204\, e^{-1.2758t} \sin 1.0309t$$

B-2-26. $\ddot{x} + 4x = 0, \qquad x(0) = 5, \qquad \dot{x}(0) = 0$

The Laplace transform of the given differential equation is

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 4X(s) = 0$$

Substitution of the initial conditions into this last equation gives

$$(s^2 + 4)X(s) = 5s$$

Solving for X(s), we obtain

$$X(s) = \frac{5s}{s^2 + 4}$$

The inverse Laplace transform of X(s) is

$$x(t) = 5 \cos 2t$$

This is the solution of the given differential equation.

B-2-27. $\ddot{x} + \omega_n^2 x = t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$

The Laplace transform of this differential equation is

$$s^2X(s) + \omega_n^2 X(s) = \frac{1}{s^2}$$

Solving this equation for X(s), we obtain

$$X(s) = \frac{1}{s^2(s^2 + \omega_n^2)} = \left(\frac{1}{s^2} - \frac{1}{s^2 + \omega_n^2}\right)\frac{1}{\omega_n^2}$$

The inverse Laplace transform of X(s) is

$$x(t) = \frac{1}{\omega_n^2}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right)$$

This is the solution of the given differential equation.

B-2-28. $2\ddot{x} + 2\dot{x} + x = 1, \qquad x(0) = 0, \qquad \dot{x}(0) = 2$

The Laplace transform of this differential equation is

$$2[s^2X(s) - sx(0) - \dot{x}(0)] + 2[sX(s) - x(0)] + X(s) = \frac{1}{s}$$

Substitution of the initial conditions into this equation gives

$$2[s^2X(s) - 2] + 2[sX(s)] + X(s) = \frac{1}{s}$$

or

$$(2s^2 + 2s + 1)X(s) = 4 + \frac{1}{s}$$

Solving this last equation for X(s), we get

$$X(s) = \frac{4s + 1}{s(2s^2 + 2s + 1)}$$

$$= \frac{4}{2s^2 + 2s + 1} + \frac{1}{s(2s^2 + 2s + 1)}$$

$$= \frac{2}{(s + 0.5)^2 + 0.25} + \frac{0.5}{s[(s + 0.5)^2 + 0.25]}$$

$$= \frac{4 \times 0.5}{(s + 0.5)^2 + 0.5^2} + \frac{1}{s} - \frac{(s + 0.5) + 0.5}{(s + 0.5)^2 + 0.5^2}$$

The inverse Laplace transform of X(s) gives

$$x(t) = 4\, e^{-0.5t} \sin 0.5t + 1 - e^{-0.5t} \cos 0.5t - e^{-0.5t} \sin 0.5t$$

$$= 1 + 3\, e^{-0.5t} \sin 0.5t - e^{-0.5t} \cos 0.5t$$

B-2-29. $2\ddot{x} + 7\dot{x} + 3x = 0, \qquad x(0) = 3, \qquad \dot{x}(0) = 0$

Taking the Laplace transform of this differential equation, we obtain

$$2[s^2X(s) - sx(0) - \dot{x}(0)] + 7[sX(s) - x(0)] + 3X(s) = 0$$

By substituting the given initial conditions into this last equation,

$$2[s^2X(s) - 3s] + 7[sX(s) - 3] + 3X(s) = 0$$

or

$$(2s^2 + 7s + 3)X(s) = 6s + 21$$

Solving for X(s) yields

$$X(s) = \frac{6s + 21}{2s^2 + 7s + 3} = \frac{6s + 21}{(2s + 1)(s + 3)}$$

$$= \frac{7.2}{2s + 1} - \frac{0.6}{s + 3} = \frac{3.6}{s + 0.5} - \frac{0.6}{s + 3}$$

Finally, taking the inverse Laplace transform of X(s), we obtain

$$x(t) = 3.6\, e^{-0.5t} - 0.6\, e^{-3t}$$

B-2-30. $\ddot{x} + x = \sin 3t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$

The Laplace transform of this differential equation is

$$s^2X(s) + X(s) = \frac{3}{s^2 + 3^2}$$

Solving this equation for X(s), we get

$$X(s) = \frac{3}{(s^2 + 1)(s^2 + 9)} = \frac{3}{8}\frac{1}{s^2 + 1} - \frac{1}{8}\frac{3}{s^2 + 9}$$

The inverse Laplace transform of X(s) gives

$$x(t) = \frac{3}{8} \sin t - \frac{1}{8} \sin 3t$$

CHAPTER 3

B-3-1. $J = \frac{1}{2} mR^2 = \frac{1}{2} \times 100 \times 0.5^2 = 12.5$ kg-m^2

B-3-2. Assume that the body of known moment of inertia J_0 is turned through a small angle θ about the vertical axis and then released. The equation of motion for the oscillation is

$$J_0 \ddot{\theta} = - k\theta$$

where k is the torsional spring constant of the string. This equation can be written as

$$\ddot{\theta} + \frac{k}{J_0} \theta = 0$$

or

$$\ddot{\theta} + \omega_n^2 \theta = 0$$

where

$$\omega_n = \sqrt{\frac{k}{J_0}}$$

The period T_0 of this oscillation is

$$T_0 = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{k}{J_0}}} \qquad (1)$$

Next, we attach a rotating body of unknown moment of inertia J and measure the period T of oscillation. The equation for the period T is

$$T = \frac{2\pi}{\sqrt{\frac{k}{J}}} \qquad (2)$$

By eliminating the unknown torsional spring constant k from Equations (1) and (2), we obtain

$$\frac{2\pi\sqrt{J_0}}{T_0} = \frac{2\pi\sqrt{J}}{T}$$

Hence

$$J = J_0\left(\frac{T}{T_0}\right)^2 \qquad (3)$$

The unknown moment of inertia J can therefore be determined by measuring the period of oscillation T and substituting it into Equation (3).

B-3-3. Define the vertical displacement of the ball as x(t) with x(0) = 0. The positive direction is downward. The equation of motion for the system is

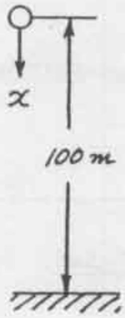
$$m\ddot{x} = mg$$

with initial conditions $x(0) = 0$ m and $\dot{x}(0) = 20$ m/s. So we have

$$\ddot{x} = g$$

$$\dot{x} = gt + \dot{x}(0)$$

$$x = \tfrac{1}{2} gt^2 + \dot{x}(0)t + x(0) = \tfrac{1}{2} gt^2 + 20\, t$$



Assume that at $t = t_1$ the ball reaches the ground. Then

$$100 = \tfrac{1}{2} \times 9.81\, t_1^{\,2} + 20\, t_1$$

from which we obtain

$$t_1 = 2.915 \text{ s}$$

The ball reaches the ground in 2.915 s.

B-3-4. Define the torque applied to the flywheel as T. The equation of motion for the system is

$$J\ddot{\theta} = T, \qquad \theta(0) = 0, \qquad \dot{\theta}(0) = 0$$

from which we obtain

$$\dot{\theta} = \frac{T}{J}\, t$$

By substituting numerical values into this equation, we have

$$20 \times 6.28 = \frac{T}{50} \times 5$$

Thus

$$T = 1256 \text{ N-m}$$

B-3-5. $$J\ddot{\theta} = -\,T \qquad (T = \text{braking torque})$$

Integrating this equation,

$$\dot{\theta} = -\frac{T}{J}\, t + \dot{\theta}(0), \qquad \dot{\theta}(0) = 100 \text{ rad/s}$$

Substituting the given numerical values,

$$20 = -\frac{T}{J} \times 15 + 100$$

Solving for T/J, we obtain

$$\frac{T}{J} = 5.33$$

Hence, the deceleration given by the brake is 5.33 rad/s^2.

The total angle rotated in 15-second period is obtained from

$$\theta(t) = -\frac{T}{J}\frac{t^2}{2} + \dot{\theta}(0)t + \theta(0), \qquad \theta(0) = 0, \qquad \dot{\theta}(0) = 100$$

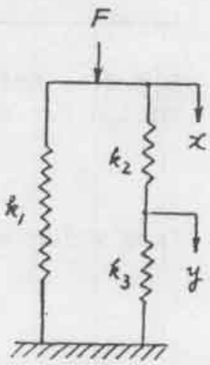
as follows;

$$\theta(15) = -5.33 \times \frac{15^2}{2} + 100 \times 15 = 900 \text{ rad}$$

B-3-6. Assume that we apply force F to the spring system. Then

$$F = k_1 x + k_2(x - y)$$

$$k_2(x - y) = k_3 y$$



Eliminating y from the preceding equations, we obtain

$$F = \frac{k_1(k_2 + k_3) + k_2 k_3}{k_2 + k_3} x$$

$$= \left(k_1 + \frac{1}{\frac{1}{k_2} + \frac{1}{k_3}} \right) x$$

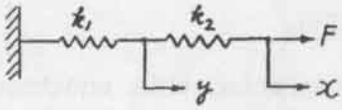
Hence the equivalent spring constant k_{eq} is given by

$$k_{eq} = k_1 + \frac{1}{\frac{1}{k_2} + \frac{1}{k_3}}$$

B-3-7. The equations for the system are

$$F = k_2(x - y)$$

$$k_2(x - y) = k_1 y$$



Eliminating y from the two equations gives

$$F = k_2 \left(x - \frac{k_2 x}{k_1 + k_2} \right) = \frac{k_1 k_2}{k_1 + k_2} x = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} x$$

The equivalent spring constant k_{eq} is then obtained as

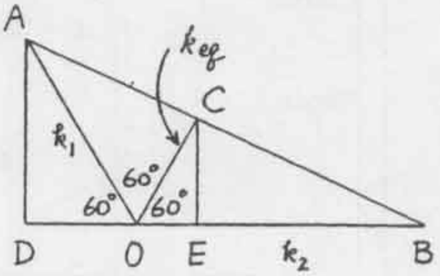
$$k_{eq} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

Next, consider the figure shown below. Note that ΔABD and ΔCBE are similar. So we have

$$\frac{\overline{CE}}{\overline{AD}} = \frac{\overline{BE}}{\overline{BD}}$$

which can be rewritten as

$$\frac{\overline{OC}\,\frac{\sqrt{3}}{2}}{\overline{OA}\,\frac{\sqrt{3}}{2}} = \frac{\overline{OB} - \overline{OC}\,\frac{1}{2}}{\overline{OB} + \overline{OA}\,\frac{1}{2}}$$



or

$$\overline{OC}(\overline{OB} + \tfrac{1}{2}\,\overline{OA}) = \overline{OA}(\overline{OB} - \tfrac{1}{2}\,\overline{OC})$$

Solving for $\overline{OC}$, we obtain

$$\overline{OC} = \frac{\overline{OA}\;\overline{OB}}{\overline{OA} + \overline{OB}} = \frac{1}{\frac{1}{\overline{OA}} + \frac{1}{\overline{OB}}} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}} = k_{eq}$$

B-3-8.

(a) The force f due to the dampers is

$$f = b_1(\dot{y} - \dot{x}) + b_2(\dot{y} - \dot{x}) = (b_1 + b_2)(\dot{y} - \dot{x})$$

In terms of the equivalent viscous friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

Hence

$$b_{eq} = b_1 + b_2$$

(b) The force f due to the dampers is

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) \qquad (1)$$

where z is the displacement of a point between damper b_1 and damper b_2. (Note that the same force is transmitted through the shaft.) From Equation (1), we have

$$(b_1 + b_2)\dot{z} = b_2\dot{y} + b_1\dot{x}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x}) \qquad (2)$$

In terms of the equivalent viscous friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

By substituting Equation (2) into Equation (1), we have

$$f = b_2(\dot{y} - \dot{z}) = b_2[\dot{y} - \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x})]$$

$$= \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

Thus,

$$f = b_{eq}(\dot{y} - \dot{x}) = b_2(\dot{y} - \dot{z}) = \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

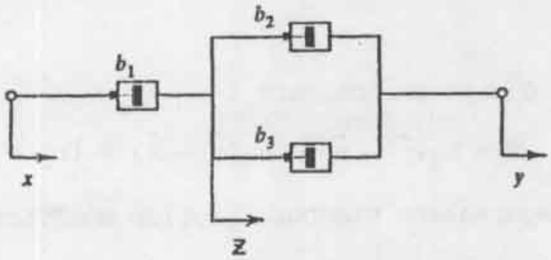
Hence,

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}$$

B-3-9. Since the same force transmits the shaft, we have

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) + b_3(\dot{y} - \dot{z}) \tag{1}$$

where displacement z is defined in the figure below.



In terms of the equivalent viscous friction coefficient, the force f is given by

$$f = b_{eq}(\dot{y} - \dot{x}) \tag{2}$$

From Equation (1) we have

$$b_1\dot{z} + b_2\dot{z} + b_3\dot{z} = b_1\dot{x} + b_2\dot{y} + b_3\dot{y}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2 + b_3}[b_1\dot{x} + (b_2 + b_3)\dot{y}] \tag{3}$$

By substituting Equation (3) into Equation (1), we have

$$f = b_1(\dot{z} - \dot{x}) = b_1 \left\{ \frac{1}{b_1 + b_2 + b_3} [b_1\dot{x} + (b_2 + b_3)\dot{y}] - \dot{x} \right\}$$

$$= b_1 \frac{b_2 + b_3}{b_1 + b_2 + b_3} (\dot{y} - \dot{x}) \qquad (4)$$

Hence, by comparing Equations (2) and (4), we obtain

$$b_{eq} = b_1 \left(\frac{b_2 + b_3}{b_1 + b_2 + b_3} \right) = \frac{1}{\dfrac{1}{b_2 + b_3} + \dfrac{1}{b_1}}$$

B-3-10. The equation for the system is

$$m\ddot{x} = -(k_1 + k_2)x - k_3 x$$

or

$$m\ddot{x} + (k_1 + k_2 + k_3)x = 0$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{k_1 + k_2 + k_3}{m}}$$

B-3-11. The density ρ of the liquid is

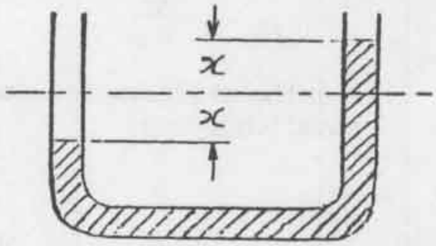
$$\rho = \frac{m}{LA}$$

where A is the cross-sectional area of the inside of the glass tube. The mathematical model for the system is

$$m\ddot{x} = -\rho A g \cdot 2x$$

or

$$\ddot{x} + \frac{2g}{L} x = 0$$



The natural frequency is

$$\omega_n = \sqrt{\frac{2g}{L}}$$

B-3-12. For a small displacement x, the torque balance equation for the system is

$$m\ddot{x}(2a) = -k(\frac{1}{2}x)a$$

or

$$m\ddot{x} + \frac{k}{4}x = 0$$

The natural frequency is

$$\omega_n = \frac{1}{2}\sqrt{\frac{k}{m}} = \frac{1}{2}\sqrt{\frac{400}{\frac{5}{9.81}}} = 14.01 \text{ rad/sec}$$

B-3-13.

(a) $$J\ddot{\theta}_o + b\dot{\theta}_o + k\theta_o = k\theta_i$$

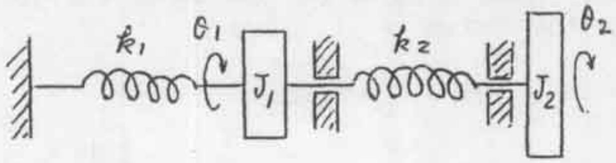
(b) $$m_1\ddot{x}_1 + b_1\dot{x}_1 + (k_1 + k_2)x_1 = k_2x_2$$

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 = k_2x_1$$

(c) $$m_1\ddot{x}_1 + b_1\dot{x}_1 + (k_1 + k_2)x_1 = k_2x_2$$

$$m_2\ddot{x}_2 + k_2x_2 = k_2x_1$$

B-3-14. A modified diagram for the system shown in Figure 3-55 is given below.



A mathematical model for the system is given by the following two equations:

$$J_1\ddot{\theta}_1 = -k_1\theta_1 - k_2(\theta_1 - \theta_2)$$

$$J_2\ddot{\theta}_2 = k_2(\theta_1 - \theta_2)$$

B-3-15. The following two equations describe the motion of the system and they are a mathematical model of the system.

$$m_1\ddot{x} = -k_1(x - y) - b_1(\dot{x} - \dot{y}) + p(t)$$

$$m_2\ddot{y} = -k_2 y - k_1(y - x) - b_1(\dot{y} - \dot{x})$$

Rewriting, we obtain

$$m_1\ddot{x} + b_1\dot{x} + k_1 x = b_1\dot{y} + k_1 y + p(t)$$

$$m_2\ddot{y} + b_1\dot{y} + k_1 y + k_2 y = b_1\dot{x} + k_1 x$$

B-3-16. A mathematical model for the system is

$$m\ddot{x} = -k_1 x - b_1\dot{x} - k_2 x - b_2\dot{x}$$

or

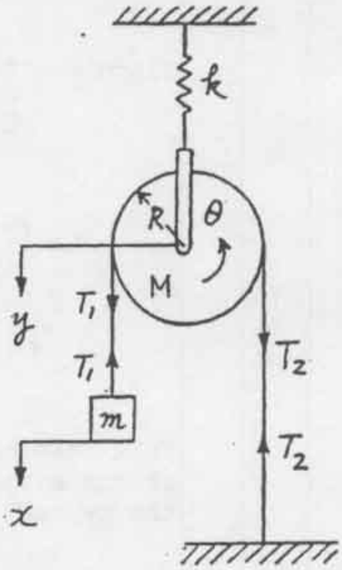
$$m\ddot{x} + (b_1 + b_2)\dot{x} + (k_1 + k_2)x = 0$$

B-3-17. The equations of motion for the system are

$$J\ddot{\theta} = (T_1 - T_2)R$$

$$m\ddot{x} = -T_1$$

$$M\ddot{y} = -ky + T_1 + T_2$$



Noting that $x = 2y$, $R\theta = x - y = y$, and $J = \frac{1}{2}MR^2$, the three equations can be rewritten as

$$\frac{1}{2}MR^2\ddot{\theta} = \frac{1}{2}MR\ddot{y} = (T_1 - T_2)R$$

$$m\ddot{x} = -T_1$$

$$M\ddot{y} + ky = T_1 + T_2$$

Eliminating T_2 from the preceding equations gives

$$\frac{1}{2}M\ddot{y} + M\ddot{y} + ky = 2T_1 = -2m\ddot{x}$$

By changing y into x,

$$\frac{3}{2}M\frac{\ddot{x}}{2} + k\frac{x}{2} = -2m\ddot{x}$$

or

$$(m + \frac{3}{8}M)\ddot{x} + \frac{1}{4}kx = 0$$

The natural frequency is

$$\omega_n = \sqrt{\frac{2k}{8m + 3M}}$$

If mass m is pulled down a distance x_o and released with zero initial velocity, the motion of mass m is

$$x(t) = x_o \cos \sqrt{\frac{2k}{8m + 3M}}\, t$$

B-3-18. Referring to the figure below, we have

$$m\ddot{x} = -T \qquad (1)$$

where T is the tension in the wire. (Note that since x is measured from the static equilibrium position, the term mg does not enter the equation.) For the rotational motion of the pulley, we have

$$J\ddot{\theta} = -k_2(y + R_2\theta)R_2 + k_2(y - R_2\theta)R_2 + TR_1 - k_1R_1x$$

or

$$J\ddot{\theta} = -k_2R_2^2\theta - k_2R_2^2\theta + TR_1 - k_1R_1x \qquad (2)$$

Eliminating T from Equations (1) and (2), we obtain

$$J\ddot{\theta} + 2k_2R_2^2\theta + mR_1\ddot{x} + k_1R_1x = 0$$

Since $x = R_1\theta$, we have

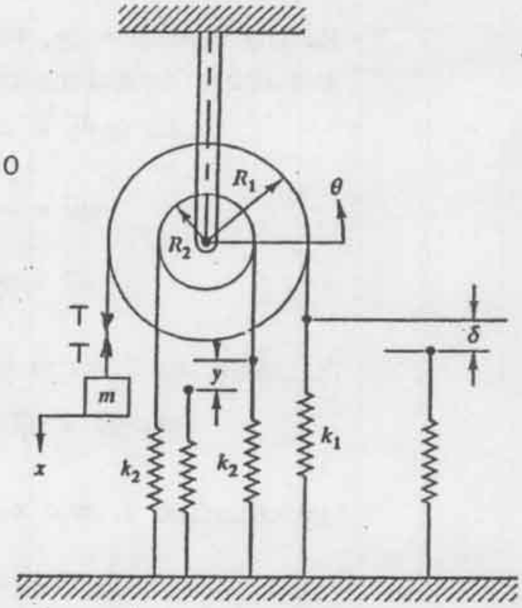
$$J\ddot{\theta} + 2k_2R_2^2\theta + mR_1^2\ddot{\theta} + k_1R_1^2\theta = 0$$

or

$$(J + mR_1^2)\ddot{\theta} + (2k_2R_2^2 + k_1R_1^2)\theta = 0$$

or

$$\ddot{\theta} + \frac{2k_2R_2^2 + k_1R_1^2}{J + mR_1^2}\theta = 0$$



This last equation is a mathematical model of the system. The natural frequency of the system is

$$\omega_n = \sqrt{\frac{2k_2R_2^2 + k_1R_1^2}{J + mR_1^2}}$$

B-3-19. The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = 0$$

Substituting the given numerical values for m, b, and k into this equation, we obtain

$$2\ddot{x} + 4\dot{x} + 20x = 0 \tag{1}$$

where $x(0) = 0.1$ and $\dot{x}(0) = 0$. The response to the given initial condition can be obtained by taking the Laplace transform of this equation, solving the resulting equation for X(s), and finding the inverse Laplace transform of X(s). The Laplace transform of Equation (1) is

$$2[s^2X(s) - sx(0) - \dot{x}(0)] + 4[sX(s) - x(0)] + 20\ X(s) = 0$$

By substituting the given initial conditions into this last equation, we get

$$2[s^2X(s) - 0.1s] + 4[sX(s) - 0.1] + 20\ X(s) = 0$$

Solving this equation for X(s) gives

$$X(s) = \frac{0.2s + 0.4}{2s^2 + 4s + 20} = \frac{0.1 + 0.1(s + 1)}{(s + 1)^2 + 3^2}$$

The inverse Laplace transform of this last equation gives

$$x(t) = 0.1(\frac{1}{3} e^{-t} \sin 3t + e^{-t} \cos 3t)$$

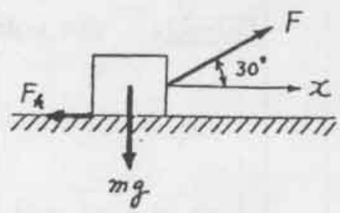
B-3-20. The equation of motion for the system is

$$m\ddot{x} = F \cos 30° - F_k$$

where $F_k = \mu_k(mg - F \sin 30°)$. Rewritting this equation,

$$m\ddot{x} = 0.866\ F - 0.3(mg - 0.5\ F)$$

For a constant speed motion, $\ddot{x} = 0$ and the last equation becomes

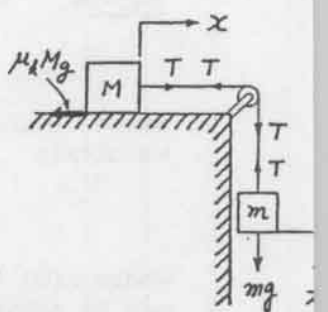


$$1.016\ F - 0.3\ mg = 0$$

or

$$F = \frac{0.3 \times 10 \times 9.81}{1.016} = 28.97\ N$$

B-3-21. The equations of motion for the system are



$$M\ddot{x} = T - \mu_k Mg$$

$$m\ddot{x} = mg - T$$

Elimination of T from these two equations gives

$$M\ddot{x} + m\ddot{x} = mg - \mu_k Mg$$

By substituting M = 2, m = 1, and $\mu_k = 0.2$ into this last equation, we get

$$3\ddot{x} = 1 \times 9.81 - 0.2 \times 2 \times 9.81 = 5.886$$

or

$$\ddot{x} = 1.962$$

Noting that $\dot{x}(0) = 0$, we have

$$\dot{x}(t) = 1.962\ t + \dot{x}(0) = 1.962\ t$$

$$x(t) = 1.962 \frac{t^2}{2} + x(0)$$

Assume that at $t = t_1$, $x(t_1) - x(0) = 0.5$ m. Then

$$\frac{1.962}{2} t_1^2 = 0.5$$

or

$$t_1 = 0.7139 \text{ s}$$

Thus, the velocity of the block when it has moved 0.5 m can be found as

$$\dot{x}(0.7139) = 1.962 \times 0.7139 = 1.401 \text{ m/s}$$

B-3-22. The equations of motion for the system are

$$m\ddot{x} = -kx - F$$

$$J\ddot{\theta} = FR$$

where $x = R\theta$ and $J = \frac{1}{2} mR^2$. So we obtain

$$m\ddot{x} = -kx - \frac{J\ddot{\theta}}{R} = -kx - \frac{1}{2} m\ddot{x}$$

or

$$\frac{3}{2} m\ddot{x} + kx = 0$$

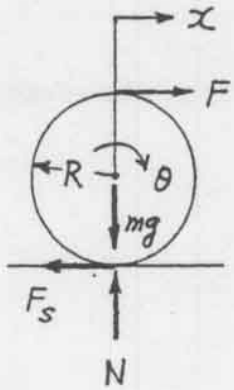
The natural frequency of the system is

$$\omega_n = \sqrt{\frac{2k}{3m}}$$

B-3-23. Assume that the direction of the static friction force F_s is to the left as shown in the diagram below. The equations for the system are

$$m\ddot{x} = F - F_s \qquad (1)$$

$$J\ddot{\theta} = FR + F_s R$$



where $J = \frac{1}{2} mR^2$. Since the cylinder rolls without sliding, we have $x = R\theta$. Consequently,

$$\frac{1}{2} mR^2\ddot{\theta} = (F + F_s)R$$

or

$$m\ddot{x} = 2(F + F_s) \qquad (2)$$

By eliminating $m\ddot{x}$ from Equations (1) and (2), we have

$$2(F + F_s) = F - F_s$$

or

$$F_s = -\frac{1}{3} F$$

Since F_s is found to be equal to $-(1/3)F$, the magnitude of F_s is one third of F and its direction is opposite to that assumed in the solution.

B-3-24. The equation of motion for the system is

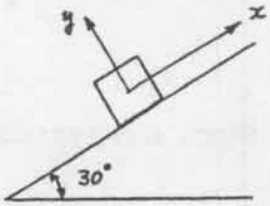
$$m\ddot{x} = F - mg \sin 30^\circ - \mu_k mg \cos 30^\circ, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

By substituting the given numerical values into this equation, we obtain

$$\ddot{x} = F - 1 \times 9.81 \times 0.5 - 0.2 \times 1 \times 9.81 \times 0.866$$

or

$$\ddot{x} = F - 4.905 - 1.699$$



and

$$\dot{x}(t) = (F - 6.604)t$$

$$x(t) = (F - 6.604)\frac{t^2}{2}$$

Assume that at $t = t_1$, $x(t_1) = 6$ m and $\dot{x}(t_1) = 5$ m/s. Then

$$6 = (F - 6.604)\frac{1}{2}t_1^2$$

$$5 = (F - 6.604)t_1$$

From the last two equations, t_1 and F are found to be

$$t_1 = 2.40 \text{ s}, \qquad F = 8.69 \text{ N}$$

Therefore,

Work done by force $F = F \times 6 = 8.69 \times 6 = 52.14$ N-m

Work done by the gravitational force $= -mg \sin 30° \times 6$

$= -9.81 \times 0.5 \times 6 = -29.43$ N-m

Work done by the sliding friction force

$= -0.2 \times mg \cos 30° \times 6$

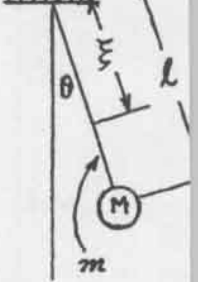
$= -1.699 \times 6 = -10.19$ N-m

B-3-25. Torque $= T = 50 \times 0.5 = 25$ N-m

Power $= T\omega = 25 \times 100 = 2500$ N-m/s $= 2500$ W

B-3-26. The kinetic energy T is

$$T = \frac{1}{2}M\ell^2\dot{\theta}^2 + \frac{1}{2}\int_0^\ell \frac{m}{\ell}\dot{\theta}^2\xi^2\, d\xi = \frac{1}{2}\left(M + \frac{m}{3}\right)\ell^2\dot{\theta}^2$$



The potential energy U is

$$U = Mg\ell(1 - \cos\theta) + \int_0^\ell \frac{m}{\ell} g\xi(1 - \cos\theta)\, d\xi$$

$$= \left(M + \frac{m}{2}\right)g\ell(1 - \cos\theta)$$

Since the system is conservative, we have

$$T + U = \frac{1}{2}(M + \frac{m}{3})\ell^2\dot{\theta}^2 + (M + \frac{m}{2})g\,\ell(1 - \cos\theta)$$

$$= \text{constant}$$

Noting that $d(T + U)/dt = 0$, we obtain

$$(M + \frac{m}{3})\,\ell^2\dot{\theta}\ddot{\theta} + (M + \frac{m}{2})g\ell\sin\theta\,\dot{\theta} = 0$$

or

$$\left[(M + \frac{m}{3})\,\ell^2\ddot{\theta} + (M + \frac{m}{2})g\ell\sin\theta\right]\dot{\theta} = 0$$

Since θ is not identically zero, we have

$$(M + \frac{m}{3})\,\ell^2\ddot{\theta} + (M + \frac{m}{2})g\ell\sin\theta = 0$$

Rewriting,

$$\ddot{\theta} + \frac{M + \frac{m}{2}}{M + \frac{m}{3}}\,\frac{g}{\ell}\sin\theta = 0$$

For small values of θ,

$$\ddot{\theta} + \frac{M + \frac{m}{2}}{M + \frac{m}{3}}\,\frac{g}{\ell}\,\theta = 0$$

So the natural frequency is

$$\omega_n = \sqrt{\frac{M + \frac{m}{2}}{M + \frac{m}{3}}\,\frac{g}{\ell}}$$

B-3-27. The kinetic energy T of the system is

$$T = \frac{1}{2}J_1\dot{\theta}_1^{\,2} + \frac{1}{2}J_2\dot{\theta}_2^{\,2}$$

and the potential energy U of the system is

$$U = \frac{1}{2}k_1\theta_1^{\,2} + \frac{1}{2}k_0(\theta_1 - \theta_2)^2 + \frac{1}{2}k_2\theta_2^{\,2}$$

Using the law of conservation of energy, we have

$$T + U = \frac{1}{2}J_1\dot{\theta}_1^{\,2} + \frac{1}{2}J_2\dot{\theta}_2^{\,2} + \frac{1}{2}k_1\theta_1^{\,2} + \frac{1}{2}k_0(\theta_1 - \theta_2)^2$$

$$+ \frac{1}{2}k_2\theta_2^{\,2} = \text{constant}$$

Noting that $d(T + U)/dt = 0$, we obtain

$$J_1\dot{\theta}_1\ddot{\theta}_1 + J_2\dot{\theta}_2\ddot{\theta}_2 + k_1\theta_1\dot{\theta}_1 + k_0(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) + k_2\theta_2\dot{\theta}_2 = 0$$

or

$$[J_1\ddot{\theta}_1 + k_1\theta_1 + k_0(\theta_1 - \theta_2)]\,\dot{\theta}_1 + [J_2\ddot{\theta}_2 + k_2\theta_2 - k_0(\theta_1 - \theta_2)]\,\dot{\theta}_2 = 0$$

Since $\dot{\theta}_1$ and $\dot{\theta}_2$ are not identically zero, we must have

$$J_1\ddot{\theta}_1 + k_1\theta_1 + k_0(\theta_1 - \theta_2) = 0$$

$$J_2\ddot{\theta}_2 + k_2\theta_2 + k_0(\theta_2 - \theta_1) = 0$$

B-3-28. At $t = 0$ (the instant the mass M is released to move) the kinetic energy T_1 and the potential energy U_1 of the system are

$$T_1 = 0$$

$$U_1 = mgx$$

The potential energy is measured relative to the floor.

At the instant mass m hits the floor the kinetic energy T_2 and the potential energy U_2 of the system are

$$T_2 = \tfrac{1}{2}\, Mv_2{}^2 + \tfrac{1}{2}\, mv_2{}^2 + \tfrac{1}{2}\, J\dot{\theta}_2{}^2$$

$$U_2 = 0$$

where v_2 is the velocity of the hanging mass m and θ_2 is the angle of rotation of the pulley both at the instant the mass hits the floor; $r\dot{\theta}_2 = v_2$, and $J = \tfrac{1}{2}\, m_p r^2$.

Using the law of conservation of energy, we obtain

$$T_1 + U_1 = T_2 + U_2$$

or

$$mgx = \tfrac{1}{2}\, Mv_2{}^2 + \tfrac{1}{2}\, mv_2{}^2 + \tfrac{1}{4}\, m_p r^2\dot{\theta}_2{}^2$$

By substituting $r\dot{\theta}_2 = v_2$ into this last equation,

$$mgx = \tfrac{1}{2}\, Mv_2{}^2 + \tfrac{1}{2}\, mv_2{}^2 + \tfrac{1}{4}\, m_p v_2^2$$

Solving this equation for v_2

$$v_2 = \frac{2\sqrt{mg}}{\sqrt{2M + 2m + m_p}} \sqrt{x}$$

B-3-29. The force F necessary to move the weight is

$$F = \frac{mg}{5} = \frac{1000 \times 9.81}{5} = 1962 \text{ N}$$

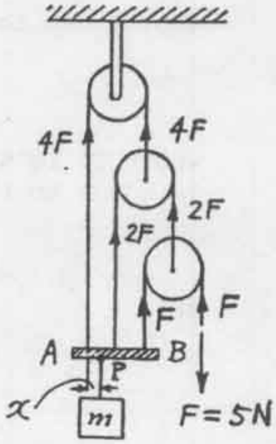
The power P is given by

$$P = \frac{dW}{dt}$$

where W = mgx. So we obtain

$$P = \frac{d(mgx)}{dt} = mg\dot{x} = 9810 \times 0.5 = 4905 \text{ W}$$

B-3-30. From the figure shown to the right, we obtain



$$7F = mg + 2$$

or

$$mg = 7F - 2 = 7 \times 5 - 2 = 33 \text{ N}$$

To keep the bar AB horizontal when pulling the weight mg, the moment about point P must balance. Thus,

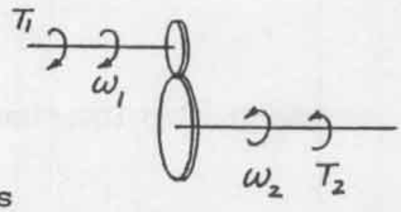
$$4Fx - 2F(0.15 - x) - F(0.3 - x) + 2(0.15 - x) = 0$$

Solving this equation for x, we obtain

$$x = \frac{2.7}{33} = 0.0818 \text{ m}$$

B-3-31. Note that

$$T_1\omega_1 = T_2\omega_2$$



Gear ratio = 1/30

where

$$\omega_1 = 60 \times 2\pi = 120\pi \text{ rad/s}$$

$$\omega_2 = 60 \times 2\pi \times \frac{1}{30} = 4\pi \text{ rad/s}$$

Since the power $T_1\omega_1$ of the motor is

$$T_1\omega_1 = 1.5 \text{ kW} = 1500 \text{ W}$$

the torque T_2 of the driven shaft is

$$T_2 = T_1 \frac{\omega_1}{\omega_2} = \frac{1500}{4\pi} = 119.4 \text{ N-m}$$

B-3-32. Assume that the stiffness of the shafts of the gear train is infinite, that there is neither backlash nor elastic deformation, and that the number of teeth on each gear is proportional to the radius of gear. Define the angular displacement of shaft 1 and shaft 2 as θ_1 and θ_2, respectively.

By applying Newton's second law to the system, we obtain for the motor shaft (shaft 1)

$$J_1\ddot{\theta}_1 + T_1 = T_m \tag{1}$$

where T_m is the torque developed by the motor and T_1 is the load torque on gear 1 due to the rest of the gear train. For shaft 2, we have

$$J_2\ddot{\theta}_2 = T_2 \tag{2}$$

where T_2 is the torque transmitted to gear 2. Since the work done by gear 1 is equal to that of gear 2,

$$T_1\theta_1 = T_2\theta_2$$

or

$$T_1 = T_2 \frac{\theta_2}{\theta_1} = T_2 \frac{n_1}{n_2}$$

Since $\theta_2/\theta_1 = n_1/n_2$, Equation (2) can be written as

$$J_2 \frac{n_1}{n_2} \ddot{\theta}_1 = \frac{n_2}{n_1} T_1$$

or

$$\left(\frac{n_1}{n_2}\right)^2 J_2\ddot{\theta}_1 = T_1 \tag{3}$$

Substituting Equation (3) into Equation (1), we get

$$J_1\ddot{\theta}_1 + \left(\frac{n_1}{n_2}\right)^2 J_2\ddot{\theta}_1 = T_m$$

or

$$\left[J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2\right]\ddot{\theta}_1 = T_m$$

The equivalent moment of inertia of the gear train referred to the motor shaft is

$$J_{eq} = J_1 + \left(\frac{n_1}{n_2}\right)^2 J_2$$

Notice that if the ratio n_1/n_2 is very much smaller than unity, then the effect of J_2 on the equivalent moment of inertia J_{eq} is negligible.

B-3-33. The equivalent moment of inertia J_m of mass m referred to the motor shaft axis can be obtained from

$$J_m \alpha = \text{Torque} = m\ddot{x}R$$

where α is the angular acceleration of the motor shaft and $\ddot{x}$ is the linear acceleration of mass m. Since $\alpha r = \ddot{x}$, we have

$$J_m \alpha = m \alpha rR$$

or

$$J_m = mrR$$

The equivalent moment of inertia J_b of the belt is obtained from

$$J_b \alpha = m_b \ddot{x} r = m_b \alpha r^2$$

or

$$J_b = m_b r^2$$

Since there is no slippage between the belt and the pulleys, the work done by the belt and the right-side pulley ($T_1\theta_1$) and that by the belt and the left-side pulley ($T_2\theta_2$) must be equal, or

$$T_1\theta_1 = T_2\theta_2$$

where T_1 is the load torque on the motor shaft and T_2 is the torque transmitted to the left-side pulley shaft, θ_1 is the angular displacement of the motor shaft, and θ_2 is the angular diaplacement of the left-side pulley shaft. Since the two pulleys are of the same size, we have $\theta_1 = \theta_2$. Hence

$$T_1 = T_2$$

For the motor shaft, we have

$$(J_r + J_p + J_m + J_b)\ddot{\theta}_1 + T_1 = T_m$$

Also, for the left-side pulley shaft, we have

$$J_p\ddot{\theta}_2 = T_2$$

Since $T_1 = T_2$, we have

$$(J_r + J_p + J_m + J_b)\ddot{\theta}_1 + J_p\ddot{\theta}_2 = T_m$$

Since $\theta_1 = \theta_2$, this last equation becomes

$$(J_r + 2J_p + J_m + J_b)\ddot{\theta}_1 = T_m$$

The equivalent moment of inertia J_{eq} of the system with respect to the motor shaft axis is

$$\begin{aligned} J_{eq} &= J_r + 2J_p + J_m + J_b \\ &= J_r + 2J_p + mrR + m_b r^2 \end{aligned}$$

CHAPTER 4

B-4-1.

$$R_{BC} = \frac{1}{\frac{1}{100} + \frac{1}{100}} = 50\,\Omega, \qquad R_{AC} = 30 + 50 = 80\,\Omega$$

$$i = \frac{E}{R_{AC}} = \frac{12}{80} = 0.15 \text{ A}$$

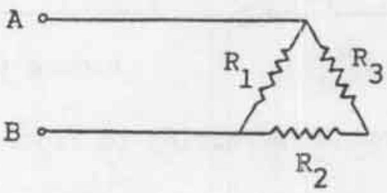
So we obtain

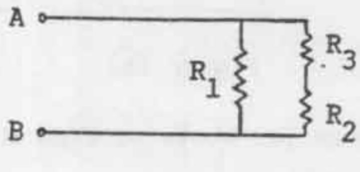
$$E_o = e_{BC} = iR_{BC} = 0.15 \times 50 = 7.5 \text{ V}$$

B-4-2.

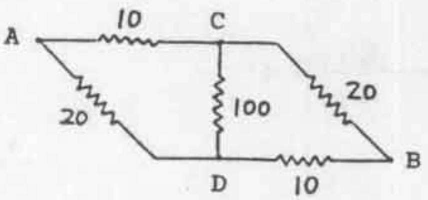
$$\frac{1}{R_{AB}} = \frac{1}{R_1} + \frac{1}{R_2 + R_3} = \frac{R_2 + R_3 + R_1}{R_1(R_2 + R_3)}$$

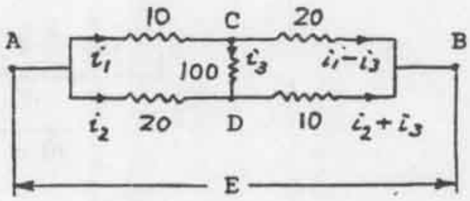
$$R_{AB} = \frac{R_1(R_2 + R_3)}{R_1 + R_2 + R_3}$$





B-4-3. Figure 4-55 can be redrawn as shown below.





From the right side diagram we obtain the following equations:

$$10\,i_1 + 20(i_1 - i_3) = E$$

$$20\,i_2 + 10(i_2 + i_3) = E$$

$$100\,i_3 + 10(i_2 + i_3) = 20(i_1 - i_3)$$

which yield

$$i_1 = \frac{12}{11} i_2, \qquad i_3 = \frac{1}{11} i_2$$

So we obtain

$$10\, i_1 + 20(i_1 - i_3) = \frac{120}{11} i_2 + 20\left(\frac{12}{11} - \frac{1}{11}\right) i_2 = E$$

Solving for i_2, we get

$$i_2 = \frac{11}{340} E$$

Thus,

$$R_{AB} = \frac{E}{i_1 + i_2} = \frac{E}{\left(\frac{12}{11} + 1\right) i_2} = \frac{E}{\frac{23}{11} \times \frac{11}{340} E} = \frac{340}{23} = 14.78\,\Omega$$

B-4-4.

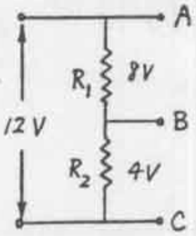


Figure (a)

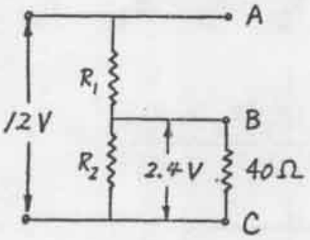


Figure (b)

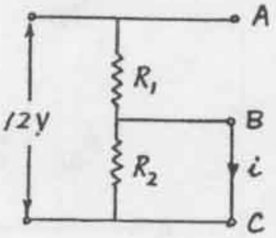


Figure (c)

From Figure (a) above we find $R_1 = 2R_2$. Referring to Figure (b) above, we get

$$\frac{1}{R_{BC}} = \frac{1}{R_2} + \frac{1}{40} = \frac{40 + R_2}{40\, R_2}$$

$$\frac{R_1}{R_{BC}} = \frac{R_1}{\frac{40\, R_2}{40 + R_2}} = \frac{9.6}{2.4} = 4$$

So R_1 is obtained as

$$R_1 = 4\, R_{BC} = \frac{160\, R_2}{40 + R_2}$$

By substituting $R_1 = 2R_2$ into this last equation, we obtain

$$R_1 = 80\,\Omega, \qquad R_2 = 40\,\Omega$$

Then, from Figure (c) above, current i is obtained as

$$i = \frac{12}{R_1} = \frac{12}{80} = 0.15 \text{ A}$$

B-4-5. When switch S is open, resistance between points A and B is 160 Ω and we have

$$i_0 = \frac{E}{160}$$

When switch S is closed, resistance between points A and B is

$$60 + \frac{1}{\frac{1}{100} + \frac{1}{R}} = 60 + \frac{100\ R}{100 + R}$$

So we have

$$2i_0 = \frac{E}{60 + \frac{100\ R}{100 + R}}$$

Consequently,

$$\frac{2E}{160} = \frac{E}{60 + \frac{100\ R}{100 + R}}$$

Solving for R, we obtain

$$R = 25\ \Omega$$

B-4-6. From the circuit diagram we obtain

$$L\frac{di_1}{dt} + R_1 i_1 + R_2(i_1 - i_2) = e$$

$$R_3 i_2 + \frac{1}{C_2}\int i_2\, dt + R_2(i_2 - i_1) = 0$$

or

$$L\frac{di_1}{dt} + (R_1 + R_2)i_1 - R_2 i_2 = e$$

$$-R_2 i_1 + (R_2 + R_3)i_2 + \frac{1}{C_2}\int i_2\, dt = 0$$

Each of the preceding two sets of equations constitutes a mathematical model for the circuit.

B-4-7. From Figure 4-55, we have for $t > 0$

$$R_1 i_1 + R_2(i_1 - i_2) = 0 \tag{1}$$

$$\frac{1}{C}\int i_2\,dt + R_2(i_2 - i_1) = 0 \tag{2}$$

Taking Laplace transforms of Equations (1) and (2), we get

$$R_1 I_1(s) + R_2[I_1(s) - I_2(s)] = 0 \tag{3}$$

$$\frac{1}{C}\left[\frac{I_2(s)}{s} + \frac{i_2^{-1}(0)}{s}\right] + R_2[I_2(s) - I_1(s)] = 0 \tag{4}$$

where $i_2^{-1}(0)$ is given by

$$i_2^{-1}(0) = \int i_2(t)\,dt\bigg|_{t=0}$$

$$= q(0) = e_0 C$$

From Equation (3) we have

$$I_2(s) = \frac{R_1 + R_2}{R_2} I_1(s) \tag{5}$$

Then Equation (4) can be rewritten as

$$\left(\frac{1}{Cs} + R_2\right) I_2(s) = R_2 I_1(s) - \frac{e_0}{s} \tag{6}$$

Substituting Equation (5) into Equation (6) we obtain

$$\left(\frac{1}{Cs} + R_2\right)\frac{R_1 + R_2}{R_2}\, I_1(s) = R_2 I_1(s) - \frac{e_0}{s}$$

or

$$I_1(s) = -\frac{e_0 R_2 C}{R_1 + R_2 + R_1 R_2 Cs}$$

$$= -\frac{e_0}{R_1}\,\frac{1}{s + \dfrac{R_1 + R_2}{R_1 R_2 C}}$$

The inverse Laplace transform of this last equation gives

$$i_1(t) = -\frac{e_0}{R_1} \exp[-(R_1 + R_2)t/(R_1R_2C)] \qquad t > 0$$

Referring to Equation (5) we have

$$i_2(t) = -\frac{(R_1 + R_2)e_0}{R_1R_2} \exp[-(R_1 + R_2)t/(R_1R_2C)] \qquad t > 0$$

B-4-8. At steady state ($t < 0$) we have

$$i_0 = \frac{E}{R_1 + R_2}$$

and

$$e_C = \frac{1}{C}\int i \, dt = R_2 i_0 = \frac{R_2}{R_1 + R_2} E$$

For $t \geqslant 0$ the equation for the circuit is

$$R_1 i + \frac{1}{C}\int i \, dt = E \qquad (1)$$

By differentiating this last equation, we obtain

$$R_1 \frac{di}{dt} + \frac{1}{C} i = 0$$

The Laplace transform of this equation is

$$R_1[sI(s) - i(0)] + \frac{1}{C} I(s) = 0$$

where

$$i(0) = i_0 = \frac{E}{R_1 + R_2}$$

Hence

$$(R_1 s + \frac{1}{C})I(s) = R_1 \frac{E}{R_1 + R_2}$$

or

$$I(s) = \frac{E}{R_1 + R_2} \frac{1}{s + \frac{1}{R_1 C}}$$

The inverse Laplace transform of I(s) gives

$$i(t) = \frac{E}{R_1 + R_2} e^{-t/(R_1 C)} , \quad t \geqslant 0$$

B-4-9. The equations for the system are

$$R_a i_a + L_a \frac{di_a}{dt} + e_b = e_a \tag{1}$$

$$e_b = K_b \frac{d\theta_1}{dt} \tag{2}$$

$$J_{eq} \frac{d^2\theta_1}{dt^2} + b_{eq} \frac{d\theta_1}{dt} = T = Ki_a \tag{3}$$

where J_{eq} is the equivalent moment of inertia of the system referred to the motor rotor axis and b_{eq} is the equivalent viscous friction coefficient of the system referred to the same axis.

By taking Laplace transforms of Equations (1) and (2), we obtain

$$(R_a + L_a s)I_a(s) + E_b(s) = E_a(s)$$

$$E_b(s) = K_b s\Theta_1(s)$$

Elimination of $E_b(s)$ from the above two equations yields

$$(R_a + L_a s)I_a(s) + K_b s\Theta_1(s) = E_a(s) \tag{4}$$

The Laplace transform of Equation (3) is

$$J_{eq}s^2\Theta_1(s) + b_{eq}s\Theta_1(s) = KI_a(s)$$

Hence

$$I_a(s) = \frac{J_{eq}s^2 + b_{eq}s}{K} \Theta_1(s) \tag{5}$$

By substituting Equation (5) into Equation (4), we obtain

$$\left[(R_a + L_a s) \frac{J_{eq}s^2 + b_{eq}s}{K} + K_b s\right] \Theta_1(s) = E_a(s)$$

or

$$\frac{\Theta_1(s)}{E_a(s)} = \frac{K}{(R_a + L_a s)(J_{eq}s^2 + b_{eq}s) + KK_b s} \tag{6}$$

The numerical values of the equivalent moment of inertia J_{eq} and equivalent viscous friction coefficient b_{eq} are, respectively,

$$J_{eq} = J_r + \left(\frac{N_1}{N_2}\right)^2 J_L = 1 \times 10^{-5} + (0.1)^2 \times 4.4 \times 10^{-3}$$

$$= 5.4 \times 10^{-5}\ \text{lb}_f\text{-ft-s}^2$$

$$b_{eq} = b_r + \left(\frac{N_1}{N_2}\right)^2 b_L = (0.1)^2 \times 4 \times 10^{-2}$$

$$= 4 \times 10^{-4}\ \text{lb}_f\text{-ft/rad/s}$$

Substituting these numerical values into Equation (6), we get

$$\frac{\Theta_1(s)}{E_a(s)} = \frac{6 \times 10^{-5}}{0.2(5.4 \times 10^{-5}s^2 + 4 \times 10^{-4}s) + 6 \times 10^{-5} \times 5.5 \times 10^{-2}s}$$

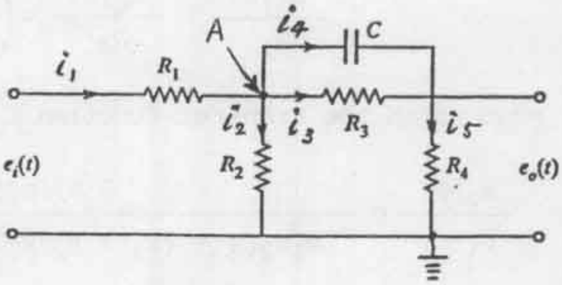
$$= \frac{6}{1.08s^2 + 8.33s}$$

$$= \frac{0.72}{s(0.1296s + 1)}$$

Since $\theta_2/\theta_1 = N_1/N_2 = n = 0.1$, we have the transfer function $\Theta_2(s)/E_a(s)$ as follows:

$$\frac{\Theta_2(s)}{E_a(s)} = \frac{n\Theta_1(s)}{E_a(s)} = \frac{0.072}{s(0.1296s + 1)}$$

B-4-10. From the circuit shown to the right, we obtain



$$i_1 = \frac{e_i - e_A}{R_1}$$

$$i_2 = \frac{e_A}{R_2}$$

$$i_3 = \frac{e_A - e_o}{R_3}$$

$$i_4 = C \frac{d}{dt}(e_A - e_o)$$

$$i_5 = \frac{e_o}{R_4}$$

Since

we have

$$i_1 = i_2 + i_5$$

$$\frac{e_i - e_A}{R_1} = \frac{e_A}{R_2} + \frac{e_o}{R_4} \qquad (1)$$

Also, since

we obtain

$$i_4 + i_3 = i_5$$

$$C\frac{d}{dt}(e_A - e_o) + \frac{e_A - e_o}{R_3} = \frac{e_o}{R_4} \qquad (2)$$

Equation (1) can be written as

$$\frac{e_i}{R_1} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)e_A + \frac{e_o}{R_4}$$

Laplace transforming this equation and simplifying, we get

$$\frac{E_i(s)}{R_1} - \frac{E_o(s)}{R_4} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)E_A(s) \qquad (3)$$

Laplace transforming Equation (2), we obtain

$$Cs[E_A(s) - E_o(s)] + \frac{1}{R_3}[E_A(s) - E_o(s)] = \frac{1}{R_4}E_o(s)$$

from which we obtain

$$E_A(s) = \frac{R_3R_4Cs + R_3 + R_4}{R_4(R_3Cs + 1)}E_o(s) \qquad (4)$$

By substituting Equation (4) into Equation (3), we have

$$\frac{E_i(s)}{R_1} - \frac{E_o(s)}{R_4} = \left(\frac{R_1 + R_2}{R_1R_2}\right)\frac{R_3R_4Cs + R_3 + R_4}{R_4(R_3Cs + 1)}E_o(s)$$

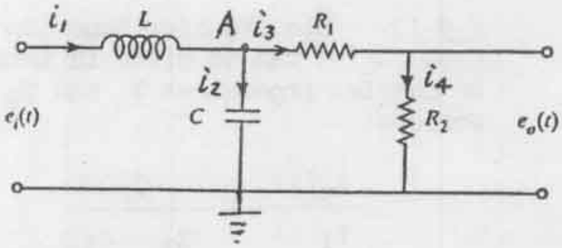
from which the transfer function $E_o(s)/E_i(s)$ can be obtained as

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2R_4(R_3Cs + 1)}{[R_1R_2R_3 + (R_1 + R_2)R_3R_4]Cs + R_1R_2 + (R_1 + R_2)(R_3 + R_4)}$$

B-4-11. From the circuit diagram shown, we obtain

$$\frac{E_i(s) - E_A(s)}{I_1(s)} = Ls$$

$$\frac{E_A(s)}{I_2(s)} = \frac{1}{Cs}$$



$$\frac{E_A(s) - E_o(s)}{I_3(s)} = R_1$$

$$\frac{E_o(s)}{I_4(s)} = R_2$$

Since

$$i_1 = i_2 + i_3$$

we have

$$\frac{E_i(s) - E_A(s)}{Ls} = CsE_A(s) + \frac{E_A(s) - E_o(s)}{R_1}$$

or

$$\frac{E_i(s)}{Ls} = \left(\frac{1}{Ls} + Cs + \frac{1}{R_1}\right)E_A(s) - \frac{E_o(s)}{R_1} \qquad (1)$$

Also, since

$$i_3 = i_4$$

we get

$$\frac{E_A(s) - E_o(s)}{R_1} = \frac{E_o(s)}{R_2}$$

Thus,

$$E_A(s) = \left(1 + \frac{R_1}{R_2}\right)E_o(s) \qquad (2)$$

Substituting Equation (2) into Equation (1) and simplifying, we obtain

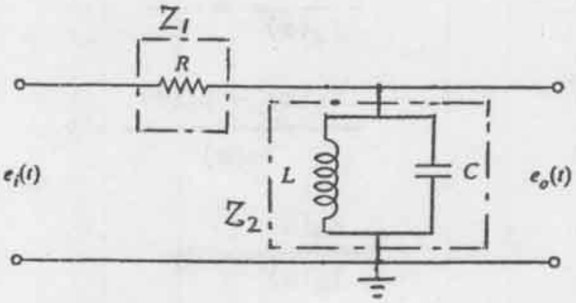
$$\frac{E_i(s)}{Ls} = \left[\left(\frac{1}{Ls} + Cs + \frac{1}{R_1}\right)R_1\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{1}{R_1}\right]E_o(s)$$

$$= \left(\frac{1}{Ls} + Cs + \frac{R_1}{LR_2 s} + \frac{R_1 Cs}{R_2} + \frac{1}{R_2}\right)E_o(s)$$

Hence the transfer function $E_o(s)/E_i(s)$ is given by

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2}{L(R_1 + R_2)Cs^2 + Ls + R_1 + R_2}$$

B-4-12. The transfer function $E_o(s)/E_i(s)$ can be given in terms of complex impedances Z_1 and Z_2 as follows:



$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2}$$

where

$$Z_1 = R$$

$$\frac{1}{Z_2} = \frac{1}{Ls} + Cs = \frac{1 + LCs^2}{Ls}$$

Hence

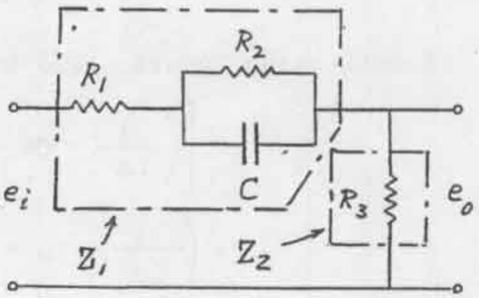
$$\frac{E_o(s)}{E_i(s)} = \frac{\dfrac{Ls}{LCs^2 + 1}}{R + \dfrac{Ls}{LCs^2 + 1}}$$

$$= \frac{Ls}{LRCs^2 + Ls + R}$$

B-4-13.

$$Z_1 = R_1 + \frac{R_2}{1 + R_2Cs} \qquad Z_2 = R_3$$

Hence



$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2}$$

$$= \frac{R_3}{R_1 + \dfrac{R_2}{1 + R_2Cs} + R_3}$$

$$= \frac{R_3(R_2Cs + 1)}{R_2(R_1 + R_3)Cs + R_1 + R_2 + R_3}$$

$$= \frac{R_3\left(s + \frac{1}{R_2C}\right)}{(R_1 + R_3)\left[s + \frac{R_1 + R_2 + R_3}{R_2(R_1 + R_3)C}\right]}$$

B-4-14.

$$Z_1 = Ls, \qquad \frac{1}{Z_2} = \frac{1}{R} + Cs = \frac{RCs + 1}{R}$$

Hence

$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2}$$

$$= \frac{\frac{R}{RCs + 1}}{Ls + \frac{R}{RCs + 1}} = \frac{1}{LCs^2 + \frac{L}{R}s + 1}$$

B-4-15.

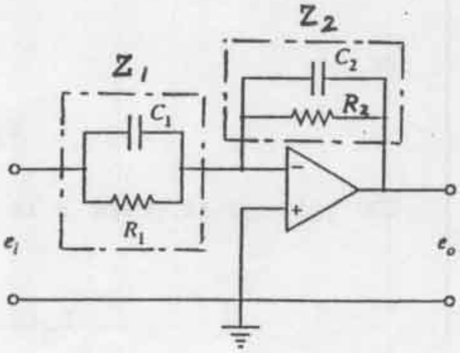
$$Z_1(s) = R, \quad Z_2(s) = \frac{1}{Cs}$$

Hence

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{1}{RCs}$$

B-4-16. Note that

$$\frac{1}{Z_1(s)} = C_1s + \frac{1}{R_1}$$



Hence

$$Z_1(s) = \frac{R_1}{R_1C_1s + 1}$$

Similarly,

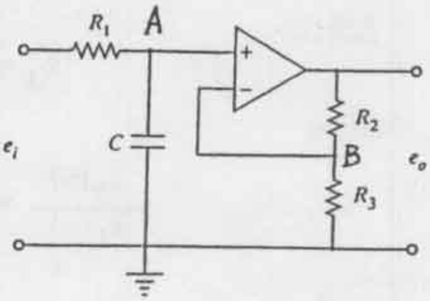
$$Z_2(s) = \frac{R_2}{R_2C_2s + 1}$$

The transfer function $E_o(s)/E_i(s)$ can be given by

$$\frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -\frac{R_2}{R_1}\frac{R_1C_1s + 1}{R_2C_2s + 1}$$

B-4-17. Define the voltage at point A as e_A. Then

$$\frac{E_A(s)}{E_i(s)} = \frac{1}{R_1Cs + 1}$$



Define the voltage at point B as e_B. Then

$$E_B(s) = \frac{R_3}{R_2 + R_3} E_o(s)$$

Noting that

$$[E_A(s) - E_B(s)]K = E_o(s)$$

and $K \gg 1$, we must have

$$E_A(s) = E_B(s)$$

Hence

$$E_A(s) = \frac{1}{R_1Cs + 1} E_i(s) = E_B(s) = \frac{R_3}{R_2 + R_3} E_o(s)$$

from which we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{R_2 + R_3}{R_3}\frac{1}{R_1Cs + 1}$$

B-4-18. The voltage at point A is

$$e_A = \frac{1}{2}(e_i + e_o)$$

or

$$E_A(s) = \frac{1}{2}[E_i(s) + E_o(s)] \qquad (1)$$

The voltage at point B is

$$E_B(s) = \frac{\frac{1}{Cs}}{R_2 + \frac{1}{Cs}} E_i(s) = \frac{1}{R_2Cs + 1} E_i(s) \qquad (2)$$

Since

$$[E_B(s) - E_A(s)]K = E_0(s), \qquad K \gg 1$$

we must have

$$E_A(s) = E_B(s)$$

Thus, equating Equations (1) and (2) we obtain

$$\frac{1}{2}\,[E_i(s) + E_o(s)] = \frac{1}{R_2Cs + 1}\,E_i(s)$$

or

$$\frac{E_o(s)}{E_i(s)} = -\,\frac{R_2Cs - 1}{R_2Cs + 1}$$

B-4-19. Define the displacement of midpoint between k_3 and b as x_3. Then the equations for the system are

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) + k_3(x_1 - x_3) = p(t)$$

$$m_2\ddot{x}_2 + b(\dot{x}_2 - \dot{x}_3) + k_2(x_2 - x_1) = 0$$

$$b(\dot{x}_3 - \dot{x}_2) = k_3(x_1 - x_3)$$

Using the force-voltage analogy, the preceding equations may be converted to

$$L_1\ddot{q}_1 + \frac{1}{C_1}\,q_1 + \frac{1}{C_2}\,(q_1 - q_2) + \frac{1}{C_3}(q_1 - q_3) = e(t)$$

$$L_2\ddot{q}_2 + R(\dot{q}_2 - \dot{q}_3) + \frac{1}{C_2}\,(q_2 - q_1) = 0$$

$$R(\dot{q}_3 - \dot{q}_2) = \frac{1}{C_3}\,(q_1 - q_3)$$

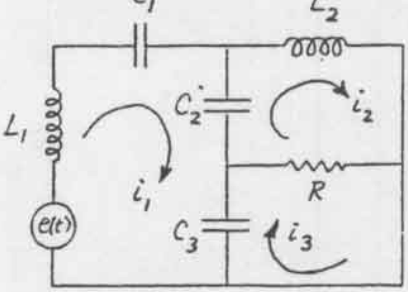
The last three equations can be modified to

$$L_1\frac{di_1}{dt} + \frac{1}{C_1}\int i_1dt + \frac{1}{C_2}\int(i_1 - i_2)dt + \frac{1}{C_3}\int(i_1 - i_3)dt = e(t)$$

$$L_2\frac{di_2}{dt} + R(i_2 - i_3) + \frac{1}{C_2}\int(i_2 - i_1)dt = 0$$

$$R(i_3 - i_2) = \frac{1}{C_3}\int(i_1 - i_3)dt$$

From these three equations we can obtain the analogous electrical system as shown to the right.



B-4-20. Define the cyclic current in the left loop as i_1 and that in the right loop as i_2. Then the equations for the circuit are

$$L_1 \frac{di_1}{dt} + \frac{1}{C_2}\int(i_1 - i_2)\,dt + R(i_1 - i_2) + \frac{1}{C_1}\int i_1 dt = 0$$

$$L_2 \frac{di_2}{dt} + \frac{1}{C_3}\int i_2\,dt + R(i_2 - i_1) + \frac{1}{C_2}\int(i_2 - i_1)\,dt = 0$$

which can be rewritten as

$$L_1\ddot{q}_1 + \frac{1}{C_2}(q_1 - q_2) + R(\dot{q}_1 - \dot{q}_2) + \frac{1}{C_1}q_1 = 0$$

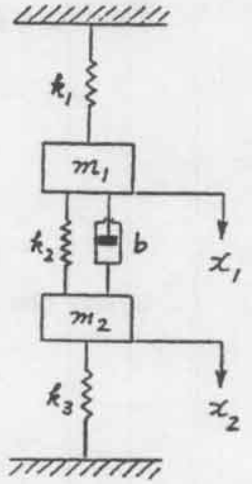
$$L_2\ddot{q}_2 + \frac{1}{C_3}q_2 + R(\dot{q}_2 - \dot{q}_1) + \frac{1}{C_2}(q_2 - q_1) = 0$$

Using the force-voltage analogy, we can convert the last two equations as follows:

$$m_1\ddot{x}_1 + k_2(x_1 - x_2) + b(\dot{x}_1 - \dot{x}_2) + k_1x_1 = 0$$

$$m_2\ddot{x}_2 + k_3x_2 + b(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$

From these equations an analogous mechanical system can be obtained as shown to the right.



CHAPTER 5

B-5-1. By substituting the given numerical values into

$$R_\ell = \frac{dh}{dQ} = \frac{128 \nu L}{g \pi D^4}$$

we obtain

$$R_\ell = \frac{128 \times 1.004 \times 10^{-6} \times 2}{9.81 \times 3.14 \times (4 \times 10^{-3})^4} = 3.2594 \times 10^4 \text{ s/m}^2$$

B-5-2. We shall solve this problem by using two different approaches: one based on the exact method and the other based on the use of an average resistance.

(1) Solution by the exact method. For the liquid-level system we have

$$C\, dH = (Q_i - Q)\, dt$$

By substituting $C = 2 \text{ m}^2$, $Q_i = 0.05 \text{ m}^3\text{/s}$, and $Q = 0.02\sqrt{H}$ into this last equation, we obtain

$$2\, dH = (0.05 - 0.02\sqrt{H})\, dt$$

or

$$dt = \frac{2}{0.05 - 0.02\sqrt{H}}\, dH$$

Let $\sqrt{H} = x$. Then $H = x^2$ and $dH = 2x\, dx$. So we have

$$dt = \frac{200}{5 - 2x}\, 2x\, dx = 200 \left(\frac{-5 + 2x + 5}{5 - 2x} \right) dx$$

Assume that at $t = t_1$ the level reaches 2.5 m. Then, t_1 is obtained as

$$t_1 = \int_0^{t_1} dt = \int_1^{\sqrt{2.5}} 200\left(-1 + \frac{5}{5 - 2x}\right) dx$$

$$= -200\, x \Big|_1^{\sqrt{2.5}} + 1000 \int_1^{\sqrt{2.5}} \frac{dx}{5 - 2x}$$

$$= -200(\sqrt{2.5} - 1) + 1000\left(-\frac{1}{2}\right) \ln(5 - 2x) \Big|_1^{\sqrt{2.5}}$$

$$= -200 \times 0.581 - 500(\ln 1.838 - \ln 3)$$

$$= -200 \times 0.581 - 500(0.6087 - 1.0986)$$

$$= 128.8 \text{ s}$$

(2) Solution by use of an average resistance. Since $Q = 0.02\sqrt{H}$, the average resistance R is obtained from

$$R = \frac{dH}{dQ} = \frac{2.5 - 1.0}{0.03162 - 0.02} = 129$$

Define $h = H - 1$. Then $q_i = Q_i - 0.02$ and $q_o = h/R$. For the liquid-level system,

$$C\,dh = (q_i - q_o)dt$$

or

$$C\frac{dh}{dt} = q_i - q_o = q_i - \frac{h}{R}$$

which can be rewritten as

$$CR\frac{dh}{dt} + h = q_i R$$

Substituting $C = 2\ m^2$, $R = 129\ s/m^2$, and $q_i = 0.05 - 0.02 = 0.03\ m^3/s$ into this last equation yields

$$258\frac{dh}{dt} + h = 3.87, \qquad h(0) = 0$$

Taking Laplace transforms of both sides of this last equation, we obtain

$$258[sH(s) - h(0)] + H(s) = \frac{3.87}{s}$$

or

$$(258s + 1)H(s) = \frac{3.87}{s}$$

Solving this equation for H(s),

$$H(s) = \frac{3.87}{s(258s + 1)}$$

$$= 3.87\left(\frac{1}{s} - \frac{258}{258s + 1}\right)$$

The inverse Laplace transform of this last equation gives

$$h(t) = 3.87(1 - e^{-\frac{1}{258}t})$$

Assume that at $t = t_1$, $h(t_1) = 1.5$. The value of t_1 can be determined from

$$1.5 = 3.87(1 - e^{-\frac{1}{258}t_1})$$

Rewriting,

$$e^{-\frac{1}{258}t_1} = 0.6124$$

or

$$\frac{t_1}{258} = 0.4904$$

So we have

$$t_1 = 0.4904 \times 258 = 126.5 \text{ s}$$

This solution has been obtained by use of an average resistance.

B-5-3. The equations for the liquid-level system are

$$C_1 dh_1 = (\bar{Q} + q - \bar{Q} - q_1)dt$$

$$C_2 dh_2 = (\bar{Q} + q_1 - \bar{Q} - q_2)dt$$

Since $R_1 = h_1/q_1$ and $R_2 = h_2/q_2$, the system equations can be rewritten as

$$C_1 \frac{dh_1}{dt} = q - q_1 = q - \frac{h_1}{R_1} \tag{1}$$

$$C_2 \frac{dh_2}{dt} = q_1 - q_2 = \frac{h_1}{R_1} - \frac{h_2}{R_2} \tag{2}$$

From Equations (1) and (2) we obtain

$$C_1 \frac{dh_1}{dt} + C_2 \frac{dh_2}{dt} = q - \frac{h_2}{R_2} \tag{3}$$

By differentiating Equation (2) with respect to t, we get

$$C_2 \frac{d^2h_2}{dt^2} + \frac{1}{R_2}\frac{dh_2}{dt} = \frac{1}{R_1}\frac{dh_1}{dt} \tag{4}$$

By eliminating dh_1/dt from Equations (3) and (4), we obtain

$$R_1C_1R_2C_2 \frac{d^2h_2}{dt^2} + (R_1C_1 + R_2C_2)\frac{dh_2}{dt} + h_2 = R_2 q$$

Substitution of $h_2 = R_2 q_2$ into this last equation yields

$$R_1C_1R_2C_2 \frac{d^2q_2}{dt^2} + (R_1C_1 + R_2C_2)\frac{dq_2}{dt} + q_2 = q$$

Hence the transfer function of the system when q is the input and q_2 is the output is given by

$$\frac{Q_2(s)}{Q(s)} = \frac{1}{(R_1C_1s + 1)(R_2C_2s + 1)}$$

B-5-4. The equations for the system are

$$C_1dh_1 = q_1dt$$

$$C_2dh_2 = (q_i - q_1 - q_2)dt$$

$$C_3dh_3 = (q_2 - q_0)dt$$

where

$$q_1 = \frac{h_2 - h_1}{R_1}$$

$$q_2 = \frac{h_2 - h_3}{R_2}$$

$$q_0 = \frac{h_3}{R_3}$$

Thus, we have

$$C_1 \frac{dh_1}{dt} = \frac{h_2 - h_1}{R_1} \qquad (1)$$

$$C_2 \frac{dh_2}{dt} = q_i - \frac{h_2 - h_1}{R_1} - \frac{h_2 - h_3}{R_2} \qquad (2)$$

$$C_3 \frac{dh_3}{dt} = \frac{h_2 - h_3}{R_2} - \frac{h_3}{R_3} \qquad (3)$$

From Equation (1) we obtain

$$C_1sH_1(s) = \frac{1}{R_1}[H_2(s) - H_1(s)]$$

or

$$H_1(s) = \frac{1}{R_1C_1s + 1} H_2(s) \qquad (4)$$

From Equation (3) we get

$$C_3R_3sH_3(s) + H_3(s) = \frac{R_3}{R_2} H_2(s) - \frac{R_3}{R_2} H_3(s)$$

or

$$H_2(s) = \frac{R_2}{R_3}\left(R_3C_3s + 1 + \frac{R_3}{R_2}\right)H_3(s) \qquad (5)$$

By adding Equations (1), (2), and (3), and taking the Laplace transform of the resulting equation, we obtain

$$C_1sH_1(s) + C_2sH_2(s) + C_3sH_3(s) = Q_i(s) - \frac{1}{R_3}H_3(s) \qquad (6)$$

By substituting Equations (4) and (5) into Equation (6), we get

$$\left[\left(\frac{C_1s}{R_1C_1s + 1} + C_2s\right)\left(\frac{R_2}{R_3}\right)\left(R_3C_3s + 1 + \frac{R_3}{R_2}\right) + \left(C_3s + \frac{1}{R_3}\right)\right]H_3(s) = Q_i(s)$$

Since $H_3(s) = R_3Q_0(s)$, this last equation can be written as

$$\left[\frac{(C_1 + C_2)s + R_1C_1C_2s^2}{R_1C_1s + 1}(R_3R_2C_3s + R_2 + R_3) + (R_3C_3s + 1)\right]Q_0(s) = Q_i(s)$$

from which we obtain

$$\frac{Q_0(s)}{Q_i(s)} = \frac{R_1C_1s + 1}{[(C_1 + C_2)s + R_1C_1C_2s^2](R_3R_2C_3s + R_2 + R_3) + (R_3C_3s + 1)(R_1C_1s + 1)}$$

This is the transfer function relating $Q_0(s)$ and $Q_i(s)$.

B-5-5. For this system

$$CdH = -Q\,dt, \qquad H = 3r, \qquad C = r^2\pi = \left(\frac{H}{3}\right)^2\pi$$

Hence

$$\left(\frac{H}{3}\right)^2\pi\,dH = -0.005\sqrt{H}\,dt$$

or

$$H^{1.5}\,dH = -0.005\,\frac{9}{\pi}\,dt$$

Assume that the head moves down from H = 2m to x for the 60 second period. Then

$$\int_2^x H^{1.5}\,dH = -0.005\,\frac{9}{\pi}\int_0^{60} dt$$

or

$$\frac{2}{5}(x^{2.5} - 2^{2.5}) = -0.01432(60 - 0)$$

which can be rewritten as

$$x^{2.5} - (1.414213)^5 = - 2.1480$$

or

$$x^{2.5} = 5.6569 - 2.1480 = 3.5089$$

Taking logarithm of both sides of this last equation, we obtain

$$2.5 \log_{10} x = \log_{10} 3.5089$$

or

$$x = 1.652 \text{ m}$$

B-5-6. From Figure 5-32 we obtain

$$C_1 \frac{dh_1}{dt} = q - q_1 \tag{1}$$

$$C_2 \frac{dh_2}{dt} = q_1 - q_2 \tag{2}$$

$$q_1 = \frac{h_1}{R_1} \tag{3}$$

$$q_2 = \frac{h_2}{R_2} \tag{4}$$

Using the electrical-liquid-level analogy given below, equations for an analogous electrical system can be obtained.

Electrical systems	Liquid-level systems
e (voltage)	q (flow rate)
q (charge)	h (head)
i (current)	dh/dt
C (capacitance)	R (resistance)
R (resistance)	C (capacitance)

Analogous equations for the electrical system are

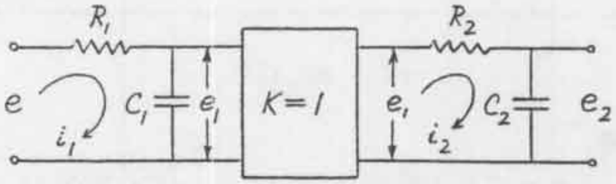
$$R_1 i_1 = e - e_1 \tag{5}$$

$$R_2 i_2 = e_1 - e_2 \tag{6}$$

$$e_1 = \frac{\int i_1 \, dt}{C_1} \tag{7}$$

$$e_2 = \frac{\int i_2 \, dt}{C_2} \tag{8}$$

Based on Equations (5) through (8), we obtain the analogous electrical system shown below.



B-5-7. The equations for the liquid-level system of Figure 5-20 are

$$C_1 \frac{dh_1}{dt} = q - q_1 \tag{1}$$

$$C_2 \frac{dh_2}{dt} = q_1 - q_2 \tag{2}$$

$$q_1 = \frac{h_1 - h_2}{R_1} \tag{3}$$

$$q_2 = \frac{h_2}{R_2} \tag{4}$$

Using the table of electrical-liquid-level analogy shown in the solution of Problem B-5-6, we can obtain an analogous electrical system. The analogous electrical equations are

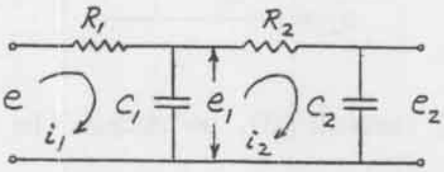
$$R_1 i_1 = e - e_1 \tag{5}$$

$$R_2 i_2 = e_1 - e_2 \tag{6}$$

$$e_1 = \frac{\int (i_1 - i_2) \, dt}{C_1} \tag{7}$$

$$e_2 = \frac{\int i_2 \, dt}{C_2} \tag{8}$$

Based on Equations (5) through (8), we obtain the analogous electrical system shown on next page.



B-5-8. $$pV = mR_{air}T$$

In this problem

$$p = 7 \times 10^5 + 1.0133 \times 10^5 = 8.0133 \times 10^5 \text{ N/m}^2 \text{ abs}$$

$$T = 273 + 20 = 293 \text{ K}$$

The mass m of the air in the tank is

$$m = \frac{pV}{R_{air}T} = \frac{8.0133 \times 10^5 \times 10}{287 \times 293} = 95.29 \text{ kg}$$

If the temperature of compressed air is raised to 40°C, then T = 273 + 40 = 313 K and the pressure p becomes

$$p = \frac{mR_{air}T}{V} = \frac{95.29 \times 287 \times 313}{10} = 8.560 \times 10^5 \text{ N/m}^2 \text{ abs}$$

$$= 7.547 \times 10^5 \text{ N/m}^2 \text{ gage} = 7.695 \text{ kg}_f/\text{cm}^2 \text{ gage}$$

$$= 109.4 \text{ lb}_f/\text{in.}^2 \text{ gage}$$

B-5-9. Note that

$$C\,dp_o = q\,dt$$

where q is the flow rate through the valve and is given by

$$q = \frac{p_i - p_o}{R}$$

Hence

$$C\frac{dp_o}{dt} = \frac{p_i - p_o}{R}$$

from which we obtain

$$\frac{P_o(s)}{P_i(s)} = \frac{1}{RCs + 1}$$

For the bellows and spring, we have the following equation:

$$Ap_o = kx$$

The transfer function $X(s)/P_i(s)$ is then given by

$$\frac{X(s)}{P_i(s)} = \frac{X(s)}{P_o(s)}\frac{P_o(s)}{p_i(s)} = \frac{A}{k}\frac{1}{RCs + 1}$$

B-5-10. Note that

$$p_1 = 0.5 \times 10^5 \text{ N/m}^2 \text{ gage} = 1.5133 \times 10^5 \text{ N/m}^2 \text{ abs}$$

$$p_2 = 0 \text{ N/m}^2 \text{ gage} = 1.0133 \times 10^5 \text{ N/m}^2 \text{ abs}$$

If $p_2 > 0.528p_1$, the speed of air flow is subsonic. So the flow throughout the system is subsonic. The flow rate through the inlet valve is

$$q_1 = K_1\sqrt{p_1 - p_2}$$

The flow rate through the outlet valve is

$$q_2 = K_2\sqrt{p_2 - p_3}$$

Since both valves have identical flow characteristics, we have $K_1 = K_2 = K$. The equation for the system is

$$C\,dp_2 = (q_1 - q_2)dt$$

or

$$C\frac{dp_2}{dt} = K\sqrt{p_1 - p_2} - K\sqrt{p_2 - p_3}$$

At steady state, we have $dp_2/dt = 0$ and this last equation becomes

$$K\sqrt{p_1 - p_2} = K\sqrt{p_2 - p_3}$$

or

$$p_1 - p_2 = p_2 - p_3$$

Hence

$$p_2 = \frac{p_1 + p_3}{2} = \frac{1.5133 \times 10^5 + 1.0133 \times 10^5}{2}$$

$$= 1.2633 \times 10^5 \text{ N/m}^2 \text{ abs} = 0.25 \times 10^5 \text{ N/m}^2 \text{ gage}$$

B-5-11. For the toggle joint shown in Figure 5-37, we have

$$\frac{R}{\frac{1}{2}F} = \frac{\ell_2}{\ell_1}$$

Hence

$$F = 2\frac{\ell_1}{\ell_2}R$$

B-5-12.

$$Q = 0.1\sqrt{H} = f(H)$$

$$= f(\bar{H}) + \left.\frac{df}{dH}\right|_{H=\bar{H}}(H - \bar{H}) + \frac{1}{2!}\left.\frac{d^2f}{dH^2}\right|_{H=\bar{H}}(H - \bar{H})^2 + \cdots$$

Neglecting the higher-order terms, a linearized equation for the system can be written as

$$Q - f(\bar{H}) = a(H - \bar{H})$$

where

$$f(\bar{H}) = f(4) = 0.2$$

$$a = \left.\frac{df}{dH}\right|_{H=\bar{H}=4} = 0.1\frac{1}{2\sqrt{\bar{H}}} = 0.025$$

Thus, a linearized equation becomes

$$Q - 0.2 = 0.025(H - 4)$$

B-5-13.

$$z = 5x^2 = f(x)$$

$$= f(\bar{x}) + \frac{df}{dx}(x - \bar{x}) + \frac{1}{2!}\frac{d^2f}{dx^2}(x - \bar{x})^2 + \cdots$$

A linearized equation for the system is

$$z - \bar{z} = a(x - \bar{x})$$

where $\bar{x} = 2$, $\bar{z} = 20$, and

$$a = \left.\frac{df}{dx}\right|_{x=2,\ z=20} = 10x\Big|_{x=2,\ z=20} = 20$$

Thus, a linearized equation becomes

$$z - 20 = 20(x - 2)$$

or

$$z - 20x = -20$$

B-5-14.

$$z = x^2 + 2xy + 5y^2 = f(x, y)$$

A linearized mathematical model is

$$z - \bar{z} = \frac{\partial f}{\partial x}(x - \bar{x}) + \frac{\partial f}{\partial y}(y - \bar{y})$$

where $\bar{x} = 11$, $\bar{y} = 5$, $\bar{z} = 356$, and

$$\frac{\partial f}{\partial x} = 2x + 2y \bigg|_{x = 11,\ y = 5} = 22 + 10 = 32$$

$$\frac{\partial f}{\partial y} = 2x + 10y \bigg|_{x = 11,\ y = 5} = 22 + 50 = 72$$

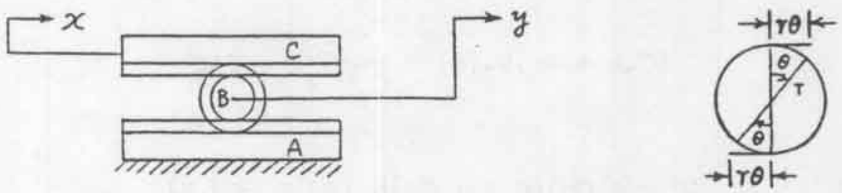
Thus, the linearized equation is

$$z - 356 = 32(x - 11) + 72(y - 5)$$

or

$$32x + 72y - z = 356$$

B-5-15. Define the radius and angle of rotation of the pinion as r and θ, respectively. Then, relative displacement between rack C and pinion B is $r\theta$.



Relative displacement between rack A and rack C is $2r\theta$ and this must equal displacement x. Therefore, we have

$$2r\theta = x$$

Since $x = r\theta + y$, we obtain

$$y = \frac{x}{2}$$

B-5-16.

$$P_2 = \frac{A_2}{A_1} P_1, \qquad x_2 = \frac{A_1}{A_2} x_1$$

B-5-17. The heat balance equations for the system are

$$C_1 d\theta_1 = (u - q_1)dt \tag{1}$$

$$C_2 d\theta_2 = (q_1 - q_2)dt \tag{2}$$

Noting that

$$q_1 = Gc\theta_1, \qquad q_2 = Gc\theta_2$$

Equations (1) and (2) can be modified to

$$C_1 \frac{d\theta_1}{dt} = u - Gc\theta_1 \tag{3}$$

$$C_2 \frac{d\theta_2}{dt} = Gc\theta_1 - Gc\theta_2 \tag{4}$$

from which we get

$$C_1 s\Theta_1(s) = U(s) - Gc\Theta_1(s)$$

$$C_2 s\Theta_2(s) = Gc\Theta_1(s) - Gc\Theta_2(s)$$

By eliminating $\Theta_1(s)$ from the preceding two equations, we obtain

$$(C_2 s + Gc)\Theta_2(s) = \frac{Gc}{C_1 s + Gc} U(s)$$

or

$$(C_1 s + Gc)(C_2 s + Gc)\Theta_2(s) = GcU(s)$$

Thus the transfer function $\Theta_2(s)/U(s)$ can be given by

$$\frac{\Theta_2(s)}{U(s)} = \frac{Gc}{(C_1 s + Gc)(C_2 s + Gc)}$$

CHAPTER 6

B-6-1. Define the current in the circuit as $i(t)$, where $t \geqslant 0$. The equation for the circuit for $t \geqslant 0$ is

$$(R_1 + R_2)i + \frac{1}{C_2}\int i\,dt = E$$

Since the capacitor is not charged for $t < 0$, the Laplace transform of this equation becomes

$$(R_1 + R_2)I(s) + \frac{1}{C_2 s} I(s) = \frac{E}{s}$$

Hence

$$I(s) = \frac{E}{R_1 + R_2 + \frac{1}{C_2 s}} \frac{1}{s} = \frac{EC_2}{(R_1 + R_2)C_2 s + 1}$$

Since

$$E_o(s) = (R_2 + \frac{1}{C_2 s})I(s)$$

we obtain

$$E_o(s) = \frac{R_2 C_2 s + 1}{C_2 s} \frac{EC_2}{(R_1 + R_2)C_2 s + 1}$$

$$= E\left[\frac{1}{s} - \frac{C_2 R_1}{(R_1 + R_2)C_2 s + 1}\right]$$

$$= E\left[\frac{1}{s} - \frac{R_1}{R_1 + R_2} \frac{1}{s + \frac{1}{(R_1 + R_2)C_2}}\right]$$

The inverse Laplace transform of $E_o(s)$ gives

$$e_o(t) = E\left\{1 - \frac{R_1}{R_1 + R_2} e^{-t/[(R_1 + R_2)C_2]}\right\}$$

B-6-2. The equation for the circuit is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E$$

or

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = E$$

Since $q(0) = 0$ and $i(0) = \dot{q}(0) = 0$, the Laplace transform of this last equation gives

$$Ls^2Q(s) + RsQ(s) + \frac{1}{C} Q(s) = \frac{E}{s}$$

or

$$Q(s) = \frac{E}{s(Ls^2 + Rs + \frac{1}{C})}$$

Since the current $i(t)$ is $dq(t)/dt$, we have

$$I(s) = sQ(s) = \frac{E}{Ls^2 + Rs + \frac{1}{C}}$$

The current $i(t)$ will be oscillatory if the two roots of the characteristic equation

$$s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$$

are complex conjugate. If two roots are real, then the current is not oscillatory.

Case 1 (Two roots of the characteristic equation are complex conjugate):

For this case, define

$$\omega_n = \sqrt{\frac{1}{LC}} \quad , \qquad \zeta = \frac{R\sqrt{C}}{2\sqrt{L}}$$

Then

$$I(s) = EC \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The inverse Laplace transform of $I(s)$ gives

$$i(t) = EC \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1 - \zeta^2}\, t)$$

$$= \frac{E\sqrt{C}}{\sqrt{L}} \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2}\, t) \quad (t \geqslant 0)$$

The current $i(t)$ approaches zero as t approaches infinity.

Case 2 (Two roots of the characteristic equation are real):

For this case define

$$s^2 + \frac{R}{L} s + \frac{1}{LC} = (s + a)(s + b)$$

Then

$$I(s) = \frac{E}{L} \frac{1}{(s + a)(s + b)}$$

The inverse Laplace transform of I(s) gives

$$i(t) = \frac{E}{L} \frac{1}{b - a} (e^{-at} - e^{-bt})$$

Notice that

$$a = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\ , \qquad b = \frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

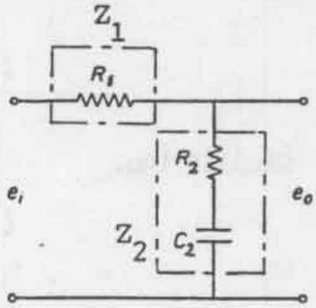
Hence

$$\frac{E}{L} \frac{1}{b - a} = - \frac{E}{2\sqrt{\left(\frac{R}{2}\right)^2 - \frac{L}{C}}}$$

The current i(t) can thus be given by

$$i(t) = \frac{E}{2\sqrt{\left(\frac{R}{2}\right)^2 - \frac{L}{C}}} \left\{ \exp\left[- \left(\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) t\right] - \exp\left[- \left(\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right) t\right] \right\} \qquad (t \geqslant 0)$$

<u>B-6-3</u>. Referring to the circuit diagram shown to the right, we have



$$Z_1 = R_1\ , \qquad Z_2 = R_2 + \frac{1}{C_2 s}$$

Hence

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2 C_2 s + 1}{(R_1 + R_2)C_2 s + 1}$$

Next, we shall find the response $e_o(t)$ when the input $e_i(t)$ is the unit step function of magnitude E_i. Since

$$E_o(s) = \frac{R_2 C_2 s + 1}{(R_1 + R_2)C_2 s + 1} \frac{E_i}{s}$$

$$= E_i \left[\frac{1}{s} - \frac{R_1 C_2}{(R_1 + R_2)C_2 s + 1}\right]$$

the inverse Laplace transform of $E_o(s)$ is

$$e_o(t) = E_i \left\{1 - \frac{R_1}{R_1 + R_2} e^{-t/[(R_1 + R_2)C_2]}\right\}$$

which gives the response to the step input of magnitude E_i.

B-6-4. The system equations are

$$k_1(x_i - y) = b_1(\dot{y} - \dot{x}_o)$$

$$b_1(\dot{y} - \dot{x}_o) = k_2 x_o$$

which can be rewritten as

$$b_1\dot{y} + k_1 y = k_1 x_i + b_1\dot{x}_o$$

$$b_1\dot{x}_o + k_2 x_o = b_1\dot{y}$$

Noting that $x_o(0-) = 0$ and $y(0-) = 0$, by taking the $\mathcal{L}_-$ transform of these two equations we obtain

$$(b_1 s + k_1)Y(s) = k_1 X_i(s) + b_1 s X_o(s)$$

$$(b_1 s + k_2)X_o(s) = b_1 s Y(s)$$

By eliminating Y(s) from the preceding two equations, we get

$$(b_1 s + k_1)\frac{b_1 s + k_2}{b_1 s} X_o(s) = k_1 X_i(s) + b_1 s X_o(s)$$

Simplifying,

$$[(k_1 + k_2)b_1 s + k_1 k_2]\, X_o(s) = k_1 b_1 s X_i(s)$$

or

$$\frac{X_o(s)}{X_i(s)} = \frac{k_1 b_1 s}{(k_1 + k_2)b_1 s + k_1 k_2} = \frac{\frac{b_1}{k_2} s}{\left(\frac{1}{k_1} + \frac{1}{k_2}\right) b_1 s + 1}$$

The response of the system to $x_i(t) = X_i 1(t)$ can be obtained by taking the inverse Laplace transform of

$$X_o(s) = \frac{(b_1/k_2)s}{\left(\frac{1}{k_1} + \frac{1}{k_2}\right) b_1 s + 1} \frac{X_i}{s}$$

$$= \frac{k_1 X_i}{k_1 + k_2} \frac{1}{s + \dfrac{1}{\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) b_1}}$$

as follows:

$$x_o(t) = \frac{k_1 X_i}{k_1 + k_2} \exp\left\{-t/[(\frac{1}{k_1} + \frac{1}{k_2})b_1]\right\} \qquad t \geqslant 0$$

B-6-5. The equations of motion for the system are

$$k_1(x_i - x_o) = b_2(\dot{x}_o - \dot{y})$$

$$b_2(\dot{x}_o - \dot{y}) = k_2 y$$

Rewriting these equations,

$$b_2\dot{x}_o + k_1 x_o = k_1 x_i + b_2\dot{y}$$

$$b_2\dot{y} + k_2 y = b_2\dot{x}_o$$

Noting that $x(0-) = 0$ and also $y(0-) = 0$, $\mathcal{L}_-$ transforms of these two equations become

$$(b_2 s + k_1)X_o(s) = k_1 X_i(s) + b_2 sY(s)$$

$$(b_2 s + k_2)Y(s) = b_2 sX_o(s)$$

Eliminating $Y(s)$ from the last two equations, we obtain

$$(b_2 s + k_1)X_o(s) = k_1 X_i(s) + b_2 s \frac{b_2 sX_o(s)}{b_2 s + k_2}$$

which can be simplified as

$$[(k_1 + k_2)b_2 s + k_1 k_2]X_o(s) = k_1(b_2 s + k_2)X_i(s)$$

Hence

$$\frac{X_o(s)}{X_i(s)} = \frac{k_1(b_2 s + k_2)}{(k_1 + k_2)b_2 s + k_1 k_2}$$

Since the input $x_i(t)$ is given as

$$x_i(t) = X_i \qquad 0 < t < t_1$$
$$= 0 \qquad \text{elsewhere}$$

we have

$$X_i(s) = \frac{X_i}{s}(1 - e^{-t_1 s})$$

The response $X_o(s)$ is then obtained as

$$X_o(s) = \frac{k_1(b_2 s + k_2)}{(k_1 + k_2)b_2 s + k_1 k_2}\frac{X_i}{s}(1 - e^{-t_1 s})$$

Since

$$\frac{k_1(b_2 s + k_2)}{(k_1 + k_2)b_2 s + k_1 k_2}\frac{1}{s} = \frac{1}{s} + \frac{-\frac{b_2}{k_1}}{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)b_2 s + 1}$$

we have

$$\mathcal{L}^{-1}\left[\frac{k_1(b_2 s + k_2)}{(k_1 + k_2)b_2 s + k_1 k_2}\frac{1}{s}\right] = 1 - \frac{k_2}{k_1 + k_2}\exp\left\{-t/[(\frac{1}{k_1} + \frac{1}{k_2})b_2]\right\}$$

Hence

$$x_o(t) = X_i\left[1 - \frac{k_2}{k_1 + k_2}\exp\left\{-t/[(\frac{1}{k_1} + \frac{1}{k_2})b_2]\right\}\right]$$
$$- X_i\left[1 - \frac{k_2}{k_1 + k_2}\exp\left\{-(t - t_1)/[(\frac{1}{k_1} + \frac{1}{k_2})b_2]\right\}\right]1(t - t_1)$$

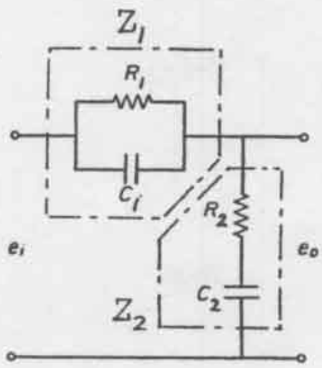
B-6-6. First note that

$$\frac{1}{Z_1} = \frac{1}{R_1} + C_1 s, \qquad Z_2 = R_2 + \frac{1}{C_2 s} = \frac{R_2 C_2 s + 1}{C_2 s}$$

Then

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{(R_2 C_2 s + 1)(R_1 C_1 s + 1)}{R_1 C_2 s + (R_2 C_2 s + 1)(R_1 C_1 s + 1)}$$

or



$$\frac{E_o(s)}{E_i(s)} = \frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}$$

Since $R_2 = 1.5\ R_1$, $C_2 = C_1$, and $R_1C_1 = 1$, we obtain

$$R_1C_2 = R_1C_1 = 1, \qquad R_2C_2 = 1.5\ R_1C_1 = 1.5$$

and the transfer function $E_o(s)/E_i(s)$ becomes

$$\frac{E_o(s)}{E_i(s)} = \frac{(s + 1)(1.5s + 1)}{1.5\ s^2 + 3.5s + 1} = \frac{(s + 1)[s + (2/3)]}{(s + 2)[s + (1/3)]}$$

Since the input $e_i(t)$ is given by

$$\begin{aligned} e_i(t) &= E_i \qquad 0 < t < t_1 \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

we have

$$e_i(t) = E_i[1(t) - 1(t - t_1)]$$

Hence, the response $e_o(t)$ can be obtained as follows:

$$E_o(s) = \frac{(s + 1)[s + (2/3)]}{(s + 2)[s + (1/3)]}\ E_i \left(\frac{1}{s} - \frac{1}{s}\ e^{-st_1}\right)$$

Since

$$\mathcal{L}^{-1}\left[\frac{(s + 1)[s + (2/3)]}{(s + 2)[s + (1/3)]s}\right] = \mathcal{L}^{-1}\left[\frac{0.4}{s + 2} - \frac{0.4}{s + (1/3)} + \frac{1}{s}\right]$$
$$= 0.4\ e^{-2t} - 0.4\ e^{-(1/3)t} + 1$$

we obtain the response $e_o(t)$ as follows:

$$\begin{aligned} e_o(t) = E_i\ &(0.4\ e^{-2t} - 0.4\ e^{-(1/3)t} + 1) \\ &- E_i\ [0.4\ e^{-2(t - t_1)} - 0.4\ e^{-(t - t_1)/3} + 1] \\ &\cdot 1(t - t_1) \end{aligned}$$

B-6-7.

$$\text{logarithmic decrement} = \ln \frac{x_o}{x_1} = \frac{1}{n} \ln \frac{x_o}{x_n} = \zeta \omega_n T$$

$$= \zeta \omega_n \frac{2\pi}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Thus

$$\frac{1}{n} \ln \frac{x_o}{x_n} = \frac{2\zeta\pi}{\sqrt{1-\zeta^2}}$$

or

$$\left(\frac{1}{n} \ln \frac{x_o}{x_n}\right)^2 (1 - \zeta^2) = 4\zeta^2\pi^2$$

Solving for ζ,

$$\zeta = \frac{\frac{1}{n} \ln \frac{x_o}{x_n}}{\sqrt{4\pi^2 + \left[\frac{1}{n}\left(\ln \frac{x_o}{x_n}\right)\right]^2}}$$

By substituting $n = 4$ and $x_o/x_4 = 4$ into this last equation, we obtain

$$\zeta = \frac{\frac{1}{4} \ln 4}{\sqrt{4\pi^2 + \left[\frac{1}{4}(\ln 4)\right]^2}}$$

$$= \frac{\frac{1}{4} \times 1.386}{\sqrt{39.48 + 0.12}} = \frac{0.3466}{6.293} = 0.055$$

Noting that $\omega_n = \sqrt{k/m}$ and $2\zeta\omega_n = b/m$, we find

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{500}{1}} = 22.36 \text{ rad/s}$$

and

$$b = 2\zeta\omega_n m = 2 \times 0.055 \times 22.36 \times 1 = 2.46 \text{ N-s/m}$$

B-6-8. The system equation is

$$(m + 2)\ddot{x} + b\dot{x} + kx = p$$

Substituting the given numerical values m = 20 kg and p = 2g = 2 x 9.81 N into this last equation, we obtain

$$22\ddot{x} + b\dot{x} + kx = 2 \times 9.81$$

At steady state

$$kx_{ss} = 2 \times 9.81$$

From Figure 6-48 (b), $x_{ss} = 0.08$ m. Thus

$$x_{ss} = \frac{2 \times 9.81}{k} = 0.08$$

Solving for k, we obtain

$$k = \frac{2 \times 9.81}{0.08} = 245 \text{ N/m}$$

Since

$$\omega_n = \sqrt{\frac{k}{22}} = \sqrt{\frac{245}{22}} = 3.34 \text{ rad/s}, \qquad 2\zeta\omega_n = \frac{b}{22}$$

we obtain

$$b = 2\zeta\omega_n \times 22 = 2 \times 0.4 \times 3.34 \times 22 = 58.8 \text{ N-s/m}$$

B-6-9. For the x direction, the equation of motion is

$$m_1 \frac{d^2}{dt^2} x + m_2 \frac{d^2}{dt^2}(x + \ell \sin\theta) = -2kx$$

For the rotational motion of the pendulum,

$$m_2 \left[\frac{d^2}{dt^2}(x + \ell\sin\theta)\right]\ell\cos\theta + m_2\left[\frac{d^2}{dt^2}(-\ell\cos\theta)\right]\ell\sin\theta = -m_2 g\ell\sin\theta$$

Rewriting the preceding two equations,

$$m_1\ddot{x} + m_2(\ddot{x} - \ell\sin\theta\,\dot{\theta}^2 + \ell\cos\theta\,\ddot{\theta}) = -2kx$$

$$m_2(\ddot{x} - \ell\sin\theta\,\dot{\theta}^2 + \ell\cos\theta\,\ddot{\theta})\ell\cos\theta$$
$$+ m_2\ell(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta})\ell\sin\theta = -m_2 g\ell\sin\theta$$

Simplifying,

$$\ddot{x} + \frac{m_2 \ell}{m_1 + m_2} \cos\theta\, \ddot{\theta} - \frac{m_2 \ell}{m_1 + m_2} \sin\theta\, \dot{\theta}^2 + \frac{2k}{m_1 + m_2} x = 0$$

$$m_2 \ddot{x} \cos\theta + m_2 \ell \cos^2\theta\, \ddot{\theta} + m_2 \ell \sin^2\theta\, \ddot{\theta} + m_2 g \sin\theta = 0$$

Thus the equations of motion for the system are

$$\ddot{x} + \frac{m_2 \ell}{m_1 + m_2} \cos\theta\, \ddot{\theta} - \frac{m_2 \ell}{m_1 + m_2} \sin\theta\, \dot{\theta}^2 + \frac{2k}{m_1 + m_2} x = 0$$

$$\ddot{\theta} + \frac{\ddot{x}}{\ell} \cos\theta + \frac{g}{\ell} \sin\theta = 0$$

B-6-10. The equation of motion for the system is

$$m\ddot{x} + kx = p(t)$$

where $m = 1$ kg, $k = 100$ N/m, $p(t) = 10\,\delta(t)$ N, $x(0-) = 0.1$ m, and $\dot{x}(0-) = 1$ m/s. By substituting the given numerical values into the system equation, we obtain

$$\ddot{x} + 100\, x = 10\ \delta(t)$$

Taking the $\mathcal{L}_-$ transform of this last equation gives

$$[s^2 X(s) - sx(0-) - \dot{x}(0-)] + 100\, X(s) = 10$$

or

$$(s^2 + 100)X(s) = 10 + 0.1s + 1 = 11 + 0.1s$$

Solving for X(s) gives

$$X(s) = \frac{11 + 0.1s}{s^2 + 10^2}$$

The inverse Laplace transform of X(s) gives

$$x(t) = \frac{11}{10} \sin 10t + 0.1 \cos 10t$$

B-6-11. When mass m is set into motion by a unit impulse force, the system equation is

$$m\ddot{x} + kx = \delta(t)$$

Let us define another impulse force to stop the motion as $A\,\delta(t - T)$, where A is the undetermined magnitude of the impulse force and $t = T$ is the undetermined instant that this impulse is to be given to the system. Then, the equation of motion for the system when the two impulse forces are given is

$$m\ddot{x} + kx = \delta(t) + A\,\delta(t - T), \qquad x(0-) = 0, \qquad \dot{x}(0-) = 0$$

The $\mathcal{L}_-$ transform of this last equation gives

$$(ms^2 + k)X(s) = 1 + A\,e^{-sT}$$

Solving for X(s),

$$X(s) = \frac{1}{ms^2 + k} + \frac{A\,e^{-sT}}{ms^2 + k}$$

$$= \frac{1}{\sqrt{km}} \frac{\sqrt{\frac{k}{m}}}{s^2 + \frac{k}{m}} + \frac{A}{\sqrt{km}} \frac{\sqrt{\frac{k}{m}}\,e^{-sT}}{s^2 + \frac{k}{m}}$$

The inverse Laplace transform of X(s) is

$$x(t) = \frac{1}{\sqrt{km}} \sin\sqrt{\frac{k}{m}}\,t + \frac{A}{\sqrt{km}} \sin\left[\sqrt{\frac{k}{m}}\,(t - T)\right] 1(t - T)$$

If the motion of mass m is to be stopped at t = T, then x(t) must be identically zero for $t \geqslant T$.

Note that x(t) can be made identically zero for $t \geqslant T$ if we choose

$$A = 1, \qquad T = \frac{\pi}{\sqrt{\frac{k}{m}}}, \quad \frac{3\pi}{\sqrt{\frac{k}{m}}}, \quad \frac{5\pi}{\sqrt{\frac{k}{m}}}, \ldots$$

Thus, the motion of mass m can be stopped by another impulse force, such as

$$\delta\left(t - \frac{\pi}{\sqrt{k/m}}\right), \qquad \delta\left(t - \frac{3\pi}{\sqrt{k/m}}\right), \qquad \delta\left(t - \frac{5\pi}{\sqrt{k/m}}\right).$$

<u>B-6-12</u>. The system equation is

$$m\ddot{x} + b\dot{x} = \delta(t), \qquad x(0-) = 0, \qquad \dot{x}(0-) = 0$$

The $\mathcal{L}_-$ transform of this equation is

$$(ms^2 + bs)X(s) = 1$$

Solving for X(s), we get

$$X(s) = \frac{1}{ms^2 + bs} = \frac{1}{m} \frac{1}{s(s + \frac{b}{m})} = \frac{1}{b}\left(\frac{1}{s} - \frac{1}{s + \frac{b}{m}}\right)$$

The response x(t) of the system is

$$x(t) = \frac{1}{b}[1 - e^{-(b/m)t}]$$

The velocity $\dot{x}(t)$ is

$$\dot{x}(t) = \frac{1}{m} e^{-(b/m)t}$$

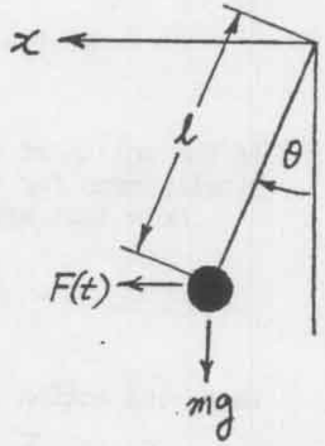
Thus

$$\dot{x}(0) = \frac{1}{m}$$

The initial velocity can also be obtained by use of the initial value theorem.

$$\dot{x}(0+) = \lim_{s \to \infty} s^2 X(s) = \lim_{s \to \infty} \frac{s^2}{ms^2 + bs} = \frac{1}{m}$$

B-6-13. The moment of inertia of the pendulum about the pivot is $J = m\ell^2$. The angle of rotation of the pendulum is θ rad. Define the force that acts on the pendulum at the time of sudden stop as $F(t)$. Then, the torque that acts on the pendulum due to the force $F(t)$ is $F(t)\ell \cos \theta$. The equation for the pendulum system can be given by



$$m \ell^2 \ddot{\theta} = F(t) \ell \cos \theta - mg \ell \sin \theta \qquad (1)$$

We shall linearize this nonlinear equation by assuming angle θ is small. (Although $\theta = 20°$ is not quite small, the resulting linearized equation will give an approximate solution.) By approximating $\cos \theta \doteqdot 1$ and $\sin \theta \doteqdot \theta$, Equation (1) can be written as

$$m \ell^2 \ddot{\theta} = F(t) \ell - mg \ell \theta$$

or

$$m \ell \ddot{\theta} + mg\theta = F(t) \qquad (2)$$

Since the velocity of the car at $t = 0-$ is 10 m/s and the car stops in 0.3 s, the deceleration is 33.3 m/s^2.

By assuming a constant acceleration of magnitude 33.3 m/s^2 to act on the mass for 0.3 seconds, $F(t)$ may be given by

$$F(t) = m\ddot{x} = 33.3\, m\, [1(t) - 1(t - 0.3)]$$

Then, Equation (2) may be written as

$$m \ell \ddot{\theta} + mg\theta = 33.3\, m\, [1(t) - 1(t - 0.3)]$$

or

$$\ddot{\theta} + \frac{g}{\ell} \theta = \frac{33.3}{\ell} [1(t) - 1(t - 0.3)]$$

Since $\ell = 0.05$ m, this last equation becomes

$$\ddot{\theta} + 196.2\,\theta = 666\,[1(t) - 1(t - 0.3)]$$

Taking Laplace transforms of both sides of this last equation, we obtain

$$(s^2 + 196.2)\Theta(s) = 666\left(\frac{1}{s} - \frac{1}{s}\,e^{-0.3s}\right) \tag{3}$$

where we used the initial conditions that $\theta(0-) = 0$ and $\dot{\theta}(0-) = 0$. Solving Equation (3) for $\Theta(s)$,

$$\Theta(s) = \frac{666}{s(s^2 + 196.2)}\,(1 - e^{-0.3s})$$

$$= \left(\frac{1}{s} - \frac{s}{s^2 + 196.2}\right)\frac{666}{196.2}\,(1 - e^{-0.3s})$$

The inverse Laplace transform of $\Theta(s)$ gives

$$\theta(t) = 3.394\,(1 - \cos 14t) - 3.394\left\{1(t - 0.3) - [\cos 14(t - 0.3)]1(t - 0.3)\right\} \tag{4}$$

Note that $1(t - 0.3) = 0$ for $0 \leqslant t < 0.3$.

Let us assume that at $t = t_1$, $\theta = 20° = 0.3491$ rad. Then by tentatively assuming that t_1 occurs before $t = 0.3$, we solve the following equation for t_1:

$$0.3491 = 3.394(1 - \cos 14t_1)$$

which can be simplified to

$$\cos 14t_1 = 0.8971$$

The result is

$$t_1 = 0.0327 \text{ s}$$

Since $t_1 = 0.0327 < 0.3$, our assumption was correct. The terms involving $1(t - 0.3)$ in Equation (4) do not affect the value of t_1. It takes approximately 33 ms for the pendulum to swing 20°.

B-6-14. The equation of motion for this system is

$$(M + 2)\ddot{x} + b\dot{x} + kx = 2g$$

By substituting the numerical values for M, b, k, and g into this equation, we obtain

$$12\ddot{x} + 40\dot{x} + 400x = 2 \times 9.81$$

By taking the Laplace transform of this last equation assuming zero initial conditions, we obtain

$$12s^2X(s) + 40sX(s) + 400X(s) = \frac{19.62}{s}$$

or

$$X(s) = \frac{19.62}{s(12s^2 + 40s + 400)}$$

$$= \frac{1.635}{s(s^2 + 3.3333s + 33.3333)}$$

$$= 1.635\left(\frac{0.03}{s} - \frac{0.03s + 0.1}{s^2 + 3.3333s + 33.3333}\right)$$

$$= 0.04905\left[\frac{1}{s} - \frac{s + 1.6666}{(s + 1.6666)^2 + (5.5277)^2}\right.$$

$$\left. - 0.3015\frac{5.5277}{(s + 1.6666)^2 + (5.5277)^2}\right]$$

The inverse Laplace transform of X(s) gives

$$x(t) = 0.04905(1 - e^{-1.6666t}\cos 5.5277t - 0.3015\,e^{-1.6666t}\sin 5.5277t)$$

Next, we shall obtain the response curve x(t) versus t with MATLAB. Note that X(s) can be written as

$$X(s) = \frac{1.635}{s^2 + 3.3333s + 33.3333}\frac{1}{s}$$

Now define

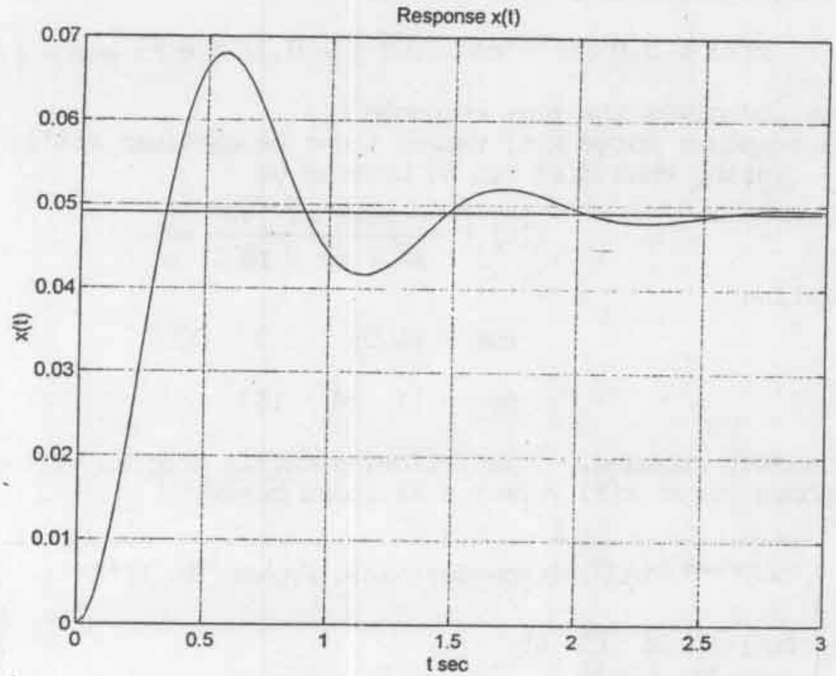
num = [0 0 1.635]

den = [1 3.3333 33.3333]

A possible MATLAB program to obtain the response curve is given below.

```
% ***** MATLAB program to solve Problem B-6-14 *****

num = [0  0  1.635];
den = [1  3.3333  33.3333];
step(num,den)
grid
title('Response x(t)')
xlabel('t sec')
ylabel('x(t)')
```

The resulting response curve x(t) versus t is shown below.



B-6-15. The equation of motion for the system is

$$m\ddot{x} + b_1\dot{x} + (k_1 + k_2)x = 0$$

By substituting the numerical values of m, k_1, k_2, and b_1 into this equation, we obtain

$$\ddot{x} + 4\dot{x} + 16x = 0$$

Laplace transforming this equation, we get

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 4[sX(s) - x(0)] + 16X(s) = 0$$

or

$$(s^2 + 4s + 16)X(s) = sx(0) + \dot{x}(0) + 4x(0)$$

Solving this equation for X(s),

$$X(s) = \frac{sx(0) + \dot{x}(0) + 4x(0)}{s^2 + 4s + 16}$$

Since $x(0) = 0.05$ and $\dot{x}(0) = 1$, X(s) becomes

$$X(s) = \frac{0.05s + 1.2}{s^2 + 4s + 16}$$

$$= \frac{0.05(s + 2)}{(s + 2)^2 + (2\sqrt{3})^2} + \frac{0.3175 \times 2\sqrt{3}}{(s + 2)^2 + (2\sqrt{3})^2}$$

The inverse Laplace transform of X(s) gives

$$x(t) = 0.05\ e^{-2t} \cos 2\sqrt{3}\ t + 0.3175\ e^{-2t} \sin 2\sqrt{3}\ t$$

This equation gives the time response x(t).

The response curve x(t) versus t can be obtained easily by use of MATLAB. Noting that X(s) can be written as

$$X(s) = \frac{0.05s^2 + 1.2s}{s^2 + 4s + 16} \frac{1}{s}$$

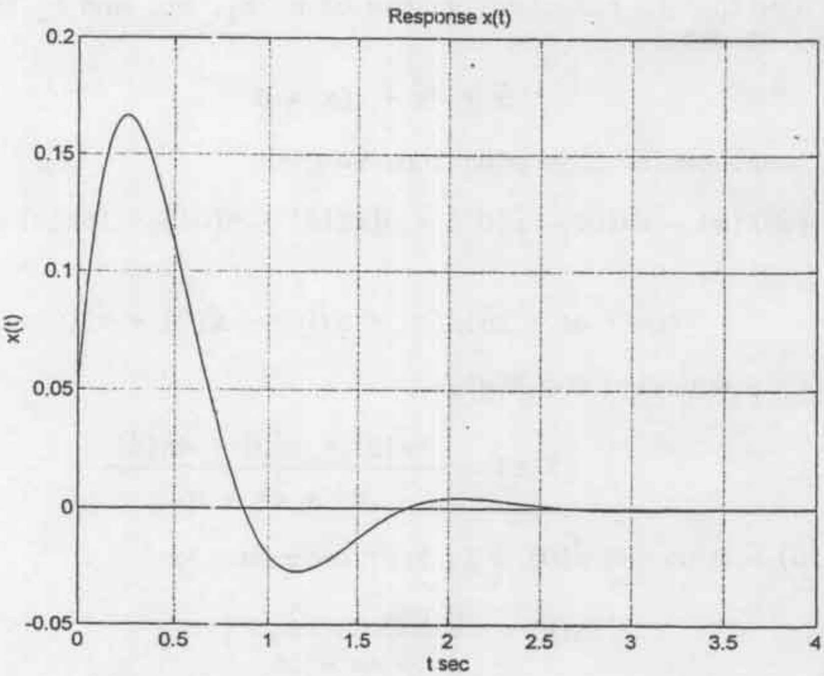
we may define

num = [0.05   1.2   0]

den = [1   4   16]

and use a step command. The following MATLAB program will generate the response curve x(t) versus t as shown below.

```
% ***** MATLAB program to solve Problem B-6-15 *****

num = [0.05   1.2   0];
den = [1   4   16];
step(num,den)
grid
title('Response x(t)')
xlabel('t sec')
ylabel('x(t)')
```



B-6-16.

$$\ddot{x} + 2\dot{x} + 10x = \dot{u} + 5u, \qquad x(0-) = 0,\ \dot{x}(0-) = 0$$

The transfer function of the system is

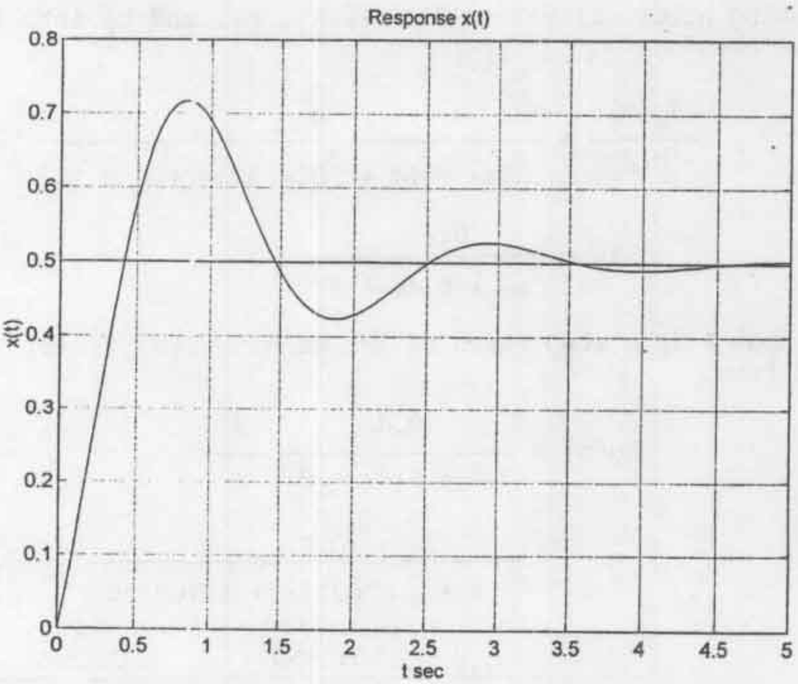
$$\frac{X(s)}{U(s)} = \frac{s + 5}{s^2 + 2s + 10}$$

Since u(t) is a unit-step function, we have U(s) = 1/s. Hence

$$X(s) = \frac{s + 5}{s^2 + 2s + 10} \frac{1}{s}$$

The following MATLAB program will generate the response curve x(t) versus t as shown in the figure below.

```
% ***** MATLAB program to solve Problem B-6-16 *****

num = [0  1  5];
den = [1  2  10];
step(num,den)
grid
title('Response x(t)')
xlabel('t sec')
ylabel('x(t)')
```



B-6-17. In this system F is the input and x_2 is the output. From Figure 6-57, we obtain the following equations:

$$b_1\dot{x}_1 + k_1(x_1 - x_2) = F$$

$$k_1(x_1 - x_2) = b_2\dot{x}_2 + k_2x_2$$

Laplace transforming these two equations, assuming zero initial conditions, we obtain

$$(b_1s + k_1)X_1(s) - k_1X_2(s) = F(s)$$

$$k_1X_1(s) = (b_2s + k_1 + k_2)X_2(s)$$

By eliminating $X_1(s)$ from these two equations, we get

$$(b_1s + k_1)\,\frac{b_2s + k_1 + k_2}{k_1}\,X_2(s) - k_1X_2(s) = F(s)$$

Simplifying this last equation, we get

$$[b_1b_2s^2 + (b_1k_1 + b_1k_2 + b_2k_1)s + k_1k_2]X_2(s) = k_1F(s)$$

from which we obtain

$$\frac{X_2(s)}{F(s)} = \frac{k_1}{b_1b_2s^2 + (b_1k_1 + b_1k_2 + b_2k_1)s + k_1k_2}$$

By substituting numerical values for k_1, k_2, b_1, and b_2 into this last equation, we obtain

$$\frac{X_2(s)}{F(s)} = \frac{4}{10s^2 + (4 + 20 + 40)s + 4 \times 20}$$

$$= \frac{0.4}{s^2 + 6.4s + 8}$$

Since the input F is a step force of 2N, we have $F(s) = 2/s$. $X_2(s)$ can be obtained from

$$X_2(s) = \frac{0.4}{s^2 + 6.4s + 8}\,\frac{2}{s}$$

$$= \frac{0.8}{(s + 4.6967)(s + 1.7033)s}$$

$$= \frac{0.1}{s} + \frac{0.0569}{s + 4.6967} + \frac{-0.1569}{s + 1.7033}$$

The inverse Laplace transform of $X_2(s)$ gives

$$x_2(t) = 0.1 + 0.0569\ e^{-4.6967t} - 0.1569\ e^{-1.7033t}$$

The response curve $x_2(t)$ versus t can be obtained with MATLAB as follows: First note that

$$X_2(s) = \frac{0.8}{s^2 + 6.4s + 8}\ \frac{1}{s}$$

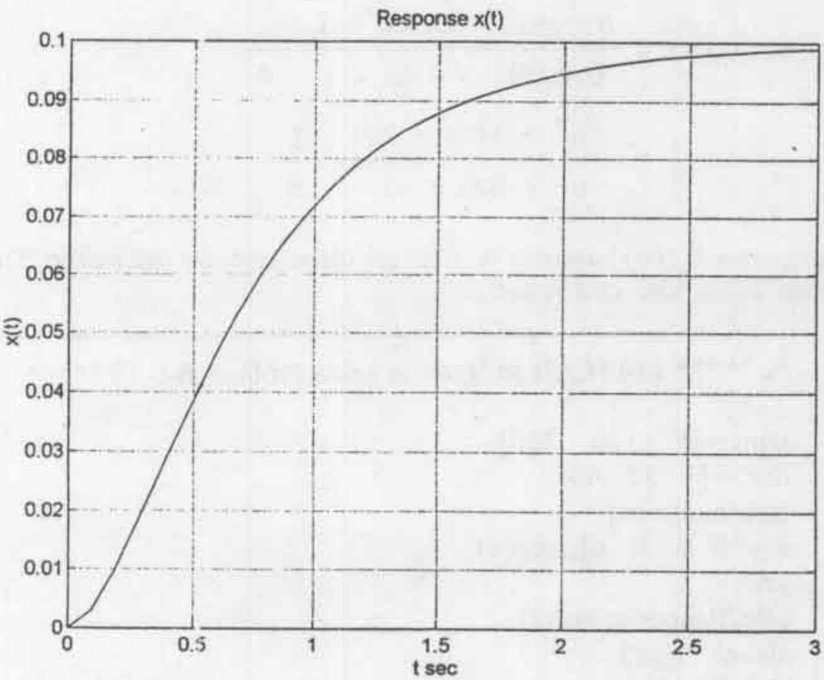
Then, define

num = [0 0 0.8]

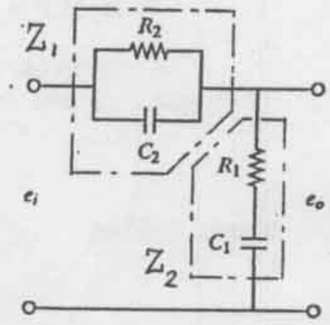
den = [1 6.4 8]

and use a step command. The following MATLAB program will yield the response curve $x_2(t)$. The resulting response curve is shown in the figure below.

```
% ***** MATLAB program to solve Problem B-6-17 *****

num = [0  0  0.8];
den = [1  6.4  8];
step(num,den)
grid
title('Response x(t)')
xlabel('t sec')
ylabel('x(t)')
```



B-6-18. Referring to the figure shown to the right, we have



$$\frac{1}{Z_1(s)} = \frac{1}{R_2} + C_2 s, \qquad Z_2(s) = R_1 + \frac{1}{C_1 s}$$

Hence

$$Z_1(s) = \frac{R_2}{R_2 C_2 s + 1}, \qquad Z_2(s) = \frac{R_1 C_1 s + 1}{C_1 s}$$

Thus

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}$$

$$= \frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_2C_1s + (R_1C_1s + 1)(R_2C_2s + 1)}$$

$$= \frac{(R_1C_1s + 1)(R_2C_2s + 1)}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_2C_1)s + 1}$$

By substituting the given numerical values for R_1, R_2, C_1, and C_2, we obtain

$$\frac{E_o(s)}{E_i(s)} = \frac{(0.5s + 1)(0.05s + 1)}{0.025s^2 + 0.8s + 1}$$

When $e_i(t) = 5$ V (step input) is applied to the system, we have

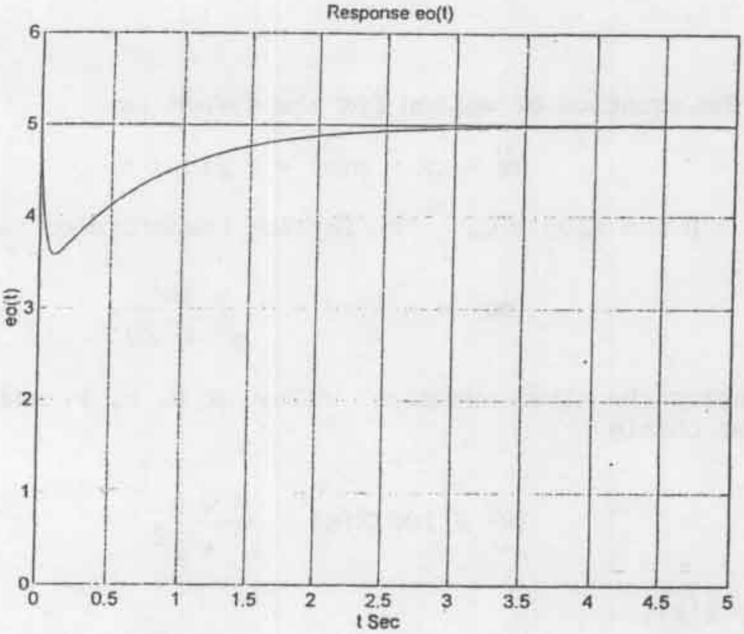
$$E_o(s) = \frac{0.025s^2 + 0.55s + 1}{0.025s^2 + 0.8s + 1}\,\frac{5}{s}$$

$$= \frac{5s^2 + 110s + 200}{s^2 + 32s + 40}\,\frac{1}{s}$$

The response curve $e_o(t)$ versus t can be obtained by entering the following MATLAB program into the computer.

```
% ***** MATLAB program to solve Problem B-6-18 *****

num = [5  110  200];
den = [1  32  40];
step(num,den);
v = [0  5  0  6]; axis(v);
grid
title('Response eo(t)')
xlabel('t Sec')
ylabel('eo(t)')
```

The resulting response curve $e_o(t)$ versus t is shown below.



Note that

$$E_o(s) = \frac{5s^2 + 110s + 200}{s^2 + 32s + 40} \frac{1}{s}$$

$$= \frac{1.7010}{s + 30.6969} - \frac{1.7010}{s + 1.3031} + \frac{5}{s}$$

Hence

$$e_o(t) = 1.7010(e^{-30.6969t} - e^{-1.3031t}) + 5$$

Notice that the response curve is a sum of two exponential curves and a step function of magnitude 5.

CHAPTER 7

B-7-1. The equation of motion for the system is

$$m\ddot{x} + kx = p(t) = P \sin \omega t$$

where $x(0) = 0$ and $\dot{x}(0) = 0$. The Laplace transform of this equation is

$$(ms^2 + k)X(s) = P \frac{\omega}{s^2 + \omega^2}$$

By substituting the given numerical values of m, k, P, and ω into this last equation, we obtain

$$(s^2 + 100)X(s) = \frac{5 \times 2}{s^2 + 2^2}$$

Solving for X(s),

$$X(s) = \frac{10}{(s^2 + 100)(s^2 + 4)} = \frac{5}{96} \frac{2}{s^2 + 4} - \frac{1}{96} \frac{10}{s^2 + 100}$$

The inverse Laplace transform of X(s) gives the response x(t).

$$x(t) = \frac{5}{96} (\sin 2t - \frac{1}{5} \sin 10t)$$

B-7-2. The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = p(t) = P \sin \omega t, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

By substituting the given numerical values into this equation, we get

$$2\ddot{x} + 24\dot{x} + 200x = 5 \sin 6t$$

or

$$\ddot{x} + 12\dot{x} + 100x = 2.5 \sin 6t$$

Taking the Laplace transform of this last equation, we obtain

$$(s^2 + 12s + 100)X(s) = 2.5 \frac{6}{s^2 + 6^2}$$

Solving for X(s),

$$X(s) = \frac{15}{(s^2 + 12s + 100)(s^2 + 36)}$$

$$= \frac{\frac{9}{464}s + \frac{15}{116}}{s^2 + 12s + 100} + \frac{-\frac{9}{464}s + \frac{3}{29}}{s^2 + 36}$$

$$= \frac{\frac{3}{232 \times 8} \times 8 + \frac{9}{464}(s + 6)}{(s + 6)^2 + 8^2} + \frac{3}{29 \times 6}\frac{6}{s^2 + 36}$$

$$- \frac{9}{464}\frac{s}{s^2 + 36}$$

The inverse Laplace transform of X(s) gives

$$x(t) = \frac{3}{1856}e^{-6t}\sin 8t + \frac{9}{464}e^{-6t}\cos 8t + \frac{1}{58}\sin 6t - \frac{9}{464}\cos 6t$$

B-7-3.

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{1}{RCs + 1}$$

Hence

$$G(j\omega) = \frac{E_o(j\omega)}{E_i(j\omega)} = \frac{1}{RCj\omega + 1}$$

Thus

$$\left|G(j\omega)\right| = \frac{1}{\sqrt{R^2C^2\omega^2 + 1}}, \quad \underline{/G(j\omega)} = -\tan^{-1} RC\omega$$

For the input $e_i(t) = E_i \sin \omega t$, the steady-state output $e_o(t)$ is given by

$$e_o(t) = \frac{E_i}{\sqrt{R^2C^2\omega^2 + 1}}\sin(\omega t - \tan^{-1} RC\omega)$$

B-7-4. The equations of motion for the system are

$$m_1\ddot{x}_1 + b(\dot{x}_1 - \dot{x}_2) + kx_1 + k(x_1 - x_2) = p(t)$$

$$m_2\ddot{x}_2 + b(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) = 0$$

which can be rewritten as

$$m_1\ddot{x}_1 + b\dot{x}_1 + 2kx_1 = b\dot{x}_2 + kx_2 + p(t)$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + kx_2 = b\dot{x}_1 + kx_1$$

Since we are interested in the steady state behavior of the system, we can assume that all initial conditions are zero. Thus, by assuming zero initial conditions and taking the Laplace transforms of the last two equations, we obtain

$$(m_1s^2 + bs + 2k)X_1(s) = (bs + k)X_2(s) + P(s) \tag{1}$$

$$(m_2s^2 + bs + k)X_2(s) = (bs + k)X_1(s) \tag{2}$$

From Equation (2) we have

$$X_2(s) = \frac{bs + k}{m_2s^2 + bs + k} X_1(s) \tag{3}$$

Substituting Equation (3) into Equation (1) and simplifying, we obtain

$$[(m_1s^2 + k)(m_2s^2 + bs + k) + (bs + k)m_2s^2]X_1(s)$$
$$= (m_2s^2 + bs + k)P(s)$$

The transfer function $G_1(s)$ between $X_1(s)$ and $P(s)$ can thus be obtained as

$$G_1(s) = \frac{X_1(s)}{P(s)} = \frac{m_2s^2 + bs + k}{(m_1s^2 + k)(m_2s^2 + bs + k) + (bs + k)m_2s^2}$$

Hence

$$G_1(j\omega) = \frac{-m_2\omega^2 + j\omega b + k}{(-m_1\omega^2 + k)(-m_2\omega^2 + j\omega b + k) + (bj\omega + k)m_2(-\omega^2)}$$

$$= \frac{(k - m_2\omega^2) + j\omega b}{(k - m_1\omega^2)(k - m_2\omega^2) - \omega^2 km_2 + j\omega[bk - (m_1 + m_2)b\omega^2]}$$

from which we obtain

$$\left|G_1(j\omega)\right| = \frac{\sqrt{(k - m_2\omega^2)^2 + \omega^2b^2}}{\sqrt{[(k - m_1\omega^2)(k - m_2\omega^2) - \omega^2km_2]^2 + \omega^2[bk - (m_1 + m_2)b\omega^2]^2}}$$

$$\underline{/G_1(j\omega)} = \tan^{-1}\left(\frac{\omega b}{k - m_2\omega^2}\right) - \tan^{-1}\left\{\frac{\omega[bk - (m_1 + m_2)b\omega^2]}{(k - m_1\omega^2)(k - m_2\omega^2) - \omega^2km_2}\right\}$$

The steady-state output $x_1(t)$ can, therefore, be given by

$$x_1(t) = \left|G_1(j\omega)\right| P \sin\left[\omega t + \underline{/G_1(j\omega)}\right]$$

Next, referring to Equation (3) we have the transfer function $G_2(s)$ between $X_2(s)$ and $P(s)$ as follows:

$$G_2(s) = \frac{X_2(s)}{P(s)}$$

$$= \frac{X_2(s)}{X_1(s)} \frac{X_1(s)}{P(s)}$$

$$= \frac{bs + k}{m_2 s^2 + bs + k} \frac{X_1(s)}{P(s)}$$

$$= \frac{bs + k}{(m_1 s^2 + k)(m_2 s^2 + bs + k) + (bs + k)m_2 s^2}$$

Hence

$$G_2(j\omega) = \frac{bj\omega + k}{(k - m_1\omega^2)(k - m_2\omega^2) - \omega^2 k m_2 + j\omega[bk - (m_1 + m_2)b\omega^2]}$$

The magnitude and angle of $G_2(j\omega)$ are given by

$$\left| G_2(j\omega) \right| = \frac{\sqrt{k^2 + b^2\omega^2}}{\sqrt{[(k - m_1\omega^2)(k - m_2\omega^2) - \omega^2 k m_2]^2 + \omega^2[bk - (m_1 + m_2)b\omega^2]^2}}$$

$$\angle G_2(j\omega) = \tan^{-1}\frac{b\omega}{k} - \tan^{-1}\left\{ \frac{\omega[bk - (m_1 + m_2)b\omega^2]}{(k - m_1\omega^2)(k - m_2\omega^2) - \omega^2 k m_2} \right\}$$

The steady-state output $x_2(t)$ can be given by

$$x_2(t) = \left| G_2(j\omega) \right| P \sin\left[\omega t + \angle G_2(j\omega)\right]$$

B-7-5. $\text{tension} = m\omega^2 r = 0.1 \times 6.28^2 \times 1 = 3.94 \text{ N}$

The tension in the cord is 3.94 N. The maximum angular speed can be obtained by solving the following equation for ω.

$$10 = 0.1 \times \omega^2 \times 1$$

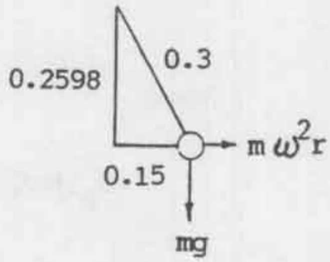
The result is

$$\omega = \sqrt{100} = 10 \text{ rad/s} = 1.59 \text{ Hz}$$

The maximum angular speed that can be attained without breaking the cord is 1.59 Hz.

B-7-6. From the diagram shown below, we obtain

$$\frac{\text{centrifugal force}}{\text{gravitational force}} = \frac{m\omega^2 r}{mg} = \frac{0.15}{0.2598}$$



or

$$\frac{\omega^2 r}{g} = 0.5774$$

Solving for ω, we obtain

$$\omega = \sqrt{\frac{0.5774 \times 9.81}{0.15}} = 6.145 \text{ rad/s}$$

B-7-7. The equations of motion for the system are

$$M\ddot{x} + b\dot{x} + kx = m\omega^2 r \sin \omega t$$

$$f(t) = b\dot{x} + kx$$

If 10 % of the excitation force is to be transmitted to the foundation, the transmissibility must be equal to 0.1. Thus

$$TR = \frac{F_t}{F_o} = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} = 0.1$$

Since ζ is desired to be 0.2, we substitute $\zeta = 0.2$ into this last equation.

$$\frac{1 + (0.4\beta)^2}{(1 - \beta^2)^2 + (0.4\beta)^2} = 0.01$$

or

$$\beta^4 - 17.84\beta^2 - 99 = 0$$

Solving for β^2, we find

$$\beta^2 = 22.28 \quad \text{or} \quad -4.443$$

Noting that $\beta > 0$, we must have $\beta^2 = 22.28$. Then

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{100}}, \qquad \beta = \frac{\omega}{\omega_n} = \frac{62.8}{\sqrt{\frac{k}{100}}} = \sqrt{22.28} = 4.72$$

So we obtain

$$\sqrt{k} = \frac{628}{4.72} = 133$$

or

$$k = 17.7 \times 10^3 \text{ N/m}$$

The amplitude of force F_t transmitted to the foundation is

$$\left|F_t\right| = (m\omega^2 r)(TR) = 0.2 \times 62.8^2 \times 0.5 \times 0.1 = 39.44 \text{ N}$$

B-7-8. The equation of motion for the system is

$$m\ddot{x} + b(\dot{x} - \dot{y}) + k(x - y) = 0$$

Rewriting,

$$m(\ddot{x} - \ddot{y}) + b(\dot{x} - \dot{y}) + k(x - y) = - m\ddot{y}$$

By substituting $x - y = z$ into this last equation, we obtain

$$m\ddot{z} + b\dot{z} + kz = - m\ddot{y}$$

The Laplace transform of this equation, assuming zero initial conditions, is

$$(ms^2 + bs + k)Z(s) = - ms^2 Y(s)$$

or

$$\frac{Z(s)}{Y(S)} = \frac{- ms^2}{ms^2 + bs + k}$$

For the sinusoidal input $y = Y \sin \omega t$,

$$\frac{Z(j\omega)}{Y(j\omega)} = \frac{m\,\omega^2}{- m\omega^2 + bj\omega + k} = \frac{\omega^2}{-\omega^2 + 2\zeta\omega_n j\omega + \omega_n^2}$$

The steady-state amplitude ratio of z to y is

$$\left|\frac{Z(j\omega)}{Y(j\omega)}\right| = \frac{m\,\omega^2}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} = \frac{\omega^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

If $\omega \gg \omega_n$,

$$\left| \frac{Z(j\omega)}{Y(j\omega)} \right| = \frac{\omega^2}{\omega^2} = 1$$

Thus, the amplitude of sinusoidal displacement y of the base is equal to the amplitude of the relative displacement z.

If $\omega \ll \omega_n$, we have

$$\frac{Z(s)}{Y(s)} = -\frac{ms^2}{k}$$

or

$$\frac{Z(s)}{s^2 Y(s)} = -\frac{m}{k} = -\frac{1}{\omega_n^2}$$

So the acceleration $\ddot{y}$ of the base is proportional to z.

B-7-9. Define the displacement of spring k_2 as y. Then the equations of motion for the system are

$$m\ddot{x} + b_2(\dot{x} - \dot{y}) + k_1 x = p(t) \tag{1}$$

$$b_2(\dot{x} - \dot{y}) = k_2 y \tag{2}$$

The force f(t) transmitted to the foundation is

$$f(t) = k_1 x + k_2 y \tag{3}$$

By taking Laplace transforms of Equations (1) and (2), assuming zero initial conditions, we obtain

$$(ms^2 + b_2 s + k_1)X(s) = b_2 sY(s) + P(s)$$

$$b_2 sX(s) = (b_2 s + k_2)Y(s)$$

By eliminating Y(s) from the last two equations and simplifying, we get

$$\frac{X(s)}{P(s)} = \frac{b_2 s + k_2}{(ms^2 + k_1)(b_2 s + k_2) + b_2 k_2 s}$$

The Laplace transform of Equation (3) is

$$F(s) = k_1 X(s) + k_2 Y(s) = k_1 X(s) + \frac{b_2 k_2 s}{b_2 s + k_2} X(s)$$

$$= \frac{k_1 b_2 s + k_1 k_2 + b_2 k_2 s}{b_2 s + k_2} X(s)$$

So we have

$$\frac{F(s)}{X(s)} = \frac{(k_1b_2 + k_2b_2)s + k_1k_2}{b_2s + k_2}$$

and

$$\frac{F(s)}{P(s)} = \frac{F(s)}{X(s)} \frac{X(s)}{P(s)} = \frac{(k_1 + k_2)b_2s + k_1k_2}{(ms^2 + k_1)(b_2s + k_2) + b_2k_2s}$$

The force transmissibility TR is

$$TR = \left| \frac{(k_1 + k_2)b_2j\omega + k_1k_2}{(- m\omega^2 + k_1)(k_2 + b_2j\omega) + b_2k_2j\omega} \right|$$

$$= \frac{\sqrt{(k_1 + k_2)^2b_2^2\omega^2 + k_1^2k_2^2}}{\sqrt{(k_1 - m\omega^2)^2k_2^2 + [b_2k_2\omega + (k_1 - m\omega^2)b_2\omega]^2}}$$

The amplitude of the force transmitted to the foundation is

$$\left|F(j\omega)\right| = \frac{P\sqrt{(k_1 + k_2)^2b_2^2\omega^2 + k_1^2k_2^2}}{\sqrt{(k_1 - m\omega^2)^2k_2^2 + [b_2k_2\omega + (k_1 - m\omega^2)b_2\omega]^2}}$$

B-7-10. Define the displacement of the top end of spring k_2 as z. Then the equations of motion for the system are

$$m\ddot{x} + b_2(\dot{x} - \dot{z}) + k_1(x - p) = 0$$

$$b_2(\dot{x} - \dot{z}) = k_2(z - p)$$

where $p = P \sin\omega t$. Rewriting these equations,

$$m\ddot{x} + b_2\dot{x} + k_1x = b_2\dot{z} + k_1p$$

$$b_2\dot{x} + k_2p = k_2z + b_2\dot{z}$$

Laplace transforming these two equations, assuming zero initial conditions, we obtain

$$(ms^2 + b_2s + k_1)X(s) = b_2sZ(s) + k_1P(s)$$

$$b_2sX(s) + k_2P(s) = (k_2 + b_2s)Z(s)$$

Eliminating Z(s) from the last two equations and simplifying gives

$$[(ms^2 + k_1)(k_2 + b_2s) + k_2b_2s]X(s) = b_2k_2sP(s) + k_1P(s)(k_2 + b_2s)$$

So the transfer function X(s)/P(s) is obtained as

$$\frac{X(s)}{P(s)} = \frac{b_2k_2s + k_1k_2 + k_1b_2s}{(ms^2 + k_1)(b_2s + k_2) + b_2k_2s}$$

The motion transmissibility TR is

$$TR = \left| \frac{X(j\omega)}{P(j\omega)} \right| = \frac{\sqrt{(k_1 + k_2)^2 b_2{}^2\omega^2 + k_1{}^2k_2{}^2}}{\sqrt{k_2{}^2(k_1 - m\omega^2)^2 + [b_2k_2\omega + b_2\omega(k_1 - m\omega^2)]^2}}$$

The vibration amplitude $|X(j\omega)|$ of the machine is

$$|X(j\omega)| = P\cdot TR$$

$$= \frac{P\sqrt{(k_1 + k_2)^2 b_2{}^2\omega^2 + k_1{}^2k_2{}^2}}{\sqrt{k_2{}^2(k_1 - m\omega^2)^2 + [b_2k_2\omega + b_2\omega(k_1 - m\omega^2)]^2}}$$

B-7-11. The equations of motion for the system are

$$m\ddot{x} + b\dot{x} + kx + k_a(x - y) = p = P\sin\omega t$$

$$m_a\ddot{y} + k_a(y - x) = 0$$

Laplace transforming these two equations, assuming zero initial conditions, we obtain

$$(ms^2 + bs + k + k_a)X(s) = k_aY(s) + P(s)$$

$$(m_as^2 + k_a)Y(s) = k_aX(s)$$

Eliminating Y(s) from the last two equations and simplifying, we obtain

$$\frac{X(s)}{P(s)} = \frac{m_as^2 + k_a}{(ms^2 + bs + k)(m_as^2 + k_a) + m_ak_as^2}$$

Hence

$$\frac{X(j\omega)}{P(j\omega)} = \frac{k_a - m_a\omega^2}{(k - m\omega^2 + bj\omega)(k_a - m_a\omega^2) - m_ak_a\omega^2}$$

Note that if $\sqrt{k_a/m_a} = \omega$, then $X(j\omega) = 0$. Since

$$\frac{Y(s)}{X(s)} = \frac{k_a}{m_a s^2 + k_a}$$

we have

$$\frac{Y(j\omega)}{P(j\omega)} = \frac{k_a}{(k - m\omega^2 + bj\omega)(k_a - m_a\omega^2) - m_a k_a \omega^2}$$

By substituting $k_a/m_a = \omega^2$ into this last equation, we get

$$\frac{Y(j\omega)}{P(j\omega)} = \frac{k_a}{- m_a k_a \omega^2} = - \frac{1}{m_a \omega^2}$$

Hence

$$\left| Y(j\omega) \right| = \frac{|P(j\omega)|}{m_a \omega^2} = \frac{P}{m_a \frac{k_a}{m_a}} = \frac{P}{k_a}$$

The amplitude of vibration of mass m_a is P/k_a.

B-7-12. Assuming small angles θ_1 and θ_2 the equations of motion for the system may be obtained as follows:

$$m_1 \ell^2 \ddot{\theta}_1 = - m_1 g \ell \theta_1 - ka^2(\theta_1 - \theta_2)$$

$$m_2 \ell^2 \ddot{\theta}_2 = - m_2 g \ell \theta_2 - ka^2(\theta_2 - \theta_1)$$

Rewriting these equations, we obtain

$$m_1 \ell^2 \ddot{\theta}_1 + m_1 g \ell \theta_1 + ka^2 \theta_1 = ka^2 \theta_2$$

$$m_2 \ell^2 \ddot{\theta}_2 + m_2 g \ell \theta_2 + ka^2 \theta_2 = ka^2 \theta_1$$

which can be simplified to

$$\ddot{\theta}_1 + \left(\frac{g}{\ell} + \frac{ka^2}{m_1 \ell^2} \right) \theta_1 = \frac{ka^2}{m_1 \ell^2} \theta_2 \qquad (1)$$

$$\ddot{\theta}_2 + \left(\frac{g}{\ell} + \frac{ka^2}{m_2 \ell^2} \right) \theta_2 = \frac{ka^2}{m_2 \ell^2} \theta_1 \qquad (2)$$

To find the natural frequencies of the free vibration, we assume the motion to be harmonic. That is, we assume

$$\theta_1 = A \sin \omega t, \qquad \theta_2 = B \sin \omega t$$

Then

$$\ddot{\theta}_1 = -A\omega^2 \sin \omega t, \qquad \ddot{\theta}_2 = -B\omega^2 \sin \omega t$$

Substituting the preceding expressions into Equations (1) and (2), we obtain

$$\left[-A\omega^2 + \left(\frac{g}{\ell} + \frac{ka^2}{m_1 \ell^2} \right) A - \frac{ka^2}{m_1 \ell^2} B \right] \sin \omega t = 0$$

$$\left[-B\omega^2 + \left(\frac{g}{\ell} + \frac{ka^2}{m_2 \ell^2} \right) B - \frac{ka^2}{m_2 \ell^2} A \right] \sin \omega t = 0$$

Since these equations must be satisfied at all times and $\sin \omega t$ cannot be zero at all times, the quantities in the brackets must be equal to zero. Thus

$$\left(\frac{g}{\ell} + \frac{ka^2}{m_1 \ell^2} - \omega^2 \right) A - \frac{ka^2}{m_1 \ell^2} B = 0 \tag{3}$$

$$-\frac{ka^2}{m_2 \ell^2} A + \left(\frac{g}{\ell} + \frac{ka^2}{m_2 \ell^2} - \omega^2 \right) B = 0 \tag{4}$$

For constants A and B to be nonzero, the determinant of the coefficients of Equations (3) and (4) must be equal to zero, or

$$\begin{vmatrix} \dfrac{g}{\ell} + \dfrac{ka^2}{m_1 \ell^2} - \omega^2 & -\dfrac{ka^2}{m_1 \ell^2} \\ -\dfrac{ka^2}{m_2 \ell^2} & \dfrac{g}{\ell} + \dfrac{ka^2}{m_2 \ell^2} - \omega^2 \end{vmatrix} = 0 \tag{5}$$

This determinant equation determines the natural frequencies of the system. Equation (5) can be rewritten as

$$\left(\frac{g}{\ell} + \frac{ka^2}{m_1 \ell^2} - \omega^2 \right) \left(\frac{g}{\ell} + \frac{ka^2}{m_2 \ell^2} - \omega^2 \right) - \frac{ka^2}{m_1 \ell^2} \frac{ka^2}{m_2 \ell^2} = 0$$

or

$$\omega^4 + \left(-\frac{2g}{\ell} - \frac{ka^2}{m_1 \ell^2} - \frac{ka^2}{m_2 \ell^2} \right) \omega^2 + \left(\frac{g}{\ell} \right)^2 + \left(\frac{ka^2}{m_1 \ell^2} + \frac{ka^2}{m_2 \ell^2} \right) \frac{g}{\ell}$$

$$= 0$$

This last equation can be factored as follows:

$$\left(\omega^2 - \frac{g}{\ell} - \frac{ka^2}{m_1\ell^2} - \frac{ka^2}{m_2\ell^2}\right)\left(\omega^2 - \frac{g}{\ell}\right) = 0$$

or

$$\omega^2 = \frac{g}{\ell}, \qquad \omega^2 = \frac{g}{\ell} + \frac{ka^2}{m_1\ell^2} + \frac{ka^2}{m_2\ell^2}$$

Thus

$$\omega_1 = \sqrt{\frac{g}{\ell}}, \quad \omega_2 = \sqrt{\frac{g}{\ell} + \frac{ka^2}{m_1\ell^2} + \frac{ka^2}{m_2\ell^2}}$$

The first natural frequency is ω_1 (first mode) and the second natural frequency is ω_2 (second mode).

At the first natural frequency $\omega = \omega_1 = \sqrt{g/\ell}$, we obtain from Equation (3) the following expression:

$$\frac{A}{B} = \frac{\dfrac{ka^2}{m_1\ell^2}}{\dfrac{g}{\ell} + \dfrac{ka^2}{m_1\ell^2} - \dfrac{g}{\ell}} = 1$$

[Note that we obtain the same result from Equation (4).] Thus, at the first mode the amplitude ratio A/B becomes unity, or A = B. This means that both masses move the same amount in the same direction. This mode is depicted in Figure (a) below.

At the second natural frequency $\omega = \omega_2$ we obtain from Equation (3)

$$\frac{A}{B} = \frac{\dfrac{ka^2}{m_1\ell^2}}{\dfrac{g}{\ell} + \dfrac{ka^2}{m_1\ell^2} - \dfrac{g}{\ell} - \dfrac{ka^2}{m_1\ell^2} - \dfrac{ka^2}{m_2\ell^2}} = -\frac{m_2}{m_1}$$

[We obtain the same result from Equation (4).] At the second mode, the amplitude ratio A/B becomes $-m_2/m_1$ or $A = -(m_2/m_1)B$. This means that masses move in the opposite direction. This mode is depicted in Figure (b) shown below.

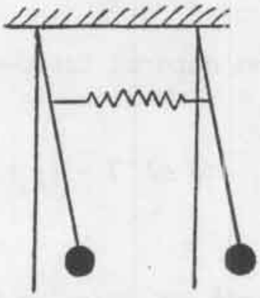


Figure (a)

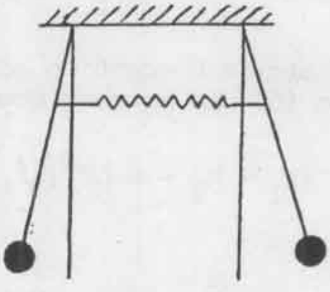


Figure (b)

B-7-13. The equations of motion for the system are

$$m\ddot{x} = -(x + \ell_1 \sin\theta)k_1 - (x - \ell_2 \sin\theta)k_2$$

$$J\ddot{\theta} = -(x + \ell_1 \sin\theta)k_1 \ell_1 \cos\theta + (x - \ell_2 \sin\theta)k_2 \ell_2 \cos\theta$$

For small angle θ, we have $\sin\theta \doteq \theta$ and $\cos\theta \doteq 1$ and the preceding two equations become

$$m\ddot{x} + (k_1 + k_2)x + (\ell_1 k_1 - \ell_2 k_2)\theta = 0 \quad (1)$$

$$J\ddot{\theta} + (\ell_1^2 k_1 + \ell_2^2 k_2)\theta + (\ell_1 k_1 - \ell_2 k_2)x = 0 \quad (2)$$

Notice that if $\ell_1 k_1 = \ell_2 k_2$, then the coupling terms become zero and two equations become independent. However, in this problem

$$\ell_1 = 1.5 \text{ m}, \quad \ell_2 = 2 \text{ m}, \quad k_1 = k_2 = 4 \times 10^4 \text{ N/m}$$

Thus,

$$\ell_1 k_1 - \ell_2 k_2 = -0.5 \times 4 \times 10^4 \neq 0$$

Therefore, coupling exists between Equation (1) and Equation (2).

To find the natural frequencies for the system, assume the following harmonic motion:

$$x = A \sin\omega t, \qquad \theta = B \sin\omega t$$

Then, from Equations (1) and (2) we obtain

$$(k_1 + k_2 - m\omega^2)A + (\ell_1 k_1 - \ell_2 k_2)B = 0 \quad (3)$$

$$(\ell_1 k_1 - \ell_2 k_2)A + (\ell_1^2 k_1 + \ell_2^2 k_2 - J\omega^2)B = 0 \quad (4)$$

For amplitudes A and B to be nonzero, the determinant of the coefficients of Equations (3) and (4) must be equal to zero, or

$$\begin{vmatrix} k_1 + k_2 - m\omega^2 & \ell_1 k_1 - \ell_2 k_2 \\ \ell_1 k_1 - \ell_2 k_2 & \ell_1^2 k_1 + \ell_2^2 k_2 - J\omega^2 \end{vmatrix} = 0 \quad (5)$$

This determinant equation determines the natural frequencies of the system. Equation (5) can be rewritten as

$$(k_1 + k_2 - m\omega^2)(\ell_1^2 k_1 + \ell_2^2 k_2 - J\omega^2) - (\ell_1 k_1 - \ell_2 k_2)^2 = 0$$

or

$$mJ\omega^4 - [(k_1 + k_2)J + (\ell_1^2 k_1 + \ell_2^2 k_2)m]\omega^2 + k_1 k_2(\ell_1 + \ell_2)^2 = 0$$

which can be simplified to

$$\omega^4 - \left(\frac{k_1 + k_2}{m} + \frac{\ell_1^2 k_1 + \ell_2^2 k_2}{J}\right)\omega^2 + \frac{k_1 k_2(\ell_1 + \ell_2)^2}{mJ} = 0 \qquad (6)$$

Notice that this last equation determines the natural frequencies of the system. By substituting the given numerical values for ℓ_1, ℓ_2, k_1, k_2, m, and J into Equation (6), we obtain

$$\omega^4 - 140\,\omega^2 + 3920 = 0$$

Solving this equation for ω^2, we get

$$\omega^2 = 38.695 \quad \text{or} \quad 101.305$$

Hence

$$\omega_1 = 6.2205, \qquad \omega_2 = 10.065$$

The first natural frequency is $\omega_1 = 6.2205$ rad/s and the second natural frequency is $\omega_2 = 10.065$ rad/s.

To determine the modes of vibration, notice that from Equations (3) and (4) we have

$$\frac{A}{B} = \frac{\ell_2 k_2 - \ell_1 k_1}{k_1 + k_2 - m\omega^2} = \frac{\ell_1^2 k_1 + \ell_2^2 k_2 - J\omega^2}{\ell_2 k_2 - \ell_1 k_1}$$

By substituting the given numerical values into this last equation, we obtain

$$\frac{A}{B} = \frac{2}{8 - 0.2\,\omega^2} = \frac{25 - 0.25\,\omega^2}{2} \qquad (7)$$

For the first mode of vibration ($\omega_1 = 6.2205$ rad/s) the amplitude ratio A/B becomes as follows:

$$\frac{A}{B} = \frac{2}{8 - 0.2 \times 6.2205^2} = \frac{25 - 0.25 \times 6.2205^2}{2}$$

$$= \frac{2}{0.261} = \frac{15.326}{2} = 7.663$$

Notice that the ratio of the displacements of springs k_1 and k_2 are

$$\frac{x + \ell_1\theta}{x - \ell_2\theta} = \frac{A + \ell_1 B}{A - \ell_2 B} = \frac{(7.663 + 1.5)B}{(7.663 - 2)B} = \frac{9.163B}{5.663B}$$

The first mode of vibration is shown in Figure (a) on next page.

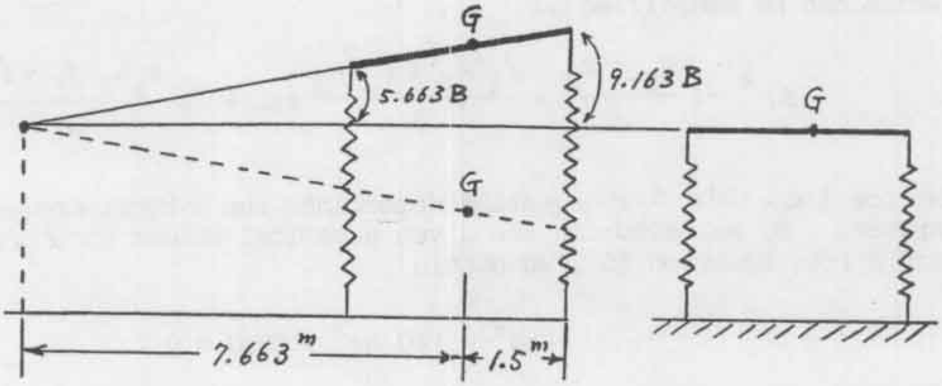


Figure (a)

For the second mode of vibration (ω_2 = 10.065 rad/s) the amplitude ratio A/B becomes as

$$\frac{A}{B} = \frac{2}{8 - 0.2 \times 10.065^2} = \frac{25 - 0.25 \times 10.065^2}{2}$$

$$= \frac{2}{-12.261} = \frac{-0.32625}{2} = -\,0.1631$$

Hence the ratio of the displacements of springs k_1 and k_2 becomes

$$\frac{x + \ell_1\theta}{x - \ell_2\theta} = \frac{A + \ell_1 B}{A - \ell_2 B} = \frac{(-0.1631 + 1.5)B}{(-0.1631 - 2)B} = \frac{1.3369B}{-2.1631B}$$

The second mode of vibration is shown in Figure (b) below.

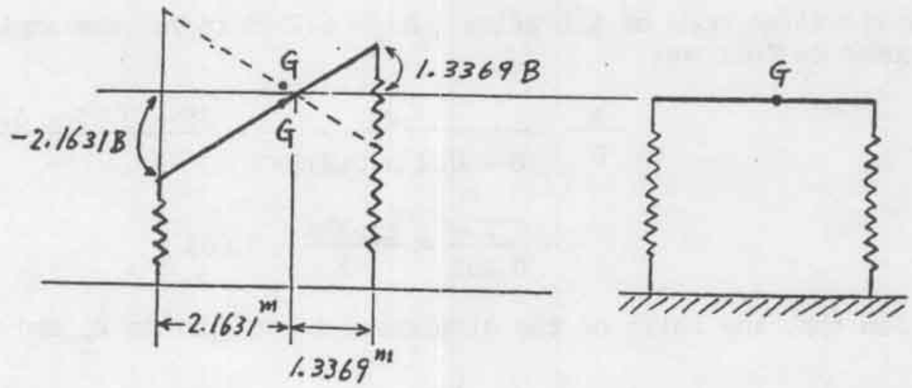


Figure (b)

B-7-14. The system shown in Figure 7-50 is a special case of the system shown in Figure 7-32 (Problem A-7-14). By defining

$$k_1 = k, \quad k_2 = 2k, \quad k_3 = k, \quad m_1 = m, \quad m_2 = m$$

Equations (7-40) and (7-41) become as follows:

$$\frac{A}{B} = \frac{2k}{-m\omega^2 + k + 2k} = \frac{2k}{-m\omega^2 + 3k} \tag{1}$$

$$\frac{A}{B} = \frac{-m\omega^2 + 2k + k}{2k} = \frac{-m\omega^2 + 3k}{2k} \tag{2}$$

Also, ω^2 that satisfies Equations (7-38) and (7-39) becomes as follows:

$$\omega^2 = \frac{1}{2}\left(\frac{k + 2k}{m} + \frac{2k + k}{m}\right) \pm \sqrt{\frac{1}{4}\left(\frac{k + 2k}{m} - \frac{2k + k}{m}\right)^2 + \frac{4k^2}{m^2}}$$

$$= \frac{3k}{m} \pm \frac{2k}{m} = \frac{k}{m} \text{ or } \frac{5k}{m}$$

Define

$$\omega_1^2 = \frac{k}{m}, \qquad \omega_2^2 = \frac{5k}{m}$$

By substituting ω_1^2 into Equation (1), we obtain

$$\frac{A_1}{B_1} = \frac{2k}{-m(k/m) + 3k} = \frac{2k}{2k} = 1$$

Similarly, by substituting ω_2^2 into Equation (1), we get

$$\frac{A_2}{B_2} = \frac{2k}{-m(5k/m) + 3k} = \frac{2k}{-2k} = -1$$

[We get the same result if we substitute ω_1^2 or ω_2^2 into Equation (2).] Hence we have

$$\frac{A_1}{B_1} = 1 > 0, \qquad \frac{A_2}{B_2} = -1 < 0$$

which means that in the first mode of vibration (with frequency ω_1), the masses move in the same direction by the same amount. In the second mode of vibration (with frequency ω_2), the masses move in opposite directions by the same amount. Figures (a) and (b) shown on next page depict the first mode of vibration and second mode of vibration, respectively.

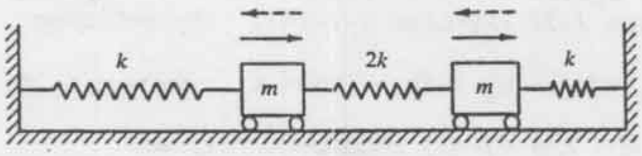


Figure (a)

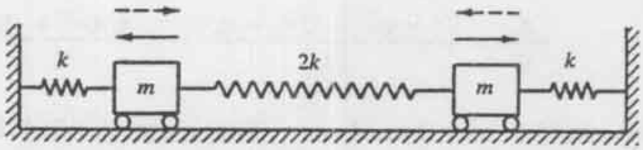


Figure (b)

B-7-15. The equations of motion for the system are

$$M\ddot{x} + k_1(x - y) + k_2 x = 0$$

$$m\ddot{y} + k_1(y - x) = 0$$

Substituting the given numerical values into these two equations and simplifying, we have

$$\ddot{x} + 11x - y = 0 \qquad (1)$$

$$\ddot{y} + 10y - 10x = 0 \qquad (2)$$

To find the natural frequencies of the free vibration, assume that the motion is harmonic, or

$$x = A \sin\omega t, \qquad y = B \sin\omega t$$

Then

$$\ddot{x} = -A\,\omega^2 \sin\omega t, \quad \ddot{y} = -B\,\omega^2 \sin\omega t$$

If the preceeding expressions are substituted into Equations (1) and (2), we obtain

$$(-A\,\omega^2 + 11A - B) \sin\omega t = 0$$

$$(-B\,\omega^2 + 10B - 10A) \sin\omega t = 0$$

Since these two equations must be satisfied at all times and $\sin\omega t$ cannot be zero at all times, we must have

$$-A\,\omega^2 + 11A - B = 0$$

$$-B\,\omega^2 + 10B - 10A = 0$$

Rearranging,

$$(11 - \omega^2)A - B = 0 \tag{3}$$

$$-10A + (10 - \omega^2)B = 0 \tag{4}$$

For constants A and B to be nonzero, the determinant of the coefficient matrix must be equal to zero, or

$$\begin{vmatrix} 11 - \omega^2 & -1 \\ -10 & 10 - \omega^2 \end{vmatrix} = 0$$

which yields

$$(11 - \omega^2)(10 - \omega^2) - 10 = 0$$

or

$$\omega^4 - 21\omega^2 + 100 = 0$$

which can be rewritten as

$$(\omega^2 - 7.2985)(\omega^2 - 13.7016) = 0$$

Hence,

$$\omega_1^2 = 7.2985, \qquad \omega_2^2 = 13.7016$$

or

$$\omega_1 = 2.7016, \qquad \omega_2 = 3.7016$$

The frequency of the first mode is 2.7016 rad/s and the frequency of the second mode is 3.7016 rad/s.

From Equations (3) and (4), we obtain

$$\frac{A}{B} = \frac{1}{11 - \omega^2}, \qquad \frac{A}{B} = \frac{10 - \omega^2}{10}$$

By substituting $\omega_1^2 = 7.2985$ into A/B, we obtain

$$\frac{A}{B} = \frac{1}{11 - 7.2985} = \frac{10 - 7.2985}{10} = 0.27016 > 0$$

Similarly, by substituting $\omega_2^2 = 13.7016$ into A/B, we get

$$\frac{A}{B} = \frac{1}{11 - 13.7016} = \frac{10 - 13.7016}{10} = -0.37016 < 0$$

Hence, at the first mode of vibration, two masses move in the same direction, while at the second mode of vibration, two masses move in opposite directions.

Next, we shall obtain the vibrations x(t) and y(t) subjected to the given initial conditions. Laplace transforming Equations (1) and (2),

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 11X(s) - Y(s) = 0$$

$$[s^2Y(s) - sy(0) - \dot{y}(0)] + 10Y(s) - 10X(s) = 0$$

Substituting the given initial conditions into the preceeding two equations we get

$$(s^2 + 11)X(s) = 0.05s + Y(s) \qquad (5)$$

$$(s^2 + 10)Y(s) = 10X(s) \qquad (6)$$

Eliminating Y(s) from Equations (5) and (6),

$$(s^2 + 11)X(s) = 0.05s + \frac{10}{s^2 + 10} X(s)$$

which can be simplified to

$$X(s) = \frac{0.05s(s^2 + 10)}{s^4 + 21s^2 + 100}$$

Similarly, we can obtain Y(s) as follows:

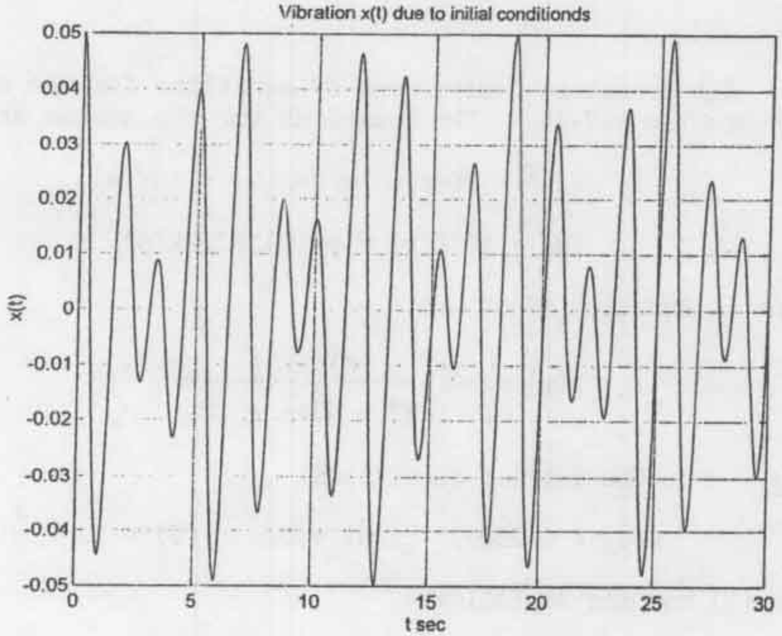
$$Y(s) = \frac{0.5s}{s^4 + 21s^2 + 100}$$

To obtain the responses x(t) and y(t) to the given initial conditions, we rewrite X(s) and Y(s) as follows:

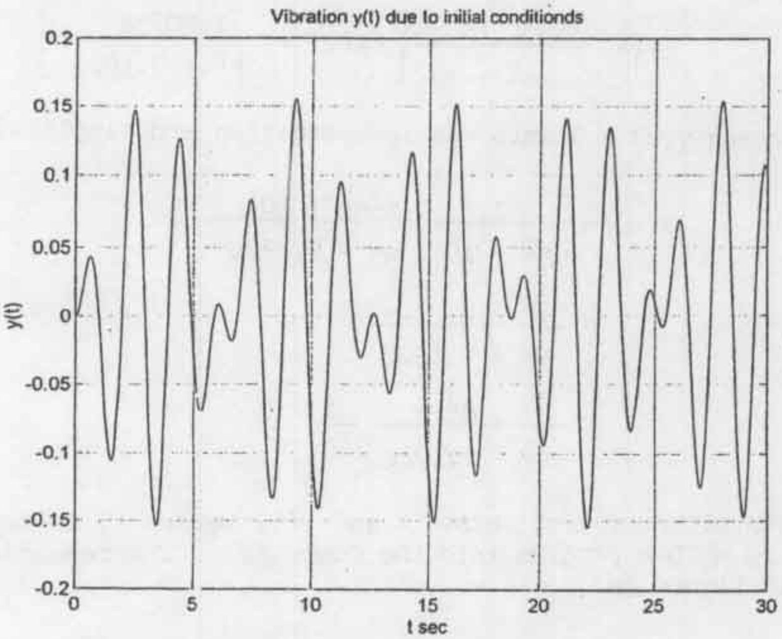
$$X(s) = \frac{0.05s^4 + 0.5s^2}{s^4 + 21s^2 + 100}\frac{1}{s}, \quad Y(s) = \frac{0.5s^2}{s^4 + 21s^2 + 100}\frac{1}{s}$$

Possible MATLAB programs to plot x(t) and y(t), respectively, are given next. The resulting plots x(t) versus t and y(t) versus t are shown on next page.

```
% ***** MATLAB program to obtain vibration x(t) *****

num = [0.05  0  0.5  0  0];
den = [1  0  21  0  100];
t = 0:0.05:30;
x = step(num,den,t);
plot(t,x)
grid
title('Vibration x(t) due to initial conditionds')
xlabel('t sec')
ylabel('x(t)')
```




```
% ***** MATLAB program to obtain vibration y(t) *****

num = [0  0  0.5  0  0];
den = [1  0  21  0  100];
t = 0:0.05:30;
y = step(num,den,t);
plot(t,y)
grid
title('Vibration y(t) due to initial conditionds')
xlabel('t sec')
ylabel('y(t)')
```



B-7-16. All necessary derivations of equations for the system are given in Problem A-7-16. The equations for the system are

$$(2s^2 + 50)X(s) = 2sx(0) + 10Y(s) \tag{1}$$

$$(s^2 + 10)Y(s) = sy(0) + 10X(s) \tag{2}$$

Referring to Equation (7-54) we have

$$X(s) = \frac{(s^2 + 10)sx(0) + 5sy(0)}{s^4 + 35s^2 + 200} \tag{3}$$

Case (a): For the initial conditions

$$x(0) = 0.2807, \quad \dot{x}(0) = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0$$

Equation (3) becomes as follows:

$$X(s) = \frac{(s^2 + 10) \times 0.2807s + 5s}{s^4 + 35s^2 + 200}$$

$$= \frac{0.2807s(s^2 + 27.808)}{(s^2 + 27.808)(s^2 + 7.1922)}$$

$$= \frac{0.2807s}{s^2 + 7.1922}$$

$$= \frac{0.2807s^2}{s^2 + 7.1922}\frac{1}{s} \tag{4}$$

By substituting Equation (4) into Equation (2) and solving for Y(s), we obtain

$$Y(s) = \frac{1}{s^2 + 10}\left[sy(0) + \frac{2.8078s}{s^2 + 7.1922}\right]$$

Substituting $y(0) = 1$ into the last equation and simplifying, we get

$$Y(s) = \frac{1}{s^2 + 10}\frac{s(s^2 + 10)}{s^2 + 7.1922}$$

$$= \frac{s}{s^2 + 7.1922}$$

$$= \frac{s^2}{s^2 + 7.1922}\frac{1}{s}$$

To obtain plots of x(t) versus t and y(t) versus t, we may enter the following MATLAB program into the computer. The resulting plots are shown in Figure (a).

```
% ***** MATLAB program to obtain x(t) and y(t), case (a) *****

num1 = [0.2807  0  0];
num2 = [1  0  0];
den = [1  0  7.1922];
step(num1,den)
hold
Current plot held
step(num2,den)
text(2,-0.5,'x(t)')
text(3,0.3,'y(t)')
title('Responses x(t) and y(t) due to initial conditions (a)')
xlabel ('t sec')
ylabel ('x(t) and y(t)')
```

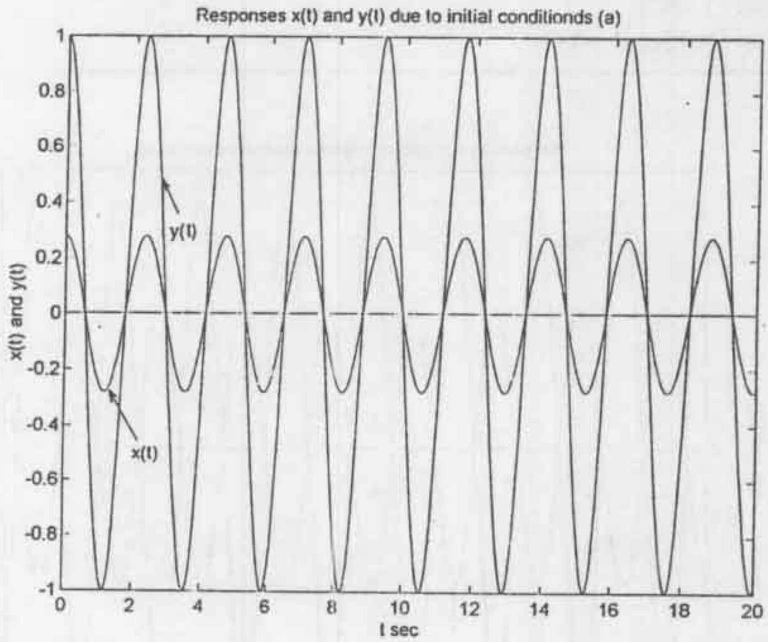


Figure (a)

Case (b): For the initial conditions

$$x(0) = 1.7808, \quad \dot{x}(0) = 0, \quad y(0) = -1, \quad \dot{y}(0) = 0$$

we obtain the following expressions for X(s) and Y(s):

$$X(s) = \frac{1.7808s}{s^2 + 27.808} = \frac{1.7808s^2}{s^2 + 27.808}\frac{1}{s}$$

$$Y(s) = -\frac{s}{s^2 + 27.808} = -\frac{s^2}{s^2 + 27.808}\frac{1}{s}$$

A MATLAB program for obtaining plots of x(t) versus t and y(t) versus t is shown below. The resulting plots are shown in Figure (b) below.

```
% ***** MATLAB program to obtain x(t) and y(t), case (b) *****

num1 = [1.7808   0  0];
num2 = [-1  0  0];
den = [1  0  27.808];
step(num1,den)
hold
Current plot held
step(num2,den)
text(1.7,1.5,'x(t)')
text(3.5,-1.5,'y(t)')
title('Responses x(t) and y(t) due to initial conditions (b)')
xlabel ('t sec')
ylabel ('x(t) and y(t)')
```

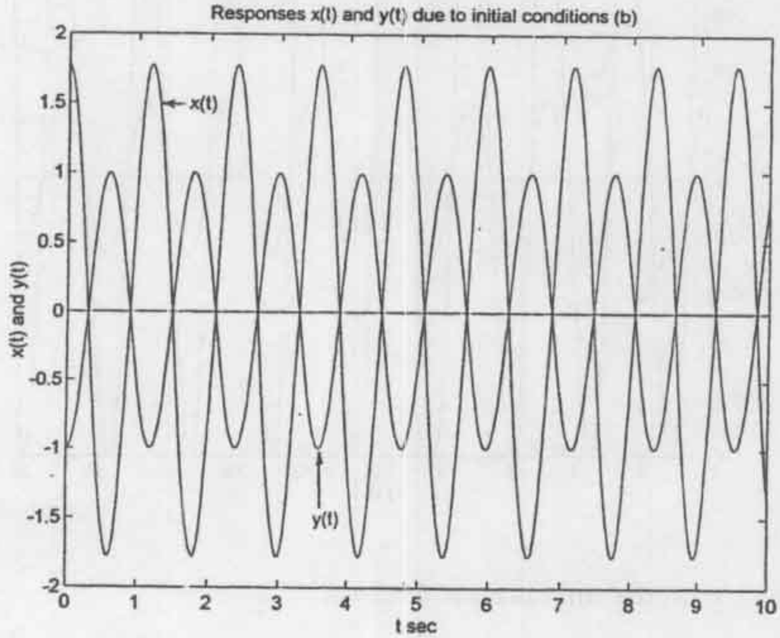


Figure (b)

Case (c): For the initial conditions

$$x(0) = 0.5, \quad \dot{x}(0) = 0, \quad y(0) = -0.5, \quad \dot{y}(0) = 0$$

we obtain the following expressions for X(s) and Y(s):

$$X(s) = \frac{(s^2 + 10)sx(0) + 5sy(0)}{s^4 + 35s^2 + 200}$$

$$= \frac{(s^2 + 10)s(0.5) + 5s(-0.5)}{s^4 + 35s^2 + 200}$$

$$= \frac{0.5s^3 + 2.5s}{s^4 + 35s^2 + 200}$$

$$= \frac{0.5s^4 + 2.5s^2}{s^4 + 35s^2 + 200} \frac{1}{s}$$

$$Y(s) = \frac{sy(0)}{s^2 + 10} + \frac{10X(s)}{s^2 + 10}$$

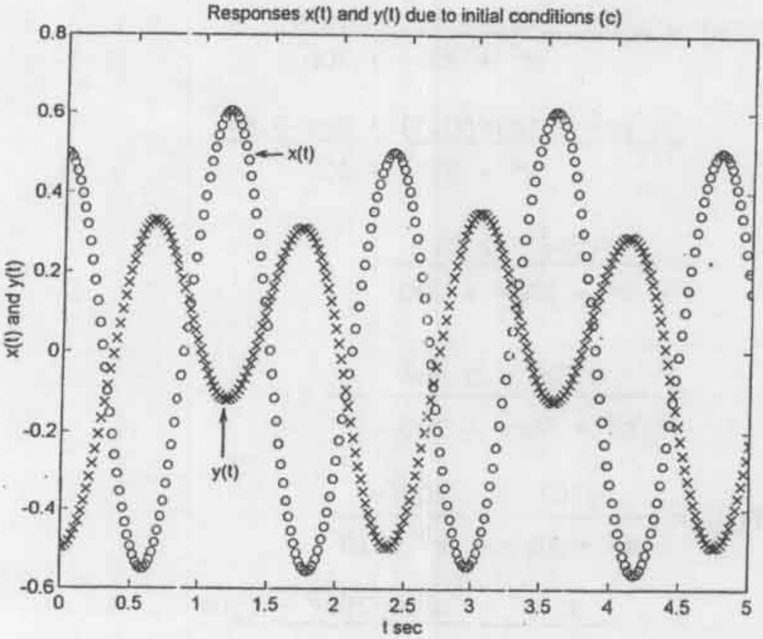
$$= - \frac{0.5}{s^2 + 10} \frac{s^5 + 25s^3 + 150s}{s^4 + 35s^2 + 200}$$

$$= - \frac{0.5s(s^2 + 15)}{s^4 + 35s^2 + 200}$$

$$= - \frac{0.5s^4 + 7.5s^2}{s^4 + 35s^2 + 200} \frac{1}{s}$$

A MATLAB program to obtain plots of x(t) versus t and y(t) versus t is shown below. The resulting plots are shown in Figure (c).

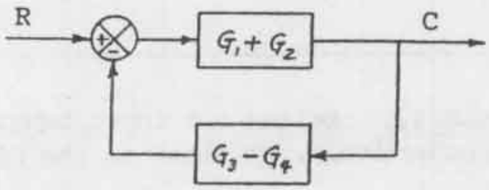
```
% ***** MATLAB program to obtain x(t) and y(t), case (c) *****

num1 = [0.5  0  2.5  0  0]
num2 = [-0.5  0  -7.5  0  0];
den = [1  0  35  0  200];
t = 0:0.02:5;
x = step(num1,den,t);
plot(t,x,'o')
hold
Current plot held
y = step(num2,den,t);
plot(t,y,'x')
text(1.6,0.5,'x(t)')
text(1.1,-0.3,'y(t)')
title('Responses x(t) and y(t) due to initial conditions (c)')
xlabel ('t sec')
ylabel ('x(t) and y(t)')
```

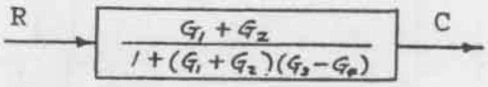

Responses x(t) and y(t) due to initial conditions (c)
x(t)
y(t)
x(t) and y(t)
t sec
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Figure (c)

CHAPTER 8

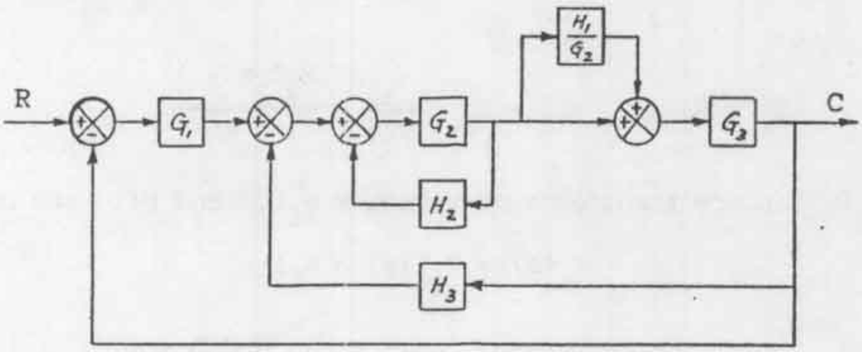
B-8-1. Simplified block diagrams are shown to the right. The transfer function C(s)/R(s) is

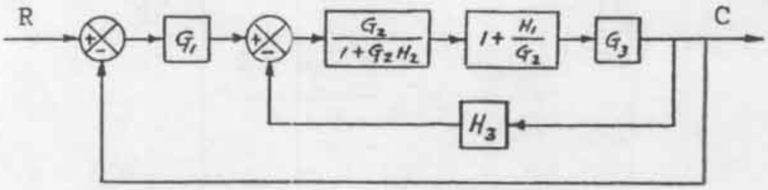
$$\frac{C(s)}{R(s)} = \frac{G_1 + G_2}{1 + (G_1 + G_2)(G_3 - G_4)}$$

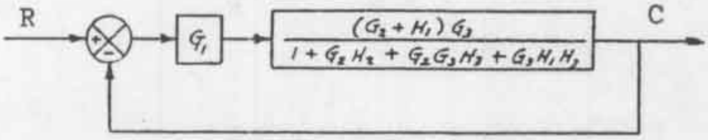


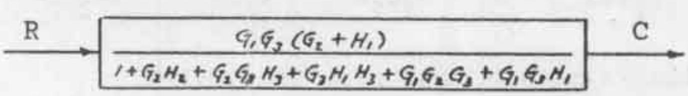


B-8-2. Simplified block diagrams for the system are shown below.





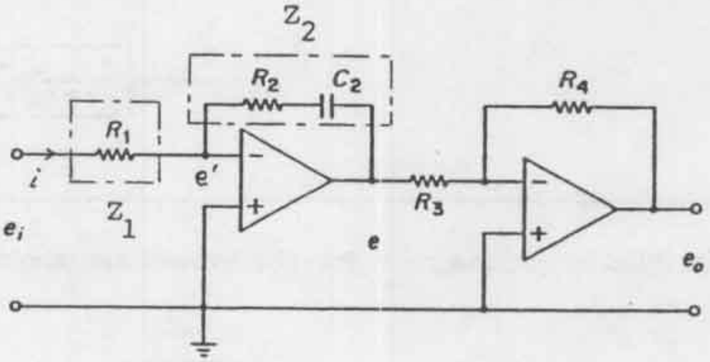




The transfer function C(s)/R(s) is

$$\frac{C(s)}{R(s)} = \frac{G_1G_3(G_2 + H_1)}{1 + G_2H_2 + G_2G_3H_3 + G_3H_1H_3 + G_1G_2G_3 + G_1G_3H_1}$$

B-8-3. Define the input impedance and feedback impedance as Z_1 and Z_2, respectively, as shown in the figure below.



Then

$$Z_1 = R_1\,, \qquad Z_2 = R_2 + \frac{1}{C_2s} = \frac{R_2C_2s + 1}{C_2s}$$

Since $e' \doteqdot 0$, Laplace transforms of voltages $e_i(t)$ and $e(t)$ are obtained as

$$E_i(s) = Z_1I(s) = R_1I(s)$$

$$E(s) = -Z_2I(s) = -\frac{R_2C_2S + 1}{C_2s}\,I(s)$$

Hence

$$\frac{E(s)}{E_i(s)} = -\frac{R_2C_2s + 1}{R_1C_2s}$$

Also,

$$\frac{E_o(s)}{E(s)} = -\frac{R_4}{R_3}$$

Therefore,

$$\frac{E_o(s)}{E_i(s)} = \frac{R_4}{R_3}\,\frac{R_2C_2s + 1}{R_1C_2s} = \frac{R_2R_4}{R_1R_3}\left(1 + \frac{1}{R_2C_2s}\right)$$

The control action is proportional plus integral.

B-8-4.

$$\frac{U(s)}{E(s)} = \frac{K}{1 + K\left(\frac{1}{K_0}\,\frac{T_1 s}{1 + T_1 s}\,\frac{1}{1 + T_2 s}\right)}$$

$$\doteqdot \frac{K}{K\left(\frac{1}{K_0}\,\frac{T_1 s}{1 + T_1 s}\,\frac{1}{1 + T_2 s}\right)}$$

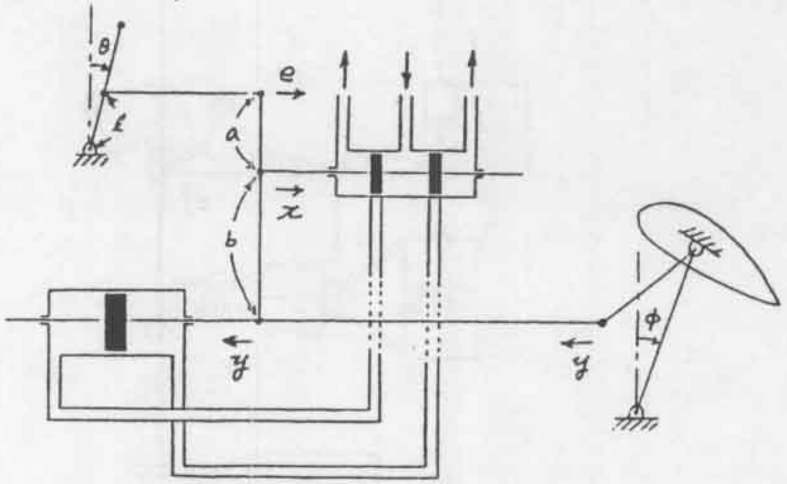
$$= \frac{K_0(1 + T_1 s)(1 + T_2 s)}{T_1 s}$$

$$= K_0\left(1 + \frac{1}{T_1 s}\right)(1 + T_2 s)$$

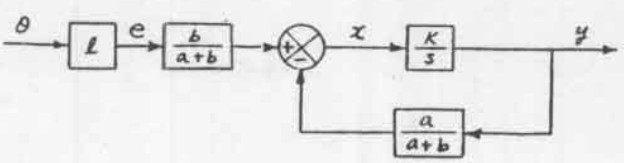
$$= K_0\left(1 + \frac{1}{T_1 s} + T_2 s + \frac{T_2}{T_1}\right)$$

$$= K_0\,\frac{T_1 + T_2}{T_1}\left[1 + \frac{1}{(T_1 + T_2)s} + \frac{T_1 T_2 s}{T_1 + T_2}\right]$$

B-8-5. Let us define displacements e, x, and y as shown in the figure below.



For relatively small angle θ we can construct a block diagram as shown below.



From the block diagram we obtain the transfer function Y(s)/Θ(s) as followws:

$$\frac{Y(s)}{\Theta(s)} = \frac{\ell \dfrac{b}{a+b} \dfrac{K}{s}}{1 + \dfrac{K}{s} \dfrac{a}{a+b}}$$

Since in such a system $\left| Ka/[s(a+b)] \right|$ is designed to be very large compared to 1, Y(s)/Θ(s) may be simplified to

$$\frac{Y(s)}{\Theta(s)} \doteq \frac{\ell b}{a+b} \frac{a+b}{a} = \ell \frac{b}{a}$$

We see that the piston displacement y is proportional to deflection angle θ of the control lever. Also, from the system diagram we see that for each small value of y, there is a corresponding value of angle ø. Therefore, for each small angle θ of the control lever, there is a corresponding steady-state elevator angle ø.

B-8-6. If the engine speed increases, the sleeve of the fly-ball governor moves upward. This movement acts as the input to the hydraulic controller. A positive error signal (upward motion of the sleeve) causes the power piston to move downward, reduces the fuel valve opening, and decreases the engine speed. Referring to Figure (a) shown below, a block diagram for the system can be drawn as shown in Figure (b) on next page.

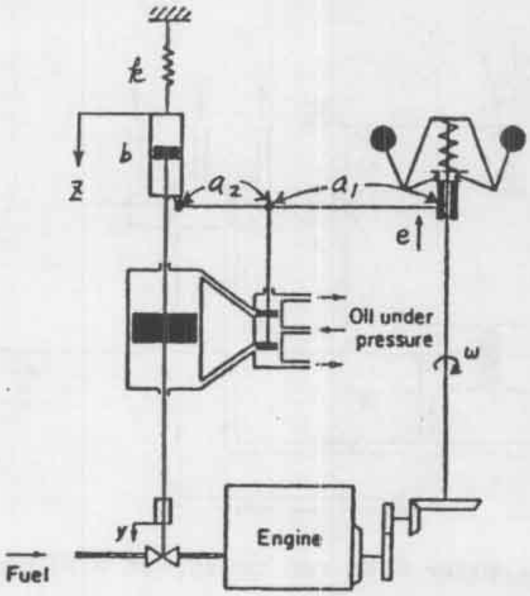


Figure (a)

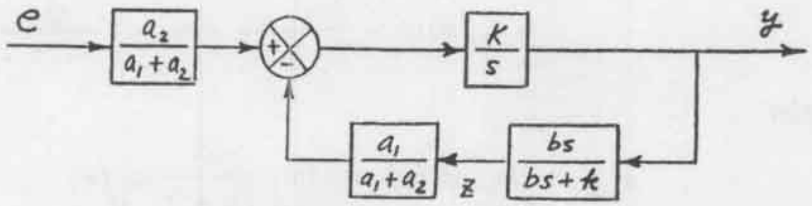


Figure (b)

From Figure (b) the transfer function Y(s)/E(s) is obtained as

$$\frac{Y(s)}{E(s)} = \frac{a_2}{a_1 + a_2} \frac{\frac{K}{s}}{1 + \frac{K}{s} \frac{a_1}{a_1 + a_2} \frac{bs}{bs + k}}$$

Since such a speed controller is usually designed such that

$$\left| \frac{K}{s} \frac{a_1}{a_1 + a_2} \frac{bs}{bs + k} \right| \gg 1$$

the transfer function Y(s)/E(s) becomes

$$\frac{Y(s)}{E(s)} \doteqdot \frac{a_2}{a_1 + a_2} \frac{a_1 + a_2}{a_1} \frac{bs + k}{bs} = \frac{a_2}{a_1} \left(1 + \frac{k}{bs}\right)$$

Thus, the control action of this speed controller is proportional-plus-integral.

B-8-7. For the first-order system

$$\frac{\Theta(s)}{\Theta_b(s)} = \frac{1}{Ts + 1}$$

the step response curve is an exponential curve. So the time constant T can be determined from such an exponential curve easily. From Figure 8-99 the time constant T is 2 s.

If this thermometer is placed in a bath, the temperature of which is increasing at a rate of 10°C/min = 1/6°C/s, or

$$\theta_b = \frac{1}{6} t + a$$

where a is a constant, then the steady-state error can be determined as follows: Noting that

$$E(s) = \Theta_b(s) - \Theta(s) = \Theta_b(s) \left[1 - \frac{\Theta(s)}{\Theta_b(s)} \right]$$

$$= \Theta_b(s)(1 - \frac{1}{2s + 1}) = \Theta_b(s) \frac{2s}{2s + 1}$$

we obtain

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{2s^2}{2s + 1} \Theta_b(s)$$

where

$$\Theta_b(s) = \frac{1}{6} \frac{1}{s^2} + \frac{a}{s} = \frac{1 + 6as}{6s^2}$$

Therefore,

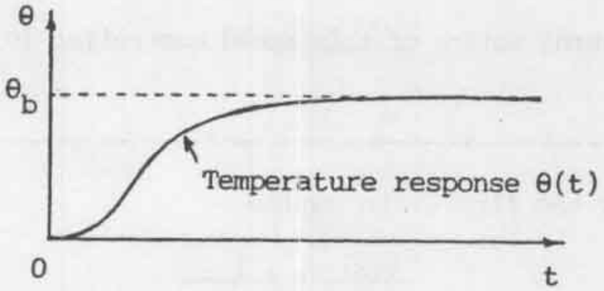
$$e_{ss} = \lim_{s\to 0} \frac{2s^2}{2s + 1} \frac{1 + 6as}{6s^2} = \frac{2}{1} \cdot \frac{1}{6} = \frac{1}{3} \text{ °C}$$

Thus, the steady-state error is 1/3°C.

For a second-order system:

$$\frac{\Theta(s)}{\Theta_b(s)} = \frac{1}{(T_1 s + 1)(T_2 s + 1)}$$

A typical response curve, when this thermometer is placed in a bath held at a constant temperature, is shown below.



B-8-8. The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{10(s + 1)}{s^2 + 10s + 10}$$

$$= \frac{10(s + 1)}{(s + 1.1270)(s + 8.8730)}$$

For the unit-step input, we have

$$C(s) = \frac{10(s + 1)}{(s + 1.1270)(s + 8.8730)} \frac{1}{s}$$

$$= \frac{1}{s} + \frac{0.1455}{s + 1.1270} - \frac{1.1455}{s + 8.8730}$$

Hence

$$c(t) = 1 + 0.1455\, e^{-1.1270t} - 1.1455\, e^{-8.8730t}$$

B-8-9. Since M_p is specified as 0.05, we have

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.05$$

or

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 2.995$$

Rewriting,

$$(\zeta\pi)^2 = (2.995)^2(1-\zeta^2)$$

Solving for the damping ratio ζ we obtain

$$\zeta = 0.69$$

The settling time t_s is specified as 2 seconds. So we have

$$t_s = \frac{4}{\zeta\omega_n} = 2$$

or

$$\zeta\omega_n = 2$$

Therefore,

$$\omega_n = \frac{2}{\zeta} = \frac{2}{0.69} = 2.90 \text{ rad/s}$$

B-8-10. The closed-loop transfer function of the system is

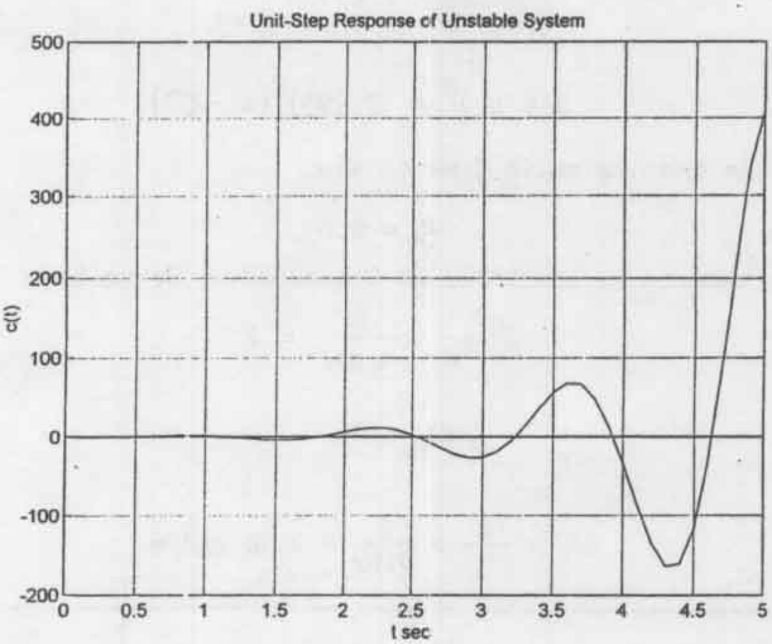
$$\frac{C(s)}{R(s)} = \frac{100}{s^3 + 2s^2 + 10s + 100}$$

$$= \frac{100}{(s + 4.5815)(s - 1.2907 + j4.4901)(s - 1.2907 - j4.4901)}$$

This system is unstable because two complex-conjugate closed-loop poles are in the right half plane. To visualize the unstable response, we may enter the following MATLAB program into the computer. The resulting unstable response curve is shown in the figure on next page.

To make the system stable, it is necessary to reduce the gain of the system or add an appropriate compensator.

```
num = [0  0  0  100];
den = [1  2  10  100];
t = 0:0.1:5;
step(num,den,t)
grid
title('Unit-Step Response of Unstable System')
xlabel('t sec')
ylabel('c(t)')
```



B-8-11.

$$\frac{C(s)}{R(s)} = \frac{16}{s^2 + (0.8 + 16k)s + 16}$$

From the characteristic polynomial, we find

$$\omega_n = 4, \quad 2\zeta\omega_n = 2 \times 0.5 \times 4 = 0.8 + 16k$$

Hence

$$k = 0.2$$

The rise timw t_r is obtained from

$$t_r = \frac{\pi - \beta}{\omega_d}$$

Since

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 4\sqrt{1 - 0.25} = 3.46$$

$$\beta = \sin^{-1}\frac{\omega_d}{\omega_n} = \sin^{-1} 0.866 = \frac{\pi}{3}$$

we have

$$t_r = \frac{\pi - \frac{1}{3}\pi}{3.46} = 0.605 \text{ s}$$

The peak time t_p is obtained as

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{3.46} = 0.907 \text{ s}$$

The maximum overshoot M_p is

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{-\frac{0.5\times3.14}{\sqrt{1-0.25}}} = e^{-1.814} = 0.163$$

The settling time t_s is

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.5 \times 4} = 2 \text{ s}$$

B-8-12. The closed-loop transfer function for the system is

$$\frac{C(s)}{R(s)} = \frac{K}{2s^2 + s + KK_hs + K} = \frac{0.5K}{s^2 + 0.5(1 + KK_h)s + 0.5K}$$

From this equation, we obtsin

$$\omega_n = \sqrt{0.5K}, \qquad 2\zeta\omega_n = 0.5(1 + KK_h)$$

Since the damping ratio ζ is specified as 0.5, we get

$$\omega_n = 0.5(1 + KK_h)$$

Therefore, we have

$$0.5(1 + KK_h) = \sqrt{0.5K}$$

The settling time is specified as

$$t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.25(1 + KK_h)} = \frac{16}{1 + KK_h} \leqslant 2$$

Since the feedforward transfer function G(s) is

$$G(s) = \frac{\frac{K}{2s + 1}}{1 + \frac{KK_h}{2s + 1}}\frac{1}{s} = \frac{K}{2s + 1 + KK_h}\frac{1}{s}$$

the static velocity error constant K_v is

$$K_v = \lim_{s\to0} sG(s) = \lim_{s\to0} s\frac{K}{2s + 1 + KK_h}\frac{1}{s} = \frac{K}{1 + KK_h}$$

This value must be equal to or greater than 50. Hence,

$$\frac{K}{1 + KK_h} \geqslant 50$$

Thus, the conditions to be satisfied can be summarized as follows:

$$0.5(1 + KK_h) = \sqrt{0.5K} \tag{1}$$

$$\frac{16}{1 + KK_h} \leqslant 2 \tag{2}$$

$$\frac{K}{1 + KK_h} \geqslant 50 \tag{3}$$

$$0 < K_h < 1 \tag{4}$$

From Equations (1) and (2), we gwt

$$8 \leqslant 1 + KK_h = \sqrt{2K}$$

or

$$32 < K$$

From Equation (3) we obtain

$$\frac{K}{50} \geqslant 1 + KK_h = \sqrt{2K}$$

or

$$K \geqslant 5000$$

If we choose K = 5000, then we get

$$1 + KK_h = \sqrt{2K} = 100$$

or

$$K_h = \frac{99}{5000} = 0.0198$$

Thus, we determined a set of values of K and K_h as follows:

$$K = 5000, \quad K_h = 0.0198$$

With these values of K and K_h, all specifications are satisfied.

B-8-13.

$$\frac{C(s)}{R(s)} = \frac{K_p(1 + T_d s)}{Js^2 + K_p(1 + T_d s)}$$

Since $R(s) = 1/s^2$, the output $C(s)$ is obtained as

$$C(s) = \frac{K_p + K_p T_d s}{Js^2 + K_p T_d s + K_p} \frac{1}{s^2} = \frac{1}{s^2} - \frac{1}{s^2 + \frac{K_p T_d}{J} s + \frac{K_p}{J}}$$

Since the system is underdamped, $C(s)$ can be written as

$$C(s) = \frac{1}{s^2} - \frac{1}{\left(s + \frac{K_p T_d}{2J}\right)^2 + \frac{K_p}{J} - \frac{K_p^2 T_d^2}{4J^2}}$$

$$= \frac{1}{s^2} - \frac{1}{\sqrt{\frac{K_p}{J} - \frac{K_p^2 T_d^2}{4J^2}}} \frac{\sqrt{\frac{K_p}{J} - \frac{K_p^2 T_d^2}{4J^2}}}{\left(s + \frac{K_p T_d}{2J}\right)^2 + \left(\sqrt{\frac{K_p}{J} - \frac{K_p^2 T_d^2}{4J^2}}\right)^2}$$

The inverse Laplace transform of $C(s)$ gives

$$c(t) = t - \frac{1}{\sqrt{\frac{K_p}{J} - \frac{K_p^2 T_d^2}{4J^2}}} e^{-\frac{K_p T_d}{2J} t} \sin \sqrt{\frac{K_p}{J} - \frac{K_p^2 T_d^2}{4J^2}}\ t$$

The steady-state error e_{ss} for a unit ramp input is

$$e_{ss} = \lim_{t \to \infty} [r(t) - c(t)] = \lim_{t \to \infty} [t - c(t)]$$

$$= \lim_{t \to \infty} \left(\frac{1}{\sqrt{\frac{K_p}{J} - \frac{K_p^2 T_d^2}{4J^2}}} e^{-\frac{K_p T_d}{2J} t} \sin \sqrt{\frac{K_p}{J} - \frac{K_p^2 T_d^2}{4J^2}}\ t \right) = 0$$

The steady-state error can also be obtained by use of the final value theorem. Since the error signal $E(s)$ is

$$E(s) = R(s) - C(s) = R(s) \left[1 - \frac{C(s)}{R(s)}\right] = \frac{1}{s^2}\left[1 - \frac{K_p(1 + T_d s)}{Js^2 + K_p(1 + T_d s)}\right]$$

we obtain the steady-state error e_{ss} as

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s\, Js^2}{s^2(Js^2 + K_p + T_d K_p s)} = 0$$

B-8-14. The characteristic equation is

$$\frac{K}{s(s + 1)(s + 5)} + 1 = 0$$

or

$$s^3 + 6s^2 + 5s + K = 0$$

The Routh array for this equation is

$$\begin{array}{cc} 1 & 5 \\ 6 & K \\ \frac{30 - K}{6} & 0 \\ K & \end{array}$$

For the system to be stable, there should be no sign changes in the first column. This requires

$$30 - K > 0, \qquad K > 0$$

Hence, we get the range of gain K for stability to be

$$30 > K > 0$$

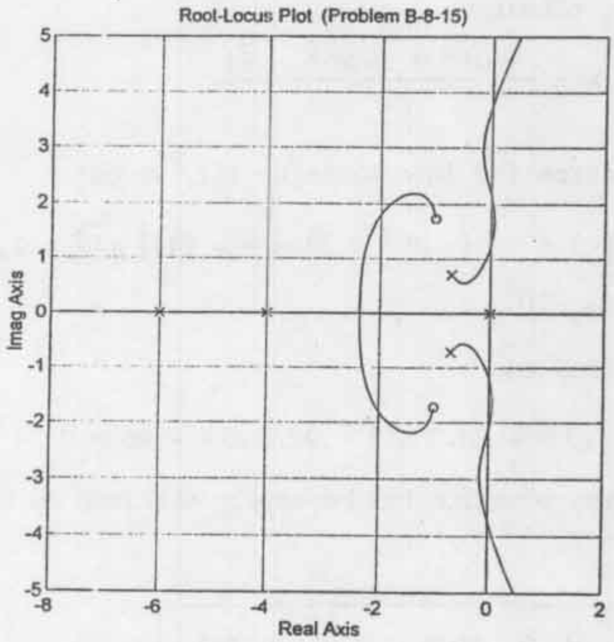
B-8-15. Since the system is of higher order (5th order), it is easier to find the range of gain K for stability by first plotting the root loci and then finding critical points (for stability) on the root loci. The open-loop transfer function G(s) can be written as

$$G(s) = \frac{K(s^2 + 2s + 4)}{s(s + 4)(s + 6)(s^2 + 1.4s + 1)}$$

$$= \frac{K(s^2 + 2s + 4)}{s^5 + 11.4s^4 + 39s^3 + 43.6s^2 + 24s}$$

The MATLAB program given on next page will generate a plot of the root loci for the system. The resulting root-locus plot is shown also on next page.

```
num = [0  0  0  1  2  4];
den = [1  11.4  39  43.6  24  0];
rlocus(num,den)

Warning: Divide by zero
v = [-8  2  -5  5]; axis(v); axis('square')
grid
title('Root-Locus Plot (Problem B-8-15)')
```



Based on this plot, it can be seen that the system is conditionally stable. All critical points for stability lie on the $j\omega$ axis.

To determine the crossing points of the root loci with the $j\omega$ axis, substitute $s = j\omega$ into the characteristic equation which is

$$s^5 + 11.4s^4 + 39s^3 + 43.6s^2 + 24s + K(s^2 + 2s + 4) = 0$$

Then

$$(j\omega)^5 + 11.4(j\omega)^4 + 39(j\omega)^3 + (43.6 + K)(j\omega)^2 + (24 + 2K)j\omega$$

$$+ 4K = 0$$

This equation can be rewritten as

$$[11.4\,\omega^4 - (43.6 + K)\omega^2 + 4K] + j[\omega^5 - 39\omega^3 + (24 + 2K)\omega] = 0$$

By equating the real part and imaginary part equal to zero, respectively, we obtain

$$11.4\,\omega^4 - (43.6 + K)\,\omega^2 + 4K = 0 \qquad (1)$$

$$\omega^5 - 39\,\omega^3 + (24 + 2K)\omega = 0 \qquad (2)$$

Equation (2) can be written as

$$\omega = 0$$

or

$$\omega^4 - 39\,\omega^2 + 24 + 2K = 0 \qquad (3)$$

From Equation (3) we obtain

$$K = \frac{-\omega^4 + 39\,\omega^2 - 24}{2} \qquad (4)$$

By substituting Equation (4) into Equation (1), we get

$$11.4\,\omega^4 - [43.6 + \tfrac{1}{2}(-\omega^4 + 39\,\omega^2 - 24)]\,\omega^2 - 2\,\omega^4$$

$$+ 78\,\omega^2 - 48 = 0$$

which can be simplified to

$$\omega^6 - 20.2\,\omega^4 + 92.8\,\omega^2 - 96 = 0$$

The roots of this last equation can be easily obtained by use of the MATLAB program given below.

```
a=[1  0  -20.2  0  92.8  0  -96];
roots(a)

ans =

  3.7553
 -3.7553
  2.1509
 -2.1509
  1.2130
 -1.2130
```

The root-locus branch in the upper half plane that goes to infinity crosses the $j\omega$ axis at $\omega = 1.2130$, $\omega = 2.1509$, and $\omega = 3.7553$. The gain values at these crossing points are obtained as follows:

$$K = \frac{-1.2130^4 + 39 \times 1.2130^2 - 24}{2} = 15.61 \qquad \text{for } \omega = 1.2130$$

$$K = \frac{-2.1509^4 + 39 \times 2.1509^2 - 24}{2} = 67.51 \qquad \text{for } \omega = 2.1509$$

$$K = \frac{-3.7553^4 + 39 \times 3.7553^2 - 24}{2} = 163.56 \qquad \text{for } \omega = 3.7553$$

Based on the K values above, we obtain the range of gain K for stability as follows: The system is stable if

$$15.61 > K > 0$$

$$163.56 > K > 67.51$$

B-8-16. A MATLAB program to plot the root loci and asymptotes for the following system

$$G(s)H(s) = \frac{K}{s(s + 0.5)(s^2 + 0.6s + 10)}$$

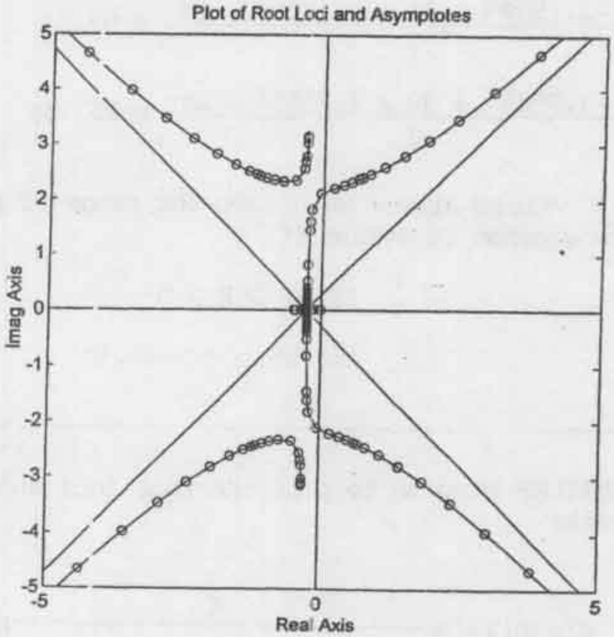
is given below and the resulting root-locus plot is shown on next page.

Note that the equation for the asymptotes is

$$G_a(s)H_a(s) = \frac{K}{(s + 0.275)^4}$$

$$= \frac{K}{s^4 + 1.1s^3 + 0.4538s^2 + 0.08319s + 0.005719}$$

```
num = [0  0  0  0  1];
den = [1  1.1  10.3  5  0];
numa = [0  0  0  0  1];
dena = [1  1.1  0.4538  0.08319  0.005719];
r = rlocus(num,den);
plot(r,'-')
hold
Current plot held
plot(r,'o')
 rlocus(numa,dena);
v = [-5  5  -5  5]; axis(v); axis('square')
title('Plot of Root Loci and Asymptotes')
```



B-8-17. The open-loop transfer function G(s) is

$$G(s) = K \frac{s + 1}{s + 5} \frac{2}{s^2(s + 2)}$$

$$= \frac{K2(s + 1)}{s^4 + 7s^3 + 10s^2}$$

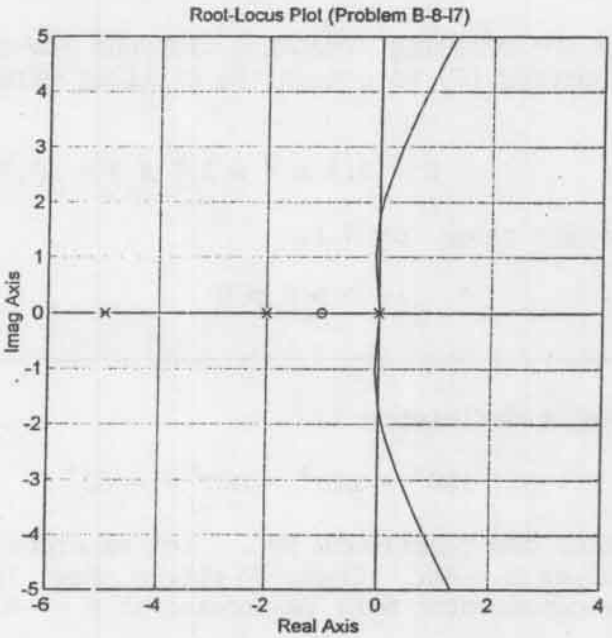
The following MATLAB program will generate a root-locus plot. The resulting plot is shown on next page.

```
num = [0  0  0  1  1];
den = [1  7  10  0  0];
rlocus(num,den)
v = [-6  4  -5  5]; axis(v); axis('square')
grid
title('Root-Locus Plot (Problem B-8-17)')
```

From the plot we find that the critical value of gain K for stability corresponds to the crossing point of the root locus branch that goes to infinity and the imaginary axis. Hence, we first find the crossing frequency and then find the corresponding gain value.

The characteristic equation for this system is

$$s^4 + 7s^3 + 10s^2 + 2Ks + 2K = 0$$



By substituting $s = j\omega$ into the characteristic equation, we obtain

$$(j\omega)^4 + 7(j\omega)^3 + 10(j\omega)^2 + 2K(j\omega) + 2K = 0$$

which can be rewritten as

$$(\omega^4 - 10\omega^2 + 2K) + j\omega(-7\omega^2 + 2K) = 0$$

By equating the real part and imaginary part of this last equation to zero, respectively, we get

$$\omega^4 - 10\omega^2 + 2K = 0 \qquad (1)$$

$$\omega(-7\omega^2 + 2K) = 0 \qquad (2)$$

Equation (2) can be rewritten as

$$\omega = 0$$

or

$$-7\omega^2 + 2K = 0 \qquad (3)$$

By substituting Equation (3) into Equation (1), we find

$$\omega^4 - 10\omega^2 + 7\omega^2 = 0$$

or

$$\omega^4 - 3\omega^2 = 0$$

which yields

$$\omega = 0, \quad \omega = 0, \quad \omega = \sqrt{3}, \quad \omega = -\sqrt{3}$$

Since $\omega = \sqrt{3}$ is the crossing frequency with the $j\omega$ axis, by substituting $\omega = \sqrt{3}$ into Equation (3) we obtain the critical value of gain K for stability.

$$K = 3.5\,\omega^2 = 3.5 \times 3 = 10.5$$

Hence the stability range for K is

$$10.5 > K > 0$$

B-8-18. The angle deficiency is

$$180^\circ - 120^\circ - 120^\circ = -60^\circ$$

A lead compensator can contribute 60°. Let us choose the zero of the lead compensator at s = -1. Then, to obtain phase lead angle of 60°, the pole of the compensator must be located at s = -4. Thus,

$$G_c(s) = K \frac{s+1}{s+4}$$

The gain K can be determined from the magnitude condition.

$$\left| K \frac{s+1}{s+4} \frac{1}{s^2} \right|_{s=-1+j\sqrt{3}} = 1$$

or

$$K = \left| \frac{(s+4)s^2}{s+1} \right|_{s=-1+j\sqrt{3}} = 8$$

Hence the lead compensator becomes as follows:

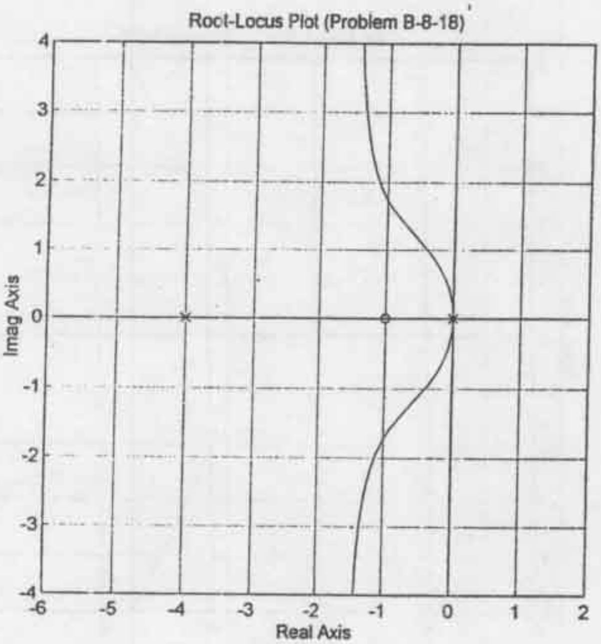
$$G_c(s) = 8 \frac{s+1}{s+4}$$

The feedforward transfer function is

$$G_c(s)G(s) = \frac{8s+8}{s^3+4s^2}$$

The following MATLAB program will generate a root-locus plot. The resulting plot is shown on next page.

```
num = [0  0  8  8];
den = [1  4  0  0];
rlocus(num,den)
v = [-6  2  -4  4]; axis(v); axis('square')
grid
title('Root-Locus Plot (Problem B-8-18)')
```



Note that the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{8s + 8}{s^3 + 4s^2 + 8s + 8}$$

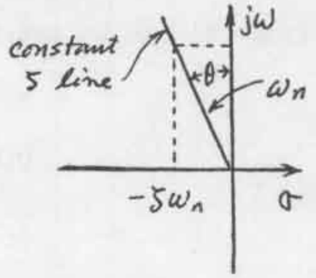
The closed-loop poles are located at $s = -1 \pm j\sqrt{3}$ and $s = -2$.

B-8-19. The MATLAB program given below generates a root-locus plot for the given system. The resulting plot is shown on next page.

```
num = [0  0  0  1];
den = [1  5  4  0];
rlocus(num,den)
v = [-6  4  -5  5]; axis(v); axis('square')
grid
title('Root-Locus Plot (Problem B-8-19)')
```

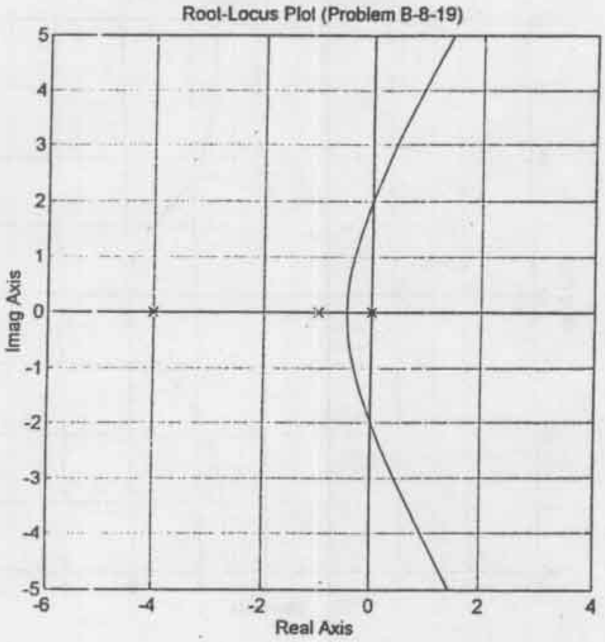
Note that constant-ζ points $(0 < \zeta < 1)$ lie on a straight line having angle θ from the $j\omega$ axis as shown in the figure below.
From the figure we obtain

$$\sin\theta = \frac{\zeta\omega_n}{\omega_n} = \zeta$$



Note also that $\zeta = 0.6$ line can be defined by

$$s = -0.75a + ja$$



where a is a variable ($0 < a < \infty$). To find the value of K such that the damping ratio ζ of the dominant closed-loop poles is 0.6 can be found by finding the intersection of the line $s = -0.75a + ja$ and the root locus. The intersection point can be determined by solving the following simultaneous equations for a.

$$s = -0.75a + ja \tag{1}$$

$$s(s + 1)(s + 4) + K = 0 \tag{2}$$

By substituting Equation (1) into Equation (2),

$$(-0.75a + ja)(-0.75a + ja + 1)(-0.75a + ja + 4) + K = 0$$

which can be rewritten as

$$(1.8281a^3 - 2.1875a^2 - 3a + K) + j(0.6875a^3 - 7.5a^2 + 4a) = 0$$

By equating the real part and imaginary part of this last equation to zero, respectively, we obtain

$$1.8281a^3 - 2.1875a^2 - 3a + K = 0 \tag{3}$$

$$0.6875a^3 - 7.5a^2 + 4a = 0 \tag{4}$$

Equation (4) can be rewritten as

$$a = 0$$

or

$$0.6875a^2 - 7.5a + 4 = 0$$

which can be written as

$$a^2 - 10.9091a + 5.8182 = 0$$

or

$$(a - 0.5623)(a - 10.3468) = 0$$

Hence

$$a = 0.5623 \quad \text{or} \quad a = 10.3468$$

From Equation (3) we find

$$K = -1.8281a^3 + 2.1875a^2 + 3a = 2.0535 \quad \text{for } a = 0.5623$$

$$K = -1.8281a^3 + 2.1875a^2 + 3a = -1759.74 \quad \text{for } a = 10.3468$$

Since the K value is positive for a = 0.5623 and negative for a = 10.3468, we choose a = 0.5623. The required gain K is 2.0535.

Since the characteristic equation with K = 2.0535 is

$$s(s + 1)(s + 4) + 2.0535 = 0$$

or

$$s^3 + 5s^2 + 4s + 2.0535 = 0$$

the closed-loop poles can be obtained by use of the following MATLAB program.

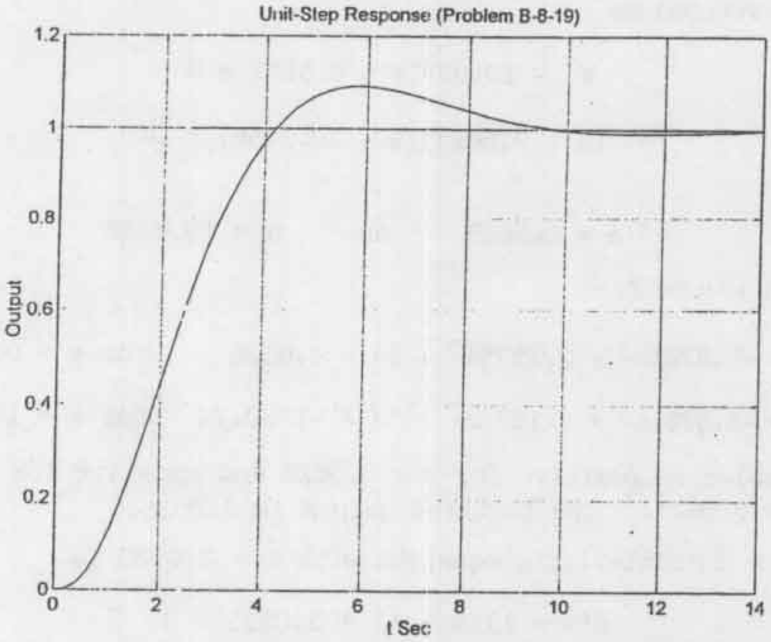
```
p = [1  5  4  2.0535];
roots(p)

ans =

  -4.1565
  -0.4217 + 0.5623i
  -0.4217 - 0.5623i
```

Thus, the closed-loop poles are located at

$$s = -0.4217 \pm j0.5623, \quad s = -4.1565$$

The unit-step response of the system with K = 2.0535 can be obtained by entering the following MATLAB program into the computer. The resulting unit-step response curve is shown on next page.

```
num = [0  0  0  2.0535];
den = [1  5  4  2.0535];
step(num,den)
grid
title('Unit-Step Response (Problem B-8-19)')
xlabel('t Sec')
ylabel('Output')
```



B-8-20. The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{K(s + a)(s + b)}{s(s^2 + 1) + K(s + a)(s + b)}$$

$$= \frac{K(s^2 + as + bs + ab)}{s^3 + s + K(s^2 + as + bs + ab)}$$

Since the dominant closed-loop poles are located at $s = -1 \pm j1$, the characteristic equation must be divisible by

$$(s + 1 + j1)(s + 1 - j1) = s^2 + 2s + 2$$

Hence

$$s^3 + Ks^2 + (1 + aK + bK)s + abK = (s^2 + 2s + 2)(s + \alpha)$$

where $s = -\alpha$ is the unknown third pole. By dividing the left side of this last equation by $s^2 + 2s + 2$, we obtain

$$s^3 + Ks^2 + (1 + aK + bK)s + abK = (s^2 + 2s + 2)(s + K - 2)$$
$$+ (aK + bK - 2K + 3)s + Kab - 2(K - 2)$$

The remainder of division must be zero. Hence we set

$$aK + bK - 2K + 3 = 0$$

$$Kab - 2(K - 2) = 0$$

Since a is specified as 0.5, by substituting a = 0.5 into these two equations, we obtain

$$bK = 1.5K - 3 \qquad (1)$$

$$0.5Kb - 2(K - 2) = 0 \qquad (2)$$

By substituting Equation (1) into Equation (2), we have

$$0.5(1.5K - 3) - 2(K - 2) = 0$$

or

$$K = 2$$

Then, by substituting K = 2 into Equation (1), we get

$$2b = 1.5 \times 2 - 3 = 0$$

Hence

$$b = 0$$

The PID controller with K = 2 and b = 0 becomes

$$G_c(s) = K \frac{(s + a)(s + b)}{s} = K \frac{(s + 0.5)s}{s} = K(s + 0.5)$$

Thus, the controller becomes a PD controller. The open-loop transfer function becomes

$$G_c(s)G(s) = \frac{K(s + 0.5)}{s^2 + 1}$$

The closed-loop transfer function (with b = 0 and K = 2) becomes as follows:

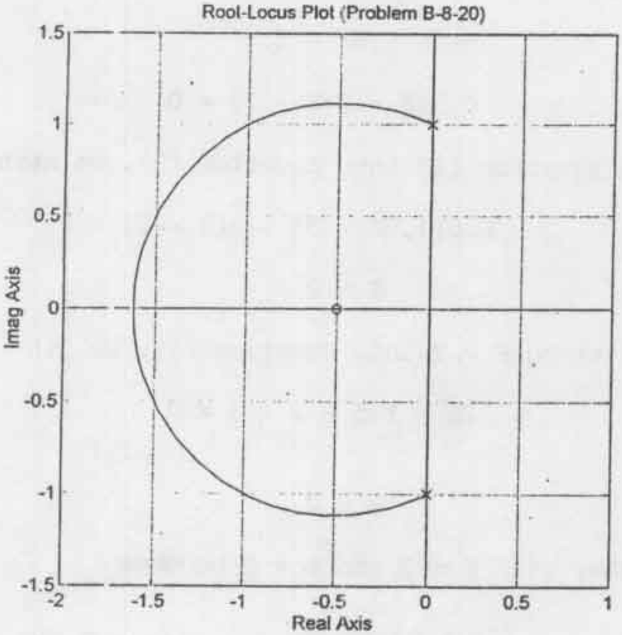
$$\frac{C(s)}{R(s)} = \frac{2(s + 0.5)}{s^2 + 2s + 2}$$

$$= \frac{2s + 1}{(s + 1 + j1)(s + 1 - j1)}$$

The root-locus plot for the designed system can be obtained by enterring the following MATLAB program into the computer.

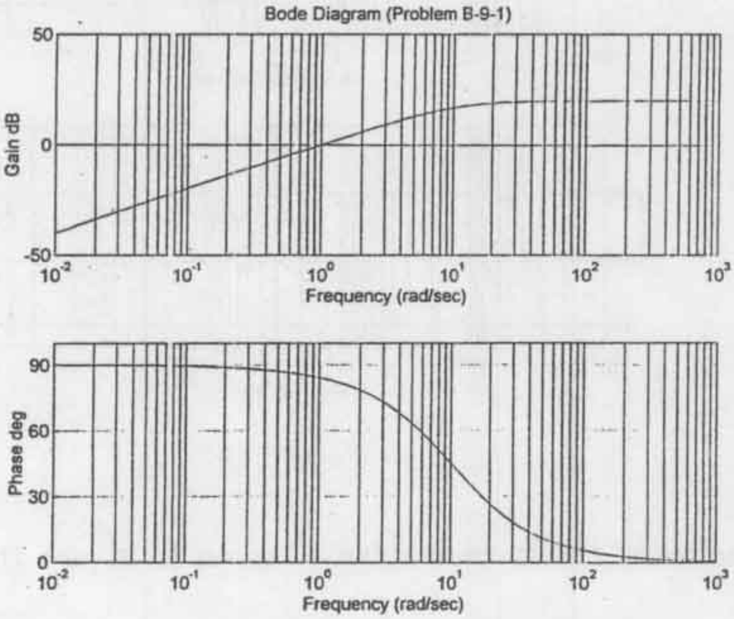
```
num = [0  1  0.5];
den = [1  0  1];
rlocus(num,den)
v = [-2  1  -1.5  1.5]; axis(v); axis('square')
grid
title('Root-Locus Plot (Problem B-8-20)')
```

The resulting root-locus plot is shown on next page.

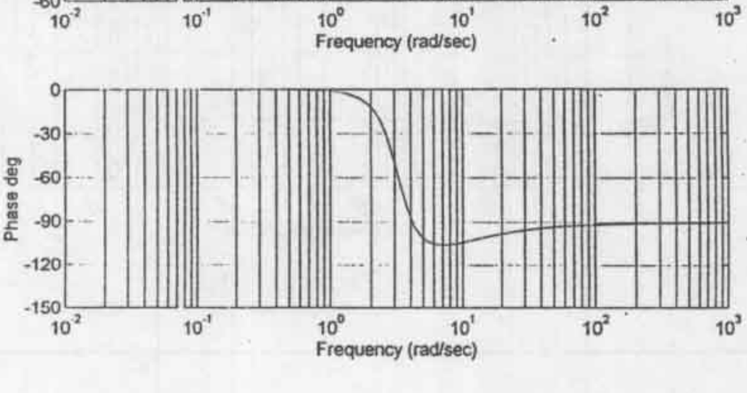

Root-Locus Plot (Problem B-8-20)
Imag Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1.5
-1
-0.5
0
0.5
1

CHAPTER 9

B-9-1.



B-9-2.



B-9-3. A Bode diagram of the PI controller is shown in Figure (a).

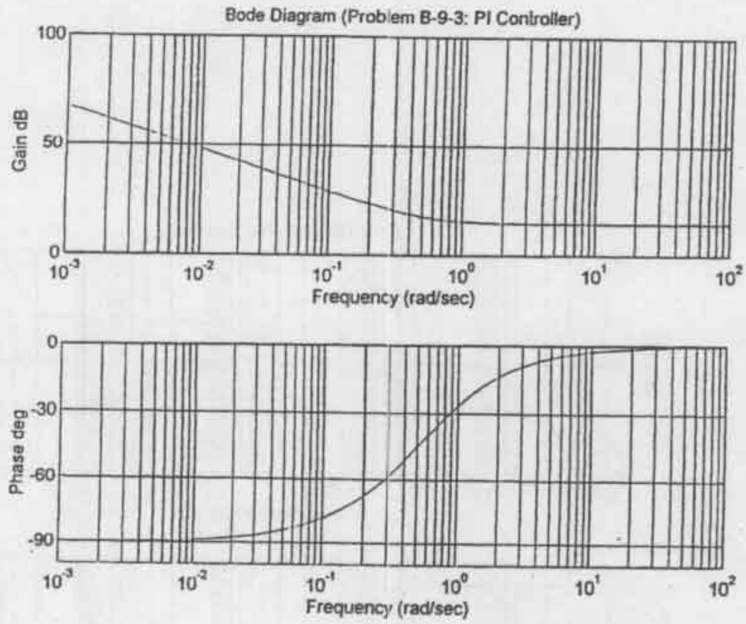


Figure (a)

A Bode diagram of the PD controller is shown in Figure (b).

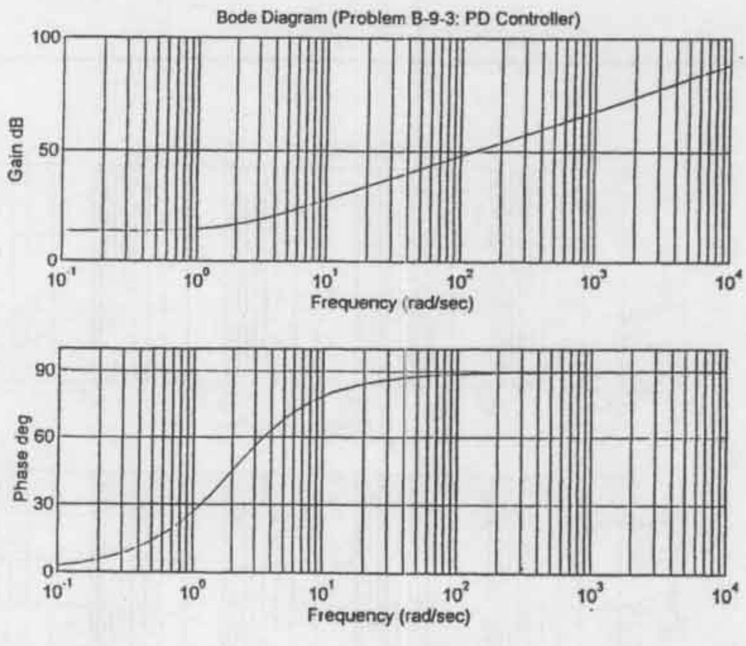
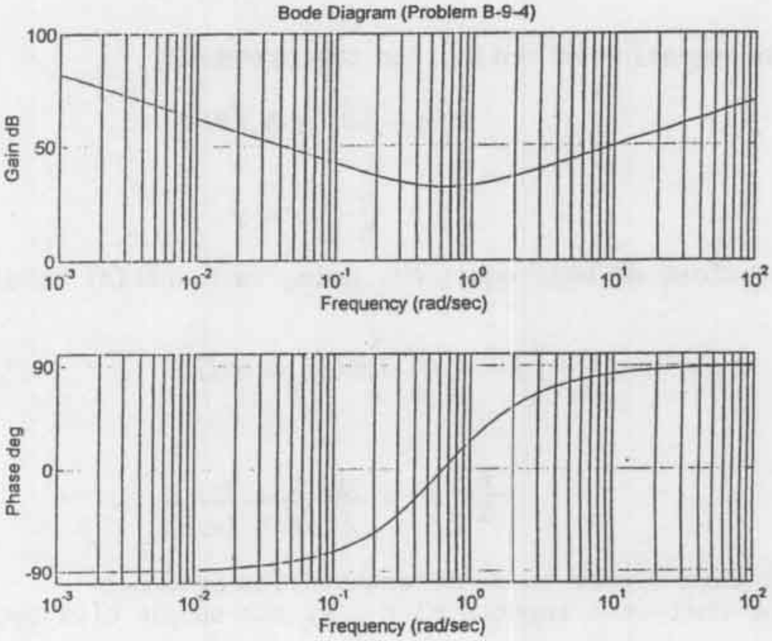


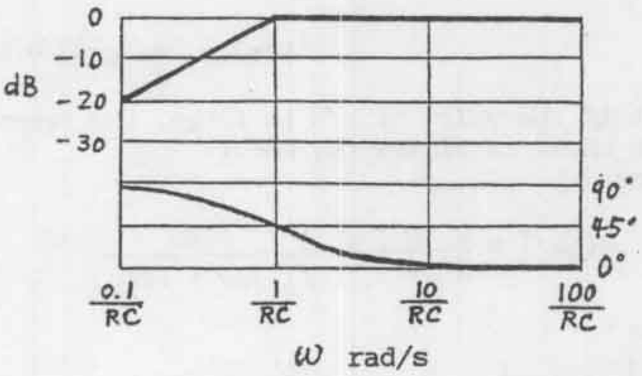
Figure (b)

B-9-4.

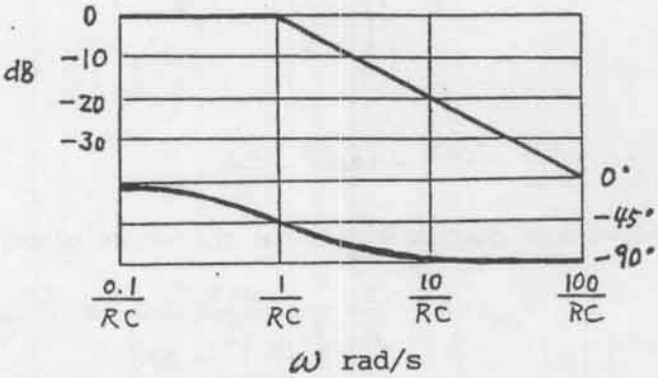


B-9-5.

Lead network



Lag network



B-9-6. The equation of motion for the system is

$$b(\dot{x} - \ell\dot{\theta}) = k\ell\theta$$

or

$$\ell\dot{\theta} + \frac{k}{b}\ell\theta = \dot{x}$$

The $\mathcal{L}$- transform of this equation, using zero initial conditions, gives

$$\left(\ell s + \frac{k}{b}\ell\right)\Theta(s) = sX(s)$$

Hence

$$\frac{\Theta(s)}{X(s)} = \frac{1}{\ell}\frac{s}{s + (k/b)}$$

Notice that this system is a differentiating system.

For the unit-step input $X(s) = 1/s$, the output $\Theta(s)$ becomes

$$\Theta(s) = \frac{1}{\ell}\frac{1}{s + (k/b)}$$

The inverse Laplace transform of $\Theta(s)$ gives

$$\theta(t) = \frac{1}{\ell}e^{-(k/b)t}$$

Note that if the value of k/b is large, the response $\theta(t)$ approaches a pulse signal as shown in Figure (a) below.

Since

$$G(j\omega) = \frac{\Theta(j\omega)}{X(j\omega)} = \frac{1}{\ell}\frac{j\omega}{j\omega + (k/b)}$$

we obtain

$$\left|G(j\omega)\right| = \frac{1}{\ell}\frac{\omega}{\sqrt{\left(\frac{k}{b}\right)^2 + \omega^2}}$$

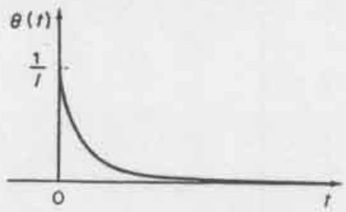


Figure (a)

and

$$\angle G(j\omega) = 90^\circ - \tan^{-1}\frac{\omega b}{k}$$

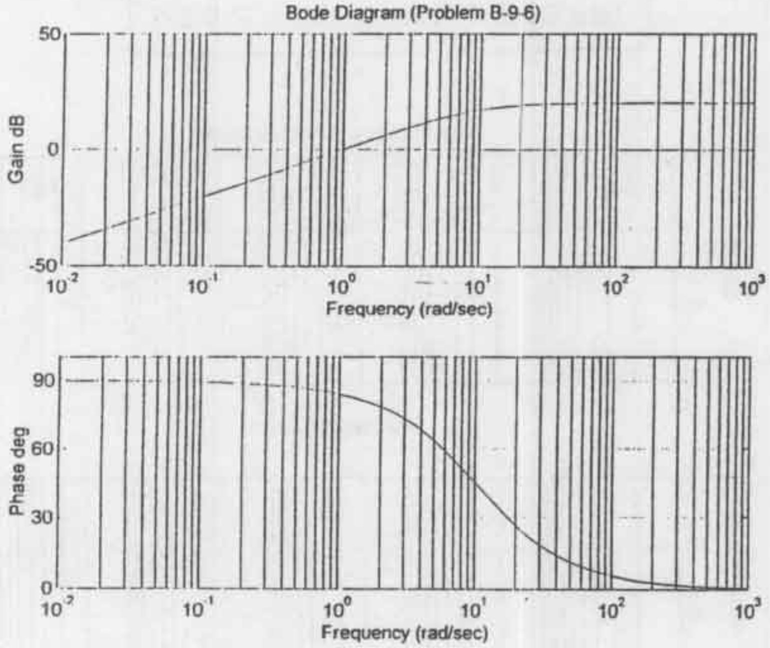
The steady-state output $\theta_{ss}(t)$ is therefore given by

$$\theta_{ss}(t) = \frac{1}{\ell}\frac{\omega X}{\sqrt{\left(\frac{k}{b}\right)^2 + \omega^2}}\sin\left(\omega t + 90^\circ - \tan^{-1}\frac{\omega b}{k}\right)$$

Next, substituting $\ell = 0.1$ m, $k = 2$ N/m, and $b = 0.2$ N-s/m into $G(j\omega)$ gives

$$G(j\omega) = 10 \frac{j\omega}{j\omega + 10}$$

A Bode diagram of $G(j\omega)$ is shown below.



B-9-7. Noting that

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

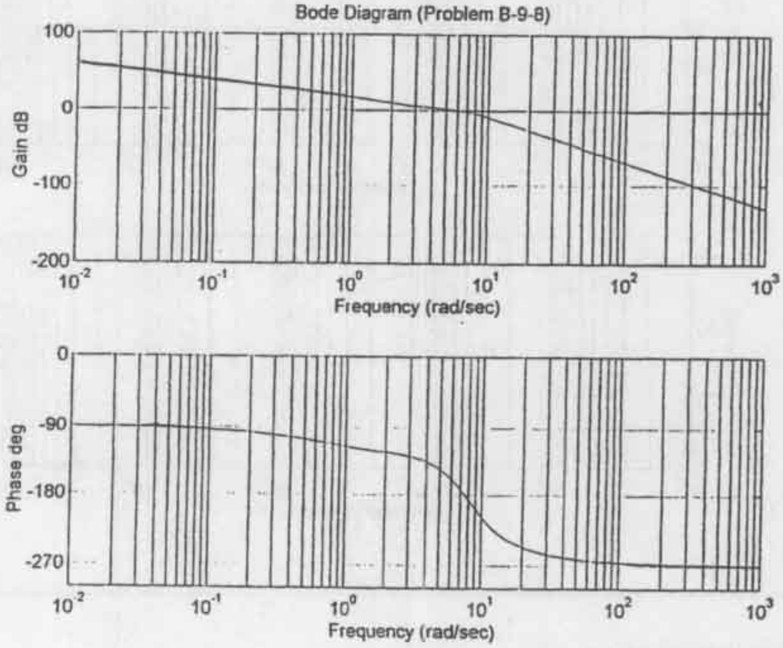
$$= \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + 1}$$

we have

$$\left|G(j\omega_n)\right| = \left|\frac{1}{-1 + 2\zeta j + 1}\right| = \frac{1}{2\zeta}$$

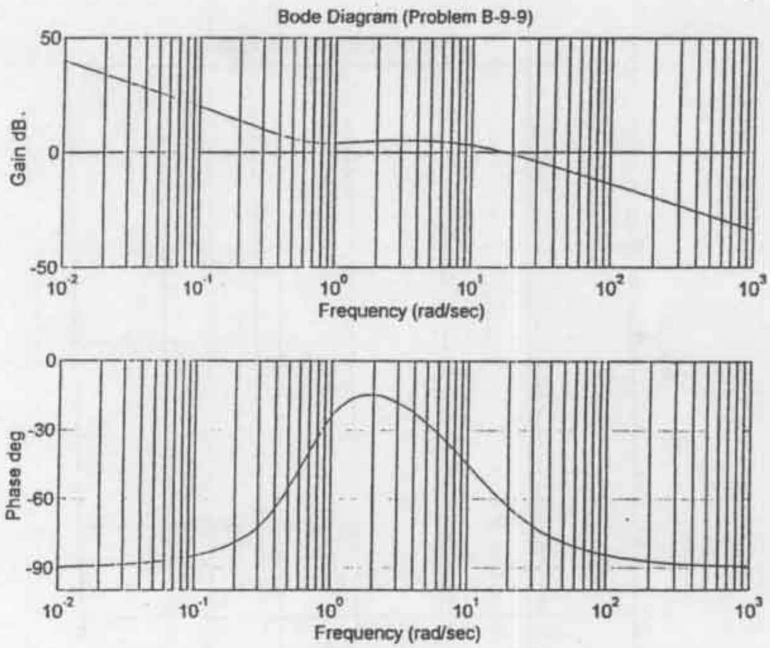
B-9-8. A possible MATLAB program for obtaining a Bode diagram of the given G(s) is shown on next page. The resulting Bode diagram is shown also on next page.

```
num = [0  0  0  320  640];
den = [1  9  72  64  0];
w = logspace(-2,3,100);
bode(num,den,w)
subplot(2,1,1);
title('Bode Diagram (Problem B-9-8)')
```



B-9-9. A possible MATLAB program to obtain a Bode diagram of the given G(s) is shown below. The resulting Bode diagram is shown on next page.

```
num = [0  20  20  10];
den = [1  11  10  0];
w = logspace(-2,3,100);
bode(num,den,w)
subplot(2,1,1);
title('Bode Diagram (Problem B-9-9)')
```

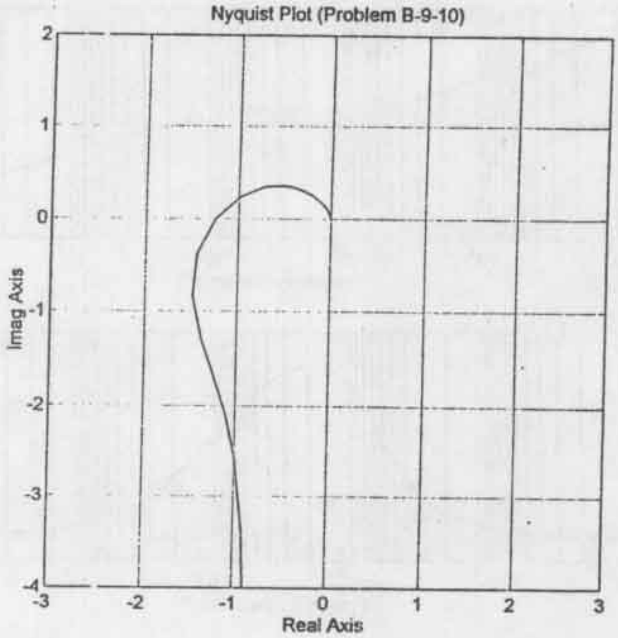


B-9-10. A possible MATLAB program for obtaining a Nyquist plot of the given G(s) is shown below. Note that to plot $G(j\omega)$ locus only for $\omega > 0$, we use the following command:

```
[re,im,w] = nyquist(num,den,w);

plot(re,im)
```

The resulting Nyquist plot is shown on next page.

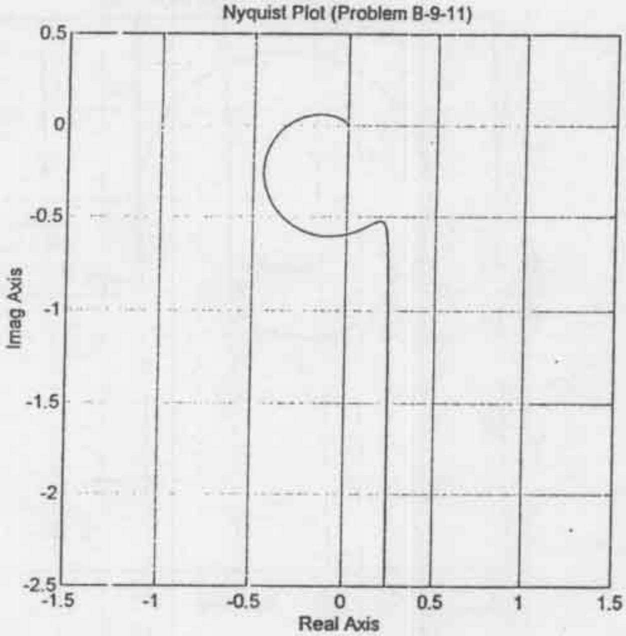
```
num = [0  0  0  1];
den = [1  0.8  1  0];
w = 0.1:0.1:100;
[re,im,w] = nyquist(num,den,w);
plot(re,im)
v = [-3  3  -4  2]; axis(v); axis('square')
grid
title('Nyquist Plot (Problem B-9-10)')
xlabel('Real Axis')
ylabel('Imag Axis')
```



Since none of the open-loop poles lie in the right-half s plane and the $G(j\omega)$ locus encircles the $-1 + j0$ point twice clockwise if $G(j\omega)$ locus is plotted from $\omega = -\infty$ to $\omega = \infty$, the closed-loop system is unstable.

B-9-11. A possible MATLAB program for obtaining a Nyquist plot of the given G(s) is shown below. The resulting Nyquist plot is shown on next page. Since none of the open-loop poles lie in the right-half s plane and from the Nyquist plot it can be seen that the $G(j\omega)$ locus does not encircle the $-1 + j0$ point, the system is stable.

```
num = [0  0  0  20  20].
den = [1  7  20  50  0];
w = 0.1:0.1:100;
[re,im,w] = nyquist(num,den,w);
plot(re,im)
v = [-1.5  1.5  -2.5  0.5]: axis(v); axis('square')
grid
title('Nyquist Plot (Problem B-9-11)')
xlabel('Real Axis')
ylabel('Imag Axis')
```



B-9-12. A possible MATLAB program for obtaining a Nyquist plot of the given G(s) is shown below. The resulting Nyquist plot is shown on next page.

```
num = [0   1   2   1];
den = [1   0.2   1   1];
w = 0:0.005:10;
[re,im,w] = nyquist(num,den,w);
plot(re,im)
v = [-3   3   -3   3]; axis(v); axis('square')
grid
title('Nyquist Plot (Problem B-9-12)')
xlabel('Real Axis')
ylabel('Imag Axis')
```

From the plot, it is seen that the $G(j\omega)$ locus encircles the $-1 + j0$ point twice as ω is varied from $\omega = -\infty$ to $\omega = 0$ to $\omega = \infty$. Referring to the Nyquist stability criterion (see page 497), we have

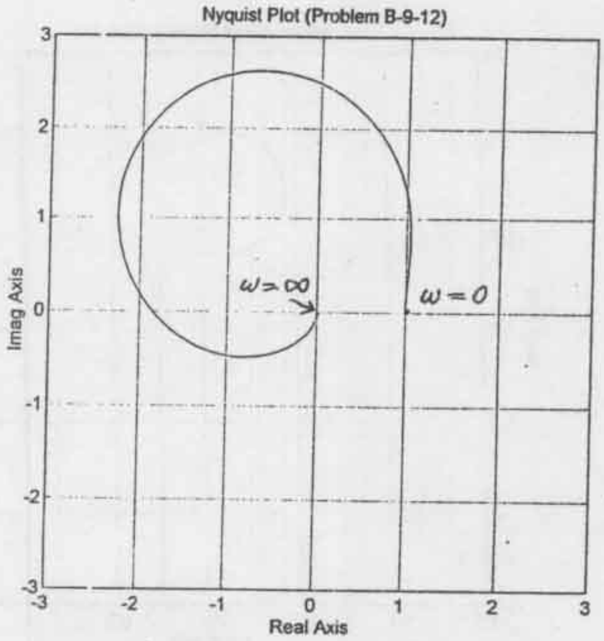
N = number of clockwise encirclement of the $-1 + j0$ point = -2

P = number of poles of G(s) in the right-half s plane = 2

Note that there are two open-loop poles in the right-half s plane, because

$$s^3 + 0.2s^2 + s + 1$$

$$= (s + 0.7246)(s - 0.2623 + j1.1451)(s - 0.2623 - j1.1451)$$



Then

$$Z = \text{number of zeros of } 1 + G(s) \text{ in the right-half s plane}$$

$$= N + P = -2 + 2 = 0$$

Thus, there are no closed-loop poles in the right-half s plane and the closed-loop system is stable.

B-9-13. A closed-loop system with the following open-loop transfer function

$$G(s)H(s) = \frac{K}{s^2(T_1 s + 1)} \qquad (T_1 > 0) \tag{1}$$

is unstable, while a closed-loop system with the following open-loop transfer function is stable.

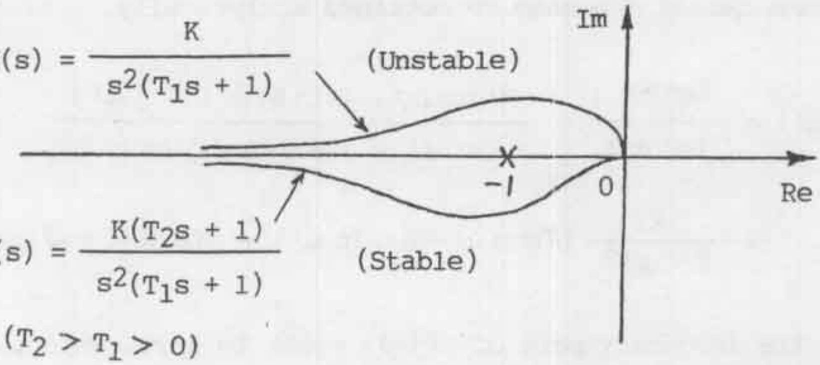
$$G(s)H(s) = \frac{K(T_2 s + 1)}{s^2(T_1 s + 1)} \qquad (T_2 > T_1 > 0) \tag{2}$$

Nyquist plots of these two systems are shown on next page. Note that $G(j\omega)H(j\omega)$ loci start from negative infinity on the real axis ($\omega = 0$) and approach the origin ($\omega = \infty$). The system with the open-loop transfer function given by Equation (1) encircles the $-1 + j0$ point twice clockwise. The system is unstable. The system with open-loop transfer function given by equation (2) does not encircle the $-1 + j0$ point. Hence, this system is stable.

$$G(s)H(s) = \frac{K}{s^2(T_1 s + 1)}$$

$$G(s)H(s) = \frac{K(T_2 s + 1)}{s^2(T_1 s + 1)}$$

$(T_2 > T_1 > 0)$

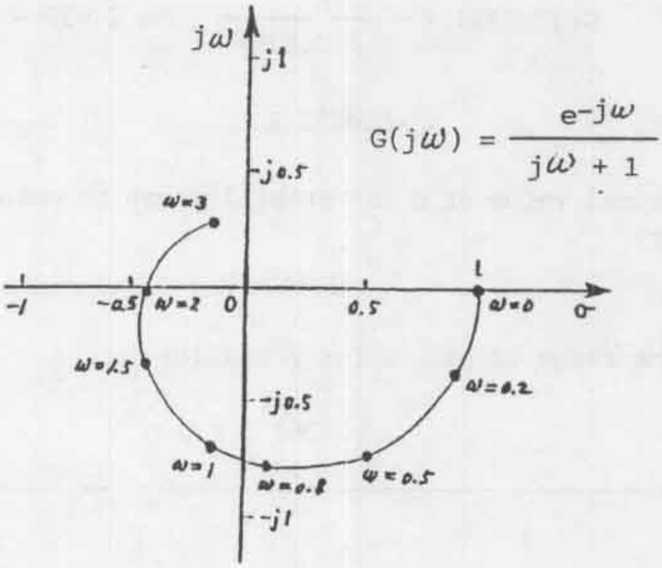


B-9-14. For this system

$$G(j\omega) = K \frac{e^{-j\omega}}{j\omega + 1}$$

By setting K = 1, we draw a Nyquist diagram as shown below. Note that

$$\angle e^{-j\omega} = -\omega \text{ (rad)} = -57.3^\circ \omega$$



The Nyquist locus crosses the negative real axis at $\sigma = -0.442$. Hence for stability, we require

$$\frac{1}{0.442} > K > 0$$

or

$$2.262 > K > 0$$

The same result can also be obtained analytically. Since

$$G(j\omega) = \frac{Ke^{-j\omega}}{j\omega + 1} = \frac{K(\cos\omega - j\sin\omega)(1 - j\omega)}{(1 + j\omega)(1 - j\omega)}$$

$$= \frac{K}{1 + \omega^2}\left[(\cos\omega - \omega\sin\omega) + j(\sin\omega + \omega\cos\omega)\right]$$

by setting the imaginary part of $G(j\omega)$ equal to zero, we obtain

$$\sin\omega + \omega\cos\omega = 0$$

or

$$\omega = -\tan\omega$$

Solving this equation for the smallest positive value of ω, we obtain

$$\omega = 2.029$$

Substituting $\omega = 2.029$ into $G(j\omega)$ yields

$$G(j2.029) = \frac{K}{1 + 2.029^2}(\cos 2.029 - 2.029 \sin 2.029)$$

$$= -0.4421\ K$$

The critical value of K for stability can be obtained by letting $G(j2.029) = -1$, or

$$0.4421\ K = 1$$

Thus, the range of gain K for stability is

$$2.262 > K > 0$$

B-9-15.

$$G(s) = \frac{K}{s(s^2 + s + 4)} = \frac{0.25K}{s(0.25s^2 + 0.25s + 1)}$$

The quadratic term in the denominator has the undamped natural frequency of 2 rad/s and the damping ratio of 0.25. Define the frequency corresponding to the angle of $-130°$ to be ω_1.

$$\underline{/G(j\omega_1)} = -\underline{/j\omega_1} - \underline{/1 - 0.25\omega_1^2 + j0.25\omega_1}$$

$$= -90° - \tan^{-1}\frac{0.25\omega_1}{1 - 0.25\omega_1^2} = -130°$$

Solving this last equation for ω_1, we find $\omega_1 = 1.491$. Thus, the phase angle becomes equal to $-130°$ at $\omega = 1.491$ rad/s. At this frequency, the magnitude must be unity, or $|G(j\omega_1)| = 1$. The required gain K can be determined from

$$|G(j1.491)| = \left| \frac{0.25K}{(j1.491)(-0.555 + j0.3725 + 1)} \right| = 0.2890K$$

Setting $|G(j1.491)| = 0.2890K = 1$, we find

$$K = 3.46$$

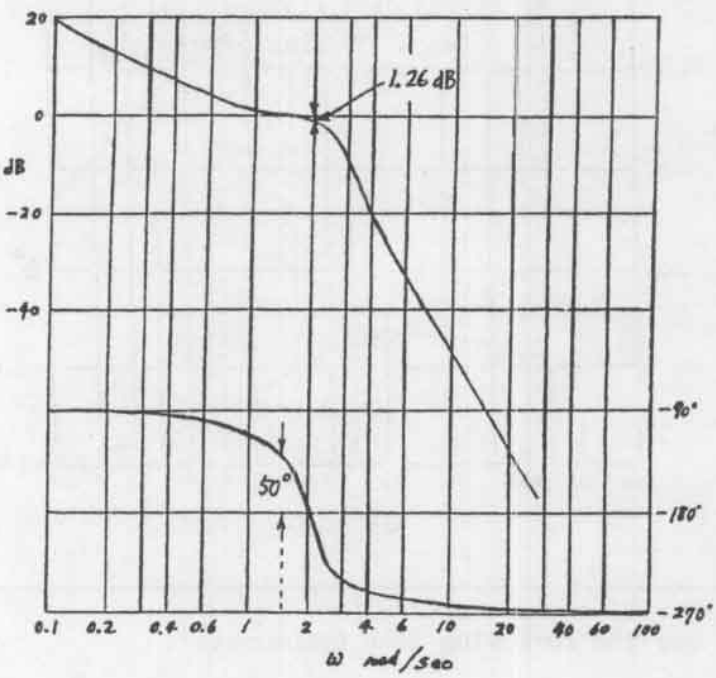
Note that the phase crossover frequency is at $\omega = 2$ rad/s, since

$$\underline{/G(j2)} = -\underline{/j2} - \underline{/-0.25 \times 2^2 + 0.25 \times j2 + 1} = -90° - 90° = -180°$$

The magnitude $|G(j2)|$ with K = 3.46 becomes

$$|G(j2)| = \left| \frac{0.865}{(j2)(-1 + j0.5 + 1)} \right| = 0.865 = -1.26 \text{ dB}$$

Thus, the gain margin is 1.26 dB. The Bode diagram of $G(j\omega)$ with K = 3.46 is shown below.



B-9-16. Note that

$$K \frac{j\omega + 0.1}{j\omega + 0.5} \frac{10}{j\omega(j\omega + 1)} = \frac{2K(10j\omega + 1)}{j\omega(2j\omega + 1)(j\omega + 1)}$$

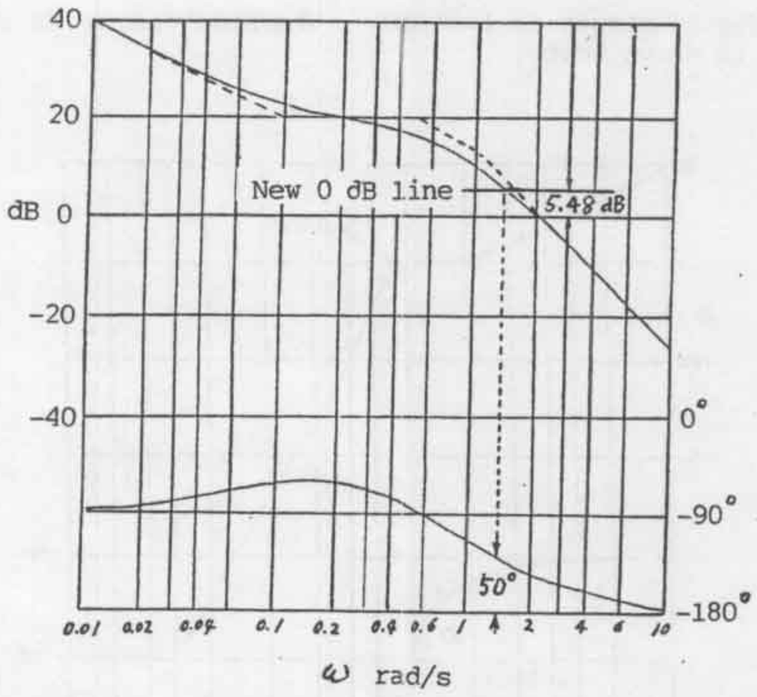
We shall plot the Bode diagram when 2K = 1. That is, we plot the Bode diagram of

$$G(j\omega) = \frac{10j\omega + 1}{j\omega(2j\omega + 1)(j\omega + 1)}$$

The diagram is shown below. The phase curve shows that the phase angle is $-130°$ at $\omega = 1.438$ rad/s. Since we require the phase margin to be $50°$, the magnitude of G(j1.438) must be equal to 1 or 0 dB. Since the Bode diagram indicates that $|G(j1.438)|$ is 5.48 dB, we need to choose 2K = -5.48 dB, or

$$K = 0.266$$

Since the phase curve lies above the $-180°$ line for all ω, the gain margin is $+\infty$ dB.



B-9-17. Let us use the following lead compensator:

$$G_c(s) = K_c \alpha \frac{Ts + 1}{\alpha Ts + 1} = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$$

Since K_v is specified as $4.0\ s^{-1}$, we have

$$K_V = \lim_{s \to 0} sK_C \alpha \frac{Ts + 1}{\alpha Ts + 1} \frac{K}{s(0.1s + 1)(s + 1)} = K_C \alpha K = 4$$

Let us set K = 1 and define $K_C \alpha = \hat{K}$. Then

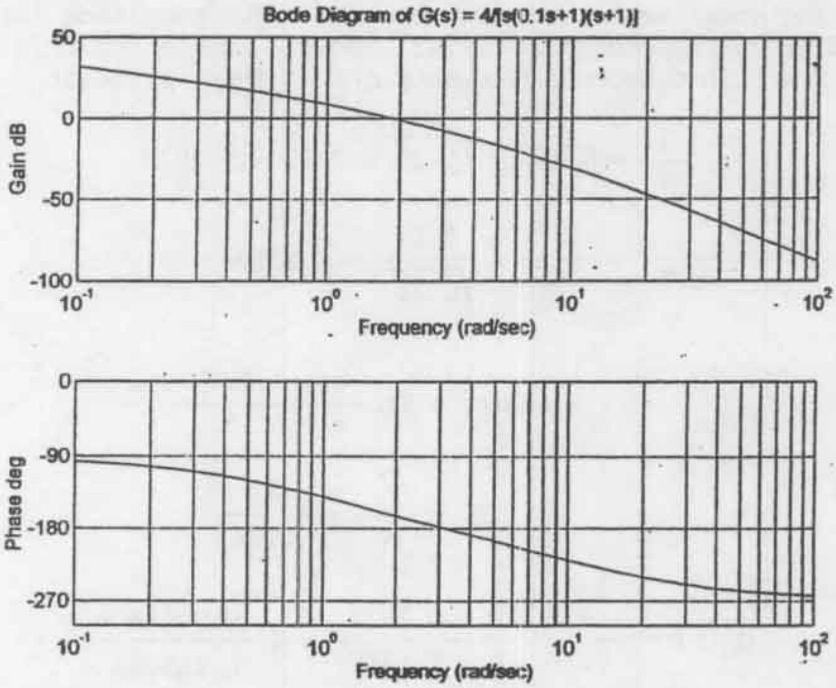
$$\hat{K} = 4$$

Next, plot a Bode diagram of

$$\frac{4}{s(0.1s + 1)(s + 1)} = \frac{4}{0.1s^3 + 1.1s^2 + s}$$

The following MATLAB program produces the Bode diagram shown below.

```
num = [0  0  0  4];
den = [0.1  1.1  1  0];
bode(num,den)
subplot(2,1,1);
title('Bode Diagram of G(s) = 4/[s(0.1s+1)(s+1)]')
```



From this plot, the phase and gain margins are 17° and 8.7 dB, respectively.

Since the specifications call for a phase margin of 45°, let us choose

$$\phi_m = 45° - 17° + 12° = 40°$$

(This means that 12° has been added to compensate for the shift in the gain crossover frequency.) The maximum phase lead is 40°. Since

$$\sin \phi_m = \frac{1 - \alpha}{1 + \alpha} \qquad (\phi_m = 40°)$$

α is determined as 0.2174. Let us choose, instead of 0.2174, α to be 0.21, or

$$\alpha = 0.21$$

Next step is to determine the corner frequencies $\omega = 1/T$ and $\omega = 1/(\alpha T)$ of the lead compensator. Note that the maximum phase-lead angle ϕ_m occurs at the geometric mean of the two corner frequencies, or $\omega = 1/(\sqrt{\alpha} T)$. The amount of the modification in the magnitude curve at $\omega = 1/(\sqrt{\alpha}T)$ due to the inclusion of the term $(Ts + 1)/(\alpha Ts + 1)$ is

$$\left| \frac{1 + j\omega T}{1 + j\omega\alpha T} \right|_{\omega = \frac{1}{\sqrt{\alpha} T}} = \frac{1}{\sqrt{\alpha}}$$

Note that

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.21}} = 2.1822 = 6.7778 \text{ dB}$$

We need to find the frequency point where, when the lead compensator is added, the total magnitude becomes 0 dB. The magnitude $|G(j\omega)|$ is -6.7778 dB corresponds to $\omega = 2.81$ rad/s. We select this frequency to be the new gain crossover frequency ω_c. Then we obtain

$$\frac{1}{T} = \sqrt{\alpha}\,\omega_c = \sqrt{0.21} \times 2.81 = 1.2877$$

$$\frac{1}{\alpha T} = \frac{\omega_c}{\sqrt{\alpha}} = \frac{2.81}{\sqrt{0.21}} = 6.1319$$

Hence

$$G_c(s) = K_c \frac{s + 1.2877}{s + 6.1319}$$

and

$$K_c = \frac{\hat{K}}{\alpha} = \frac{4}{0.21}$$

Thus

$$G_c(s) = \frac{4}{0.21} \frac{s + 1.2877}{s + 6.1319} = 4 \frac{0.7766s + 1}{0.16308s + 1}$$

The open-loop transfer function becomes as

$$G_c(s)G(s) = 4 \frac{0.7766s + 1}{0.16308s + 1} \frac{1}{s(0.1s + 1)(s + 1)}$$

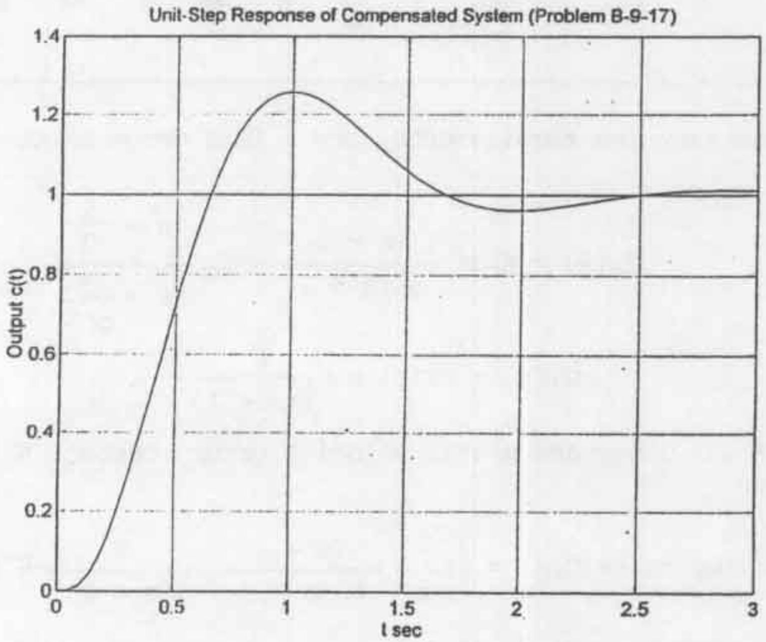
$$= \frac{3.1064s + 4}{0.01631s^4 + 0.2794s^3 + 1.2631s^2 + s}$$

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{3.1064s + 4}{0.01631s^4 + 0.2794s^3 + 1.2631s^2 + 4.1064s + 4}$$

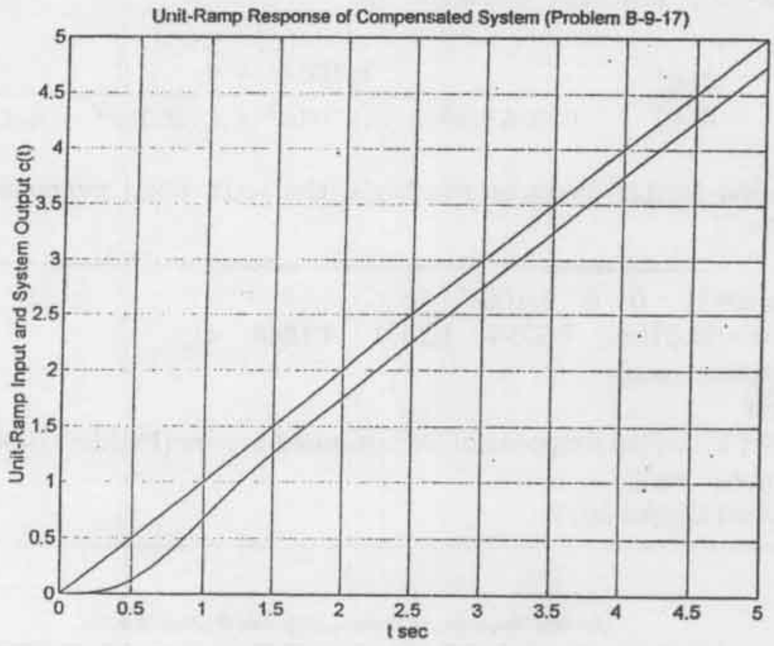
The following MATLAB program produces the unit-step response curve as shown below.

```
num = [0  0  0  3.1064  4];
den = [0.01631  0.2794  1.2631  4.1064  4];
step(num,den)
grid
title('Unit-Step Response of Compensated System (Problem B-9-17)')
xlabel('t sec')
ylabel('Output c(t)')
```



Similarly, the following MATLAB program produces the unit-ramp response curve as shown on next page.

```
num = [0  0  0  0  3.1064  4];
den = [0.01631  0.2794  1.2631  4.1064  4  0];
t = 0:0.01:5;
c = step(num,den,t);
plot(t,c,t,t)
grid
title('Unit-Ramp Response of Compensated System (Problem B-9-17)')
xlabel('t sec')
ylabel('Unit-Ramp Input and System Output c(t)')
```



B-9-18. To satisfy the requirements, try a lead compensator $G_c(s)$ of the form

$$G_c(s) = K_c \alpha \frac{Ts + 1}{\alpha Ts + 1} = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$$

Define

$$G_1(s) = KG(s) = \frac{K}{s(s + 1)}$$

where $K = K_c \alpha$. Since the static velocity error constant K_v is given as 50 s^{-1}, we have

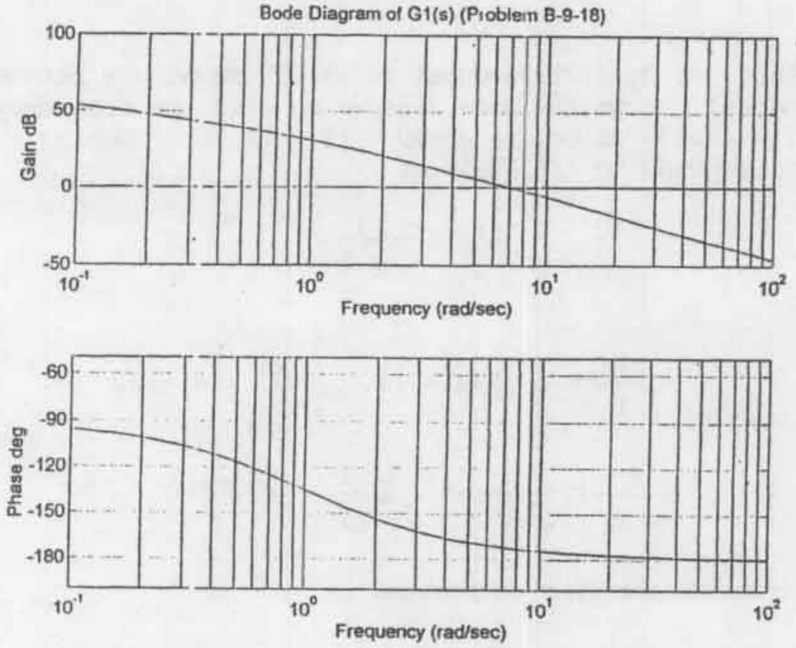
$$K_v = \lim_{s \to 0} sG_c(s)G(s) = \lim_{s \to 0} s \frac{Ts + 1}{\alpha Ts + 1} \frac{K}{s(s + 1)} = K = 50$$

We shall now plot a Bode diagram of

$$G_1(s) = \frac{50}{s(s + 1)}$$

The following MATLAB program produces the Bode diagram shown on next page.

```
num = [0  0  50];
den = [1  1  0];
w = logspace(-1,2,100);
bode(num,den,w);
subplot(2,1,1);
title('Bode Diagram of G1(s) (Problem B-9-18)')
```



From this plot, the phase margin is found to be 7.8°. The gain margin is $+\infty$ dB. Since the specifications call for a phase margin of 50°, the additional phase lead angle necessary to satisfy the phase margin requirement is 42.2°. We may assume the maximum phase lead required to be 48°. This means that 5.8° has been added to compensate for the shift in the gain crossover frequency. Since

$$\sin \phi_m = \frac{1-\alpha}{1+\alpha}$$

$\phi_m = 48°$ corresponds to $\alpha = 0.14735$. (Note that $\alpha = 0.15$ corresponds to $\phi_m = 47.657°$.) Whether we choose $\phi_m = 48°$ or $\phi_m = 47.657°$ does not make much difference in the final solution. Hence, we choose $\alpha = 0.15$.

The next step is to determine the corner frequencies $\omega = 1/T$ and $\omega = 1/(\alpha T)$ of the lead compensator. Note that the maximum phase lead angle ϕ_m occurs at the geometric mean of the two corner frequencies, or $\omega = 1/(\sqrt{\alpha}T)$. The amount of the modification in the magnitude curve at $\omega = 1/(\sqrt{\alpha}T)$ due to the inclusion of the term $(Ts + 1)/(\alpha Ts + 1)$ is

$$\left|\frac{1 + j\omega T}{1 + j\omega\alpha T}\right|_{\omega = \frac{1}{\sqrt{\alpha}T}} = \left|\frac{1 + j\frac{1}{\sqrt{\alpha}}}{1 + j\alpha\frac{1}{\sqrt{\alpha}}}\right| = \frac{1}{\sqrt{\alpha}}$$

Note that

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.15}} = 2.5820 = 8.239 \text{ dB}$$

We need to find the frequency point where, when the lead compensator is added, the total magnitude becomes 0 dB. The frequency at which the magnitude of $G_1(j\omega)$ is equal to -8.239 dB occurs between ω = 10 and 100 rad/s. From the Bode diagram we find the frequency point where $|G_1(j\omega)|$ = -8.239 dB occurs at ω = 11.4 rad/s. Noting that this frequency corresponds to $1/(\sqrt{\alpha}T)$, or

$$\omega_c = \frac{1}{\sqrt{\alpha}\,T}$$

we obtain

$$\frac{1}{T} = \omega_c\sqrt{\alpha} = 11.4\sqrt{0.15} = 4.4152$$

$$\frac{1}{\alpha T} = \frac{\omega_c}{\sqrt{\alpha}} = \frac{11.4}{\sqrt{0.15}} = 29.4347$$

The lead compensator thus determined is

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} = K_c \frac{s + 4.4152}{s + 29.4347}$$

where K_c is determined as

$$K_c = \frac{K}{\alpha} = \frac{50}{0.15} = \frac{1000}{3}$$

Thus,

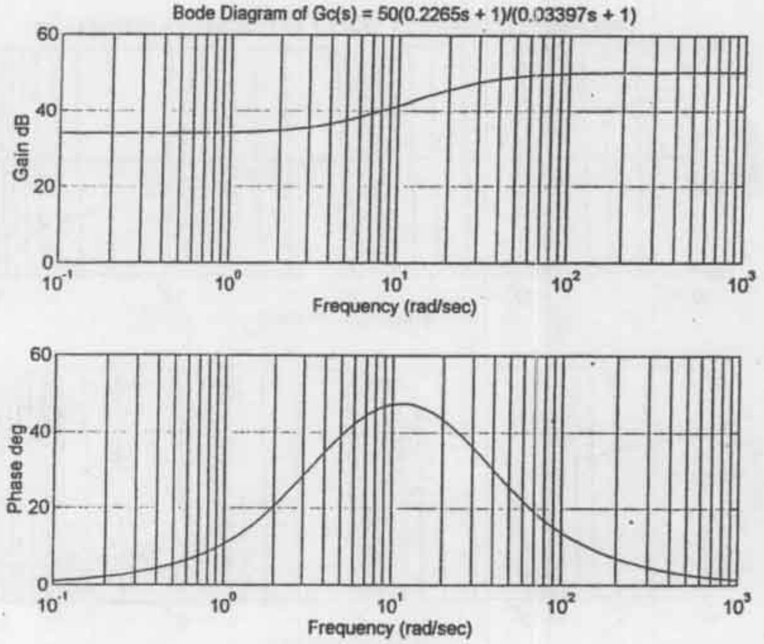
$$G_c(s) = \frac{1000}{3}\,\frac{s + 4.4152}{s + 29.4347}$$

$$= 50\,\frac{0.2265s + 1}{0.03397s + 1}$$

The following MATLAB program produces the Bode diagram of the lead compensator just designed. It is shown on next page.

```
num = [11.325   50];
den = [0.03397   1];
w = logspace(-1,3,100);
bode(num,den,w);
subplot(2,1,1);
title('Bode Diagram of Gc(s) = 50(0.2265s + 1)/(0.03397s + 1)')
```

The open-loop transfer function of the designed system is



$$G_c(s)G(s) = \frac{1000}{3}\left(\frac{s + 4.4152}{s + 29.4347}\right)\frac{1}{s(s + 1)}$$

The following MATLAB program produces the Bode diagram of $G_c(s)G(s)$ which is shown on next page.

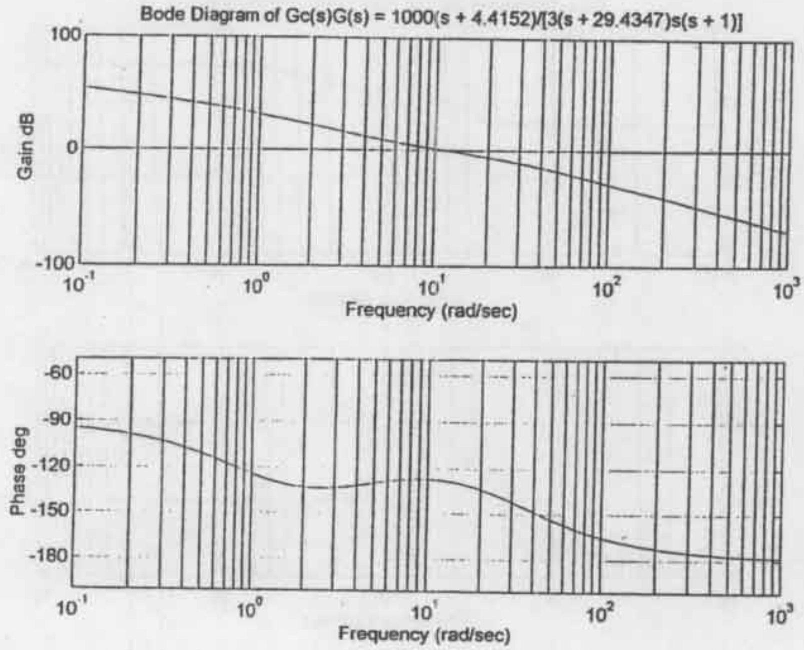
```
num = [0   0   1000   4415.2];
den = [3   91.3041   88.3041   0];
w = logspace(-1,3,100);
bode(num,den,w);
subplot(2,1,1);
title('Bode Diagram of Gc(s)G(s) = 1000(s + 4.4152)/[3(s + 29.4347)s(s + 1)]')
```

From this Bode diagram, it is clearly seen that the phase margin is approximately 52°, the gain margin is + ∞ dB, and $K_v = 50\ s^{-1}$; all specifications are met. Thus, the designed system is satisfactory.

Next, we shall obtain the unit-step and unit-ramp responses of the original uncompensated system and the compensated system. The original uncompensated system has the following closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + s + 1}$$

The closed-loop transfer function of the compensated system is



$$\frac{C(s)}{R(s)} = \frac{1000(s + 4.4152)}{3(s + 29.4347)s(s + 1) + 1000(s + 4.4152)}$$

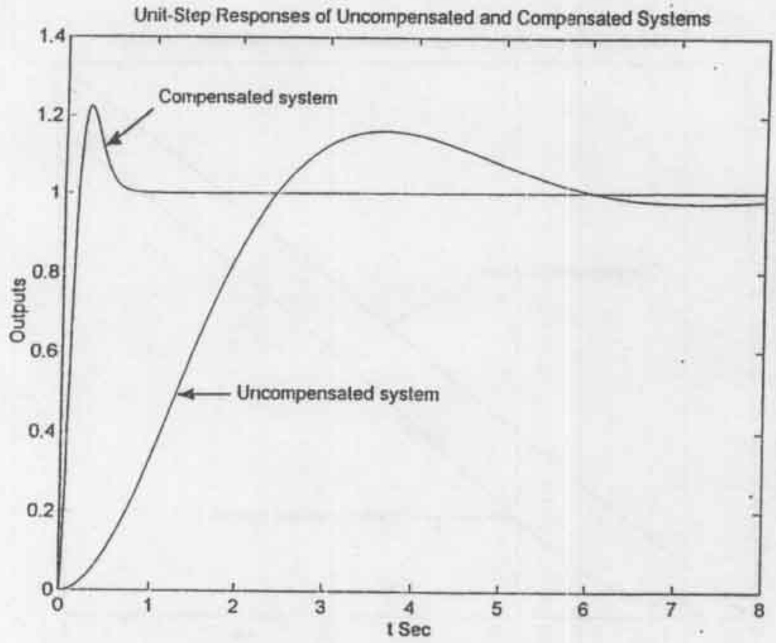
$$= \frac{1000s + 4415.2}{3s^3 + 91.3041s^2 + 1088.3041s + 4415.2}$$

The closed-loop poles of the compensated system are as follows:

$$s = -11.1772 \pm j7.5636, \qquad s = -8.0804$$

The MATLAB program given below produces the unit-step responses of the uncompensated and compensated systems. The resulting response curves are shown on next page.

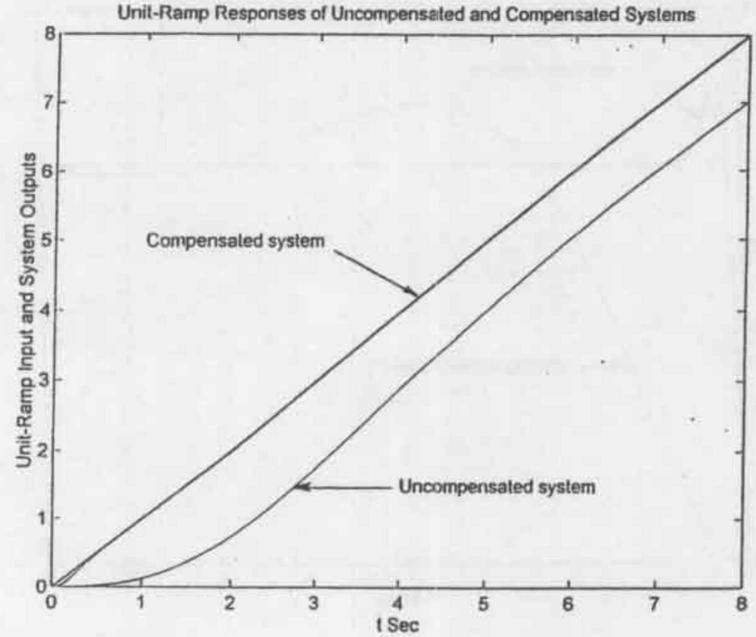
```
num = [0  0  1];
den = [1  1  1];
numc = [0  0  1000  4415.2];
denc = [3  91.3041  1088.3041  4415.2];
t = 0:0.01:8;
c1 = step(num,den,t);
c2 = step(numc,denc,t);
plot(t,c1,t,c2)
title('Unit-Step Responses of Uncompensated and Compensated Systems')
xlabel('t Sec')
ylabel('Outputs')
text(1,1.25,'Compensated system')
text(2,0.5,'Uncompensated system')
```



The MATLAB program given below produces the unit-ramp response of the uncompensated system and compensated system. The response curves are shown on next page.

```
num = [0   0   0   1];
den = [1   1   1   0];
numc = [0   0   0   1000   4415.2];
denc = [3   91.3041   1088.3041   4415.2   0];
t = 0:0.01:8;
c1 = step(num,den,t);
c2 = step(numc,denc,t);
plot(t,c1,t,c2,t,t)
title('Unit-Ramp Responses of Uncompensated and Compensated Systems')
xlabel('t Sec')
ylabel('Unit-Ramp Input and System Outputs')
text(1,5,'Compensated system')
text(4,1.5,'Uncompensated system')
```

Notice from the unit-ramp response curves that the compensated system follows the input ramp very closely. For the compensated system the error in following the input can be seen for $0 < t < 0.5$, but it is almost zero for $0.5 < t$. (The steady state error in the unit-ramp response is 0.02.)


Unit-Ramp Responses of Uncompensated and Compensated Systems
Unit-Ramp Input and System Outputs
t Sec
Compensated system
Uncompensated system

CHAPTER 10

B-10-1. The differential equation for the system is

$$m\ddot{y} + b_1\dot{y} + (k_1 + k_2)y = u$$

This is a second-order system. Therefore, we need two state variables.

Define state variables x_1 and x_2 as follows:

$$x_1 = y$$
$$x_2 = \dot{y}$$

Then we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k_1 + k_2}{m} x_1 - \frac{b_1}{m} x_2 + u$$

The output y for the system is simply x_1. Thus,

$$y = x_1$$

The state space representation for this system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_1 + k_2}{m} & -\frac{b_1}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

B-10-2. The system equations are

$$m_1\ddot{y} + k(y - z) = u$$
$$m_2\ddot{z} + k(z - y) = 0$$

Define

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = z$$
$$x_4 = \dot{z}$$

Then we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m_1} x_1 + \frac{k}{m_1} x_3 + \frac{u}{m_1}$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{k}{m_2} x_1 - \frac{k}{m_2} x_3$$

The standard state space representation for the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & 0 & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

B-10-3. The equations for the system are

$$k_1(u - z) = b_1(\dot{z} - \dot{y})$$
$$b_1(\dot{z} - \dot{y}) = k_2 y$$

or

$$k_1 u + b_1 \dot{y} = b_1 \dot{z} + k_1 z$$
$$b_1 \dot{z} = b_1 \dot{y} + k_2 y$$

By taking Laplace transforms of these two equations, assuming the zero initial conditions, we obtain

$$(b_1 s + k_1)Z(s) = b_1 sY(s) + k_1 U(s)$$

$$(b_1 s + k_2)Y(s) = b_1 sZ(s) \tag{1}$$

By eliminating Z(s) from the last two equations,

$$(b_1 s + k_1)\frac{b_1 s + k_2}{b_1 s} Y(s) = b_1 sY(s) + k_1 U(s)$$

or

$$(b_1 s + k_1)(b_1 s + k_2)Y(s) = b_1^2 s^2 Y(s) + b_1 s k_1 U(s)$$

which can be simplified to

$$[(k_1 + k_2)b_1 s + k_1 k_2]Y(s) = k_1 b_1 sU(s)$$

Hence

$$\frac{Y(s)}{U(s)} = \frac{k_1 b_1 s}{(k_1 + k_2)b_1 s + k_1 k_2} = \frac{\frac{b_1}{k_2} s}{\frac{k_1 + k_2}{k_1 k_2} b_1 s + 1} \tag{2}$$

Using Equations (1) and (2), Z(s)/U(s) can be obtained as follows:

$$\frac{Z(s)}{U(s)} = \frac{Z(s)}{Y(s)} \frac{Y(s)}{U(s)} = \frac{b_1 s + k_2}{b_1 s} \frac{k_1 b_1 s}{(k_1 + k_2)b_1 s + k_1 k_2}$$

$$= \frac{b_1 k_1 s + k_1 k_2}{(k_1 + k_2)b_1 s + k_1 k_2}$$

$$= \frac{\frac{b_1}{k_2} s + 1}{\frac{k_1 + k_2}{k_1 k_2} b_1 s + 1} \tag{3}$$

Define

$$\frac{b_1}{k_2} = T, \qquad \frac{k_1 + k_2}{k_1 k_2} b_1 = aT$$

where

$$a = \frac{K_1 + k_2}{k_1}$$

Then, Equations (2) and (3) can be written as

$$\frac{Y(s)}{U(s)} = \frac{Ts}{aTs + 1}, \qquad \frac{Z(s)}{U(s)} = \frac{Ts + 1}{aTs + 1}$$

Hence

$$aTsY(s) - TsU(s) = -Y(s) \tag{4}$$

$$aTsZ(s) - TsU(s) = -Z(s) + U(s) \tag{5}$$

Define

$$aY(s) - U(s) = X_1(s)$$

$$aZ(s) - U(s) = X_2(s)$$

Then

$$Y(s) = \frac{1}{a}[X_1(s) + U(s)] \tag{6}$$

$$Z(s) = \frac{1}{a}[X_2(s) + U(s)] \tag{7}$$

Equations (4) and (5) can be written as

$$TsX_1(s) = -Y(s) = -\frac{1}{a}[X_1(s) + U(s)]$$

$$TsX_2(s) = -Z(s) + U(s) = -\frac{1}{a}[X_2(s) + U(s)] + U(s)$$

Rewriting,

$$TsX_1(s) = -\frac{1}{a}X_1(s) - \frac{1}{a}U(s)$$

$$TsX_2(s) = -\frac{1}{a}X_2(s) - \frac{1}{a}U(s) + U(s)$$

from which we get

$$\dot{x}_1 = -\frac{1}{aT}x_1 - \frac{1}{aT}u$$

$$\dot{x}_2 = -\frac{1}{aT}x_2 + \frac{a-1}{aT}u$$

Hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{aT} & 0 \\ 0 & -\frac{1}{aT} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{aT} \\ \frac{a-1}{aT} \end{bmatrix} u$$

This is the state equation.

From Equation (6) we have

$$y = \frac{1}{a} x_1 + \frac{1}{a} u$$

From Equation (7) we get

$$z = \frac{1}{a} x_2 + \frac{1}{a} u$$

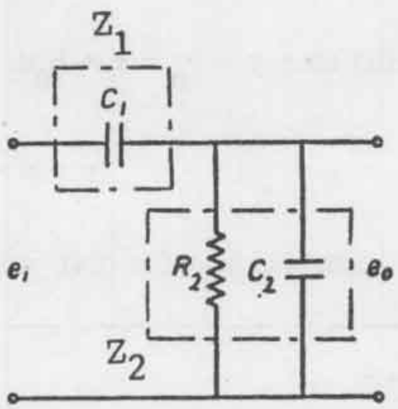
Hence

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{a} \\ \frac{1}{a} \end{bmatrix} u$$

This is the output equation.

B-10-4. Note that

$$Z_1 = \frac{1}{C_1 s}, \quad Z_2 = \frac{R_2}{R_2 C_2 s + 1}$$



Hence

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_2}$$

$$= \frac{R_2 C_1 s}{R_2(C_1 + C_2)s + 1}$$

which can be rewritten as

$$[(R_2C_1 + R_2C_2)s + 1]E_o(s) = R_2C_1 s E_i(s)$$

which is equivalent to

$$(R_2C_1 + R_2C_2)\dot{e}_o + e_o = R_2C_1\dot{e}_i$$

or

$$\dot{e}_o + \frac{1}{R_2(C_1 + C_2)} e_o = \frac{C_1}{C_1 + C_2} \dot{e}_i$$

This equation can be written as

$$\dot{e}_o + a_1 e_o = b_0 \dot{e}_i \qquad (1)$$

where

$$a_1 = \frac{1}{R_2(C_1 + C_2)}, \qquad b_0 = \frac{C_1}{C_1 + C_2}$$

Define

$$x = e_o - b_0 u, \qquad y = e_o, \qquad u = e_i$$

Then, Equation (1) becomes

$$\dot{x} + b_0\dot{u} + a_1(x + b_0 u) = b_0\dot{u}$$

or

$$\dot{x} = -a_1 x - a_1 b_0 u$$

from which the state equation can be obtained as

$$\dot{x} = -\frac{1}{R_2(C_1 + C_2)}x - \frac{C_1}{R_2(C_1 + C_2)^2}u \qquad (2)$$

Noting that $y = e_o = x + b_0 u$, the output equation can be given by

$$y = x + \frac{C_1}{C_1 + C_2}u \qquad (3)$$

Equations (2) and (3) give a state space representation for the given system.

B-10-5.

Method 1: The system differential equation can be written as

$$\dddot{y} + 18\ddot{y} + 192\dot{y} + 640\,y = 160\dot{u} + 640u$$

Comparing this equation with a standard third-order equation:

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

we find

$$a_1 = 18, \qquad a_2 = 192, \qquad a_3 = 640$$

$$b_0 = 0, \qquad b_1 = 0, \qquad b_2 = 160, \qquad b_3 = 640$$

Referring to Problem A-10-12, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta_0 u$$

where

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = 0$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = 160$$

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = 640 - 18 \times 160 = -2240$$

Thus, the state equation and output equation are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -640 & -192 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 160 \\ -2240 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Method 2: Referring to Problem A-10-13, $Y(s)/U(s)$ can be written as

$$\frac{Y(s)}{U(s)} = \frac{160(s+4)}{s^3 + 18s^2 + 192s + 640} = \frac{Z(s)}{U(s)} \frac{Y(s)}{Z(s)}$$

where

$$\frac{Z(s)}{U(s)} = \frac{1}{s^3 + 18s^2 + 192s + 640}, \qquad \frac{Y(s)}{Z(s)} = 160(s+4)$$

Then we have

$$\dddot{z} + 18\ddot{z} + 192\dot{z} + 640z = u$$

$$160\dot{z} + 640z = y$$

Define

$$x_1 = z$$

$$x_2 = \dot{z} = \dot{x}_1$$

$$x_3 = \ddot{z} = \ddot{x}_1 = \dot{x}_2$$

Then

$$\dot{x}_3 = -640z - 192\dot{z} - 18\ddot{z} + u$$

$$= -640x_1 - 192x_2 - 18x_3 + u$$

Also,

$$y = 640z + 160\dot{z} = 640x_1 + 160x_2$$

Hence, the state space representation for the system by this approach is

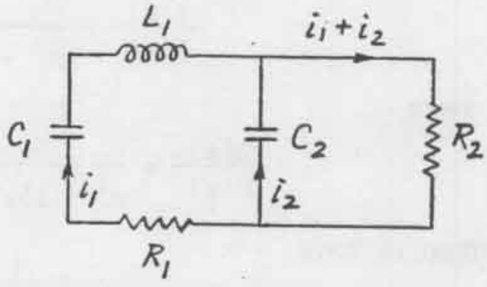
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -640 & -192 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 640 & 160 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-10-6. The equations for the circuit are

$$R_1 i_1 + \frac{1}{C_1}\int i_1\,dt + L_1\frac{di_1}{dt} = \frac{1}{C_2}\int i_2\,dt$$

$$\frac{1}{C_2}\int i_2\,dt + R_2(i_1 + i_2) = 0$$



or

$$R_1\dot{q}_1 + \frac{1}{C_1}q_1 + L_1\ddot{q}_1 = \frac{1}{C_2}q_2$$

$$\frac{1}{C_2}q_2 + R_2(\dot{q}_1 + \dot{q}_2) = 0$$

Substitution of $q_1 = x_1$, $\dot{q}_1 = x_2$, $q_2 = x_3$ into the last two equations yields

$$R_1 x_2 + \frac{1}{C_1} x_1 + L_1 \dot{x}_2 = \frac{1}{C_2} x_3$$

$$\frac{1}{C_2} x_3 + R_2(x_2 + \dot{x}_3) = 0$$

Hence we have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{L_1} \left(- \frac{1}{C_1} x_1 - R_1 x_2 + \frac{1}{C_2} x_3\right)$$

$$\dot{x}_3 = \frac{1}{R_2} \left(-R_2 x_2 - \frac{1}{C_2} x_3\right)$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{L_1 C_1} & -\frac{R_1}{L_1} & \frac{1}{L_1 C_2} \\ 0 & -1 & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-10-7. A partial-fraction expansion of Y(s)/U(s) gives

$$\frac{Y(s)}{U(s)} = \frac{5}{(s + 1)^2} - \frac{5}{s + 1} + \frac{5}{s + 2}$$

Define

$$X_1(s) = \frac{1}{(s + 1)^2} U(s)$$

$$X_2(s) = \frac{1}{s + 1} U(s)$$

$$X_3(s) = \frac{1}{s + 2} U(s)$$

Notice that

$$\frac{X_1(s)}{X_2(s)} = \frac{1}{s + 1}$$

Then, from the preceding equations for $X_1(s)$, $X_2(s)$, and $X_3(s)$ we obtain

$$sX_1(s) = - X_1(s) + X_2(s)$$

$$sX_2(s) = - X_2(s) + U(s)$$

$$sX_3(s) = - 2\, X_3(s) + U(s)$$

Also,

$$Y(s) = 5\, X_1(s) - 5\, X_2(s) + 5\, X_3(s)$$

By taking the inverse Laplace transforms of the last four equations, we get

$$\dot{x}_1 = - x_1 + x_2$$

$$\dot{x}_2 = - x_2 + u$$

$$\dot{x}_3 = - 2\, x_3 + u$$

and

$$y = 5\, x_1 - 5\, x_2 + 5\, x_3$$

In the standard state space representation, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 5 & -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

B-10-8. The equations for the system are

$$\dot{x}_1 = x_2 + u \qquad (1)$$

$$\dot{x}_2 = x_3 + u \qquad (2)$$

$$\dot{x}_3 = x_1 - 3\, x_2 + 3\, x_3 + u \qquad (3)$$

$$y = x_1$$

Notice that from Equations (1) and (2)

$$\dot{x}_3 = \ddot{x}_2 - \dot{u} = \dddot{x}_1 - \ddot{u} - \dot{u} \qquad (4)$$

Also, from Equation (3)

$$\dot{x}_3 = x_1 - 3(\dot{x}_1 - u) + 3(\dot{x}_2 - u) + u$$

$$= x_1 - 3(\dot{x}_1 - u) + 3(\ddot{x}_1 - \dot{u} - u) + u \qquad (5)$$

Thus, equating Equations (4) and (5) and simplifying, we obtain

$$\dddot{x}_1 - 3\ddot{x}_1 + 3\dot{x}_1 - x_1 = \ddot{u} - 2\dot{u} + u$$

Substituting $x_1 = y$ into this last equation, we get

$$\dddot{y} - 3\ddot{y} + 3\dot{y} - y = \ddot{u} - 2\dot{u} + u$$

[The same result can be obtained by use of Equation (10-51).]

B-10-9. The eigenvector $\underset{\sim}{x}_i$ associated with an eigenvalue λ_i is a vector that satisfies the following equation:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix} = \lambda_i \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}$$

or

$$a_{11}x_{i1} + a_{12}x_{i2} = \lambda_i \, x_{i1}$$

$$a_{21}x_{i1} + a_{22}x_{i2} = \lambda_i \, x_{i2}$$

which can be rewritten as

$$(a_{11} - \lambda_i)x_{i1} = - a_{12}x_{i2}$$

$$(a_{22} - \lambda_i)x_{i2} = - a_{21}x_{i1}$$

For $\lambda_i = \lambda_1$

$$x_{11} = - \frac{a_{12}}{a_{11} - \lambda_1} x_{12}$$

An example of eigenvectors corresponding to eigenvalue λ_1 is

$$\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} -\dfrac{a_{12}}{a_{11} - \lambda_1} \\ 1 \end{bmatrix}$$

For $\lambda_i = \lambda_2$

$$x_{22} = -\frac{a_{21}}{a_{22} - \lambda_2} x_{21}$$

An example of eigenvectors corresponding to eigenvalue λ_2 is

$$\begin{bmatrix} x_{21} \\ \\ x_{22} \end{bmatrix} = \begin{bmatrix} -\dfrac{a_{22} - \lambda_2}{a_{21}} \\ \\ 1 \end{bmatrix}$$

B-10-10. Define

$$\underset{\sim}{B} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

The eigenvectors for this matrix can be determined by solving the following equation for $\underset{\sim}{x}$.

$$\underset{\sim}{B}\underset{\sim}{x} = \lambda_1 \underset{\sim}{x}$$

or

$$\begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

which can be rewritten as

$$\lambda_1 x_1 + x_2 = \lambda_1 x_1$$

$$\lambda_1 x_2 = \lambda_1 x_2$$

$$\lambda_1 x_3 = \lambda_1 x_3$$

which, in turn, gives

x_1 = arbitrary constant

$x_2 = 0$

x_3 = arbitrary constant

Hence,

$$\underset{\sim}{x} = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$$

where a and b are arbitrary nonzero constant. Notice that

$$\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent vectors. The eigenvectors of matrix $\underset{\sim}{B}$ involve two linearly independent eigenvectors.

Next, define

$$\underset{\sim}{C} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$$

The eigenvectors for this matrix can be determined by solving the following equation for $\underset{\sim}{x}$.

$$\underset{\sim}{C}\underset{\sim}{x} = \lambda_1 \underset{\sim}{x}$$

or

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

which can be rewritten as

$$\lambda_1 x_1 = \lambda_1 x_1$$

$$\lambda_1 x_2 = \lambda_1 x_2$$

$$\lambda_1 x_3 = \lambda_1 x_3$$

from which we obtain

$$x_1 = \text{arbitrary constant}$$
$$x_2 = \text{arbitrary constant}$$
$$x_3 = \text{arbitrary constant}$$

Hence,

$$\underset{\sim}{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where a, b, and c are arbitrary nonzero constant. Notice that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the eigenvectors of matrix $\underset{\sim}{C}$ involve three linearly independent eigenvectors.

B-10-11.

$$\frac{Y(s)}{U(s)} = \frac{25.04s + 5.008}{s^3 + 5.03247s^2 + 25.1026s + 5.008}$$

A MATLAB program to obtain a state-space representation of this system is given on next page. Based on the MATLAB output we get the following state space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5.0325 & -25.1026 & -5.008 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 25.04 \quad 5.008] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

```
num = [0  0  25.04  5.008];
den = [1  5.03247  25.1026  5.008];
[A,B,C,D] = tf2ss(num,den)

A =

  -5.0325 -25.1026  -5.0080
   1.0000        0        0
        0   1.0000        0


B =

   1
   0
   0


C =

        0  25.0400   5.0080


D =

   0
```

B-10-12. Referring to Equation (10-51), we have

$$\frac{Y(s)}{U(s)} = G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 600 & 100 & s + 10 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} + 0$$

$$= \frac{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}{s^3 + 10s^2 + 100s + 600} \begin{bmatrix} s^2 + 10s + 100 & s + 10 & 1 \\ -600 & s^2 + 10s & s \\ -600s & -100s - 600 & s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

$$= \frac{[s^2 + 10s + 100 \quad s + 10 \quad 1]}{s^3 + 10s^2 + 100s + 600} \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

Thus

$$\frac{Y(s)}{U(s)} = \frac{10(s + 10)}{s^3 + 10s^2 + 100s + 600}$$

B-10-13.

$$e^{\underset{\sim}{A}t} = \mathcal{L}^{-1} [(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}] = \mathcal{L}^{-1}\left\{\begin{bmatrix} s + 2 & 1 \\ -2 & s + 5 \end{bmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 7s + 12} \begin{bmatrix} s + 5 & -1 \\ 2 & s + 2 \end{bmatrix} \right\}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} \frac{s + 5}{(s + 3)(s + 4)} & \frac{-1}{(s + 3)(s + 4)} \\ \frac{2}{(s + 3)(s + 4)} & \frac{s + 2}{(s + 3)(s + 4)} \end{bmatrix}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} \frac{2}{s + 3} - \frac{1}{s + 4} & -\frac{1}{s + 3} + \frac{1}{s + 4} \\ \frac{2}{s + 3} - \frac{2}{s + 4} & -\frac{1}{s + 3} + \frac{2}{s + 4} \end{bmatrix}$$

$$= \begin{bmatrix} 2\,e^{-3t} - e^{-4t} & -\,e^{-3t} + e^{-4t} \\ 2\,e^{-3t} - 2\,e^{-4t} & -\,e^{-3t} + 2\,e^{-4t} \end{bmatrix}$$

B-10-14.

$$e^{\underset{\sim}{A}t} = \mathcal{L}^{-1} [(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}]$$

$$= \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 3 & s-3 \end{bmatrix}^{-1}\right\}$$

where

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 3 & s-3 \end{bmatrix}^{-1}$$

$$= \frac{1}{s^2(s-3) - 1 + 3s}\begin{bmatrix} \begin{vmatrix} s & -1 \\ 3 & s-3 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 3 & s-3 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ s & -1 \end{vmatrix} \\ -\begin{vmatrix} 0 & -1 \\ -1 & s-3 \end{vmatrix} & \begin{vmatrix} s & 0 \\ -1 & s-3 \end{vmatrix} & -\begin{vmatrix} s & 0 \\ 0 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & s \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} s & -1 \\ -1 & 3 \end{vmatrix} & \begin{vmatrix} s & -1 \\ 0 & s \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{(s-1)^3}\begin{bmatrix} s^2 - 3s + 3 & s - 3 & 1 \\ 1 & s(s-3) & s \\ s & -(3s-1) & s^2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s^2 - 3s + 3}{(s-1)^3} & \frac{s-3}{(s-1)^3} & \frac{1}{(s-1)^3} \\ \frac{1}{(s-1)^3} & \frac{s(s-3)}{(s-1)^3} & \frac{s}{(s-1)^3} \\ \frac{s}{(s-1)^3} & \frac{-3s+1}{(s-1)^3} & \frac{s^2}{(s-1)^3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3} & \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} & \frac{1}{(s-1)^3} \\ \frac{1}{(s-1)^3} & \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} & \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3} \\ \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3} & -\frac{3}{(s-1)^2} - \frac{2}{(s-1)^3} & \frac{1}{s-1} + \frac{2}{(s-1)^2} + \frac{1}{(s-1)^3} \end{bmatrix}$$

Noting that

$$\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^t$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] = te^t$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)^3}\right] = \frac{1}{2}t^2e^t$$

we obtain

$$e^{\underset{\sim}{A}t} = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 3 & s-3 \end{bmatrix}^{-1}\right\}$$

$$= \begin{bmatrix} e^t - te^t + \frac{1}{2}t^2e^t & te^t - t^2e^t & \frac{1}{2}t^2e^t \\ \frac{1}{2}t^2e^t & e^t - te^t - t^2e^t & te^t + \frac{1}{2}t^2e^t \\ te^t + \frac{1}{2}t^2e^t & -3te^t - t^2e^t & e^t + 2te^t + \frac{1}{2}t^2e^t \end{bmatrix}$$

B-10-15. The solution of the state equation is

$$\underset{\sim}{x}(t) = e^{\underset{\sim}{A}t}\underset{\sim}{x}(0) + \int_0^t e^{\underset{\sim}{A}(t-\tau)}\underset{\sim}{B}u(\tau)\,d\tau$$

Since $\underset{\sim}{x}(0) = \underset{\sim}{0}$ and $u(t) = 1(t)$, we obtain

$$\underset{\sim}{x}(t) = \int_0^t e^{\underset{\sim}{A}(t-\tau)}\underset{\sim}{B}\,d\tau$$

Note that

$$e^{\underset{\sim}{A}t} = \mathcal{L}^{-1}[(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}]$$

where

$$(s\underset{\sim}{I} - \underset{\sim}{A})^{-1} = \begin{bmatrix} s & -1 \\ 6 & s + 5 \end{bmatrix}^{-1} = \frac{1}{s^2 + 5s + 6} \begin{bmatrix} s + 5 & 1 \\ -6 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s + 5}{(s + 2)(s + 3)} & \frac{1}{(s + 2)(s + 3)} \\ \frac{-6}{(s + 2)(s + 3)} & \frac{s}{(s + 2)(s + 3)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{s + 2} - \frac{2}{s + 3} & \frac{1}{s + 2} - \frac{1}{s + 3} \\ -\frac{6}{s + 2} + \frac{6}{s + 3} & -\frac{2}{s + 2} + \frac{3}{s + 3} \end{bmatrix}$$

Thus

$$e^{\underset{\sim}{A}t} = \mathcal{L}^{-1}[(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}]$$

$$= \begin{bmatrix} 3\, e^{-2t} - 2\, e^{-3t} & e^{-2t} - e^{-3t} \\ -6\, e^{-2t} + 6\, e^{-3t} & -2\, e^{-2t} + 3\, e^{-3t} \end{bmatrix}$$

Hence

$$x(t) = \int_0^t e^{\underset{\sim}{A}(t - \tau)}\, \underset{\sim}{B}\, d\tau = e^{\underset{\sim}{A}t} \int_0^t e^{-\underset{\sim}{A}\tau} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 3\, e^{-2t} - 2\, e^{-3t} & e^{-2t} - e^{-3t} \\ -6\, e^{-2t} + 6\, e^{-3t} & -2\, e^{-2t} + 3\, e^{-3t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{2\tau} - e^{3\tau} \\ -2\, e^{2\tau} + 3\, e^{3\tau} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 3\, e^{-2t} - 2\, e^{-3t} & e^{-2t} - e^{-3t} \\ -6\, e^{-2t} + 6\, e^{-3t} & -2\, e^{-2t} + 3\, e^{-3t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\, e^{2t} - \frac{1}{3}\, e^{3t} - \frac{1}{6} \\ - e^{2t} + e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} - \frac{1}{2} e^{-2t} + \frac{1}{3} e^{-3t} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

Alternate solution Referring to Example 10-9, we have for the unit-step input the following solution:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\, \mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}$$

Since $\mathbf{x}(0) = \mathbf{0}$ in the present case, we have

$$\mathbf{x}(t) = \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}$$

where

$$\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{5}{6} & -\frac{1}{6} \\ 1 & 0 \end{bmatrix}$$

and

$$e^{\mathbf{A}t} = \begin{bmatrix} 3 e^{-2t} - 2 e^{-3t} & e^{-2t} - e^{-3t} \\ -6 e^{-2t} + 6 e^{-3t} & -2 e^{-2t} + 3 e^{-3t} \end{bmatrix}$$

Thus,

$$(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B} = \begin{bmatrix} 3 e^{-2t} - 2 e^{-3t} - 1 & e^{-2t} - e^{-3t} \\ -6 e^{-2t} + 6 e^{-3t} & - 2 e^{-2t} + 3 e^{-3t} - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t} - e^{-3t} \\ -2 e^{-2t} + 3 e^{-3t} - 1 \end{bmatrix}$$

and $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \begin{bmatrix} -\frac{5}{6} & -\frac{1}{6} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-2t} - e^{-3t} \\ - 2 e^{-2t} + 3 e^{-3t} - 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} - \frac{1}{2} e^{-2t} + \frac{1}{3} e^{-3t} \\ e^{-2t} - e^{-3t} \end{bmatrix}$$

B-10-16. The solution to the state equation when $u(\tau) = \tau$ can be given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\tau d\tau$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\tau d\tau \mathbf{B}$$

Note that

$$\int_0^t e^{-A\tau}\tau d\tau = \int_0^t (I - \mathbf{A}\tau + \frac{1}{2!}\mathbf{A}^2\tau^2 - \frac{1}{3!}\mathbf{A}^3\tau^3 + \frac{1}{4!}\mathbf{A}^4\tau^4 + \cdots)\tau d\tau$$
$$= \frac{\mathbf{I}}{2}t^2 - \frac{2\mathbf{A}}{3!}t^3 + \frac{3\mathbf{A}^2}{4!}t^4 - \frac{4\mathbf{A}^3}{5!}t^5 + \frac{5\mathbf{A}^4}{6!}t^6 + \cdots$$

Hence

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\frac{\mathbf{I}}{2}t^2 - \frac{2\mathbf{A}}{3!}t^3 + \frac{3\mathbf{A}^2}{4!}t^4 - \frac{4\mathbf{A}^3}{5!}t^5 + \cdots\right)\mathbf{B}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-2}e^{\mathbf{A}t}\left(\frac{\mathbf{A}^2}{2}t^2 - \frac{2\mathbf{A}^3}{3!}t^3 + \frac{3\mathbf{A}^4}{4!}t^4 - \frac{4\mathbf{A}^5}{5!}t^5 + \cdots\right)\mathbf{B}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-2}e^{\mathbf{A}t}[\mathbf{I} - (\mathbf{I} - \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 - \frac{1}{3!}\mathbf{A}^3t^3$$
$$+ \frac{1}{4!}\mathbf{A}^4t^4 + \cdots)(\mathbf{I} + \mathbf{A}t)]\mathbf{B}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-2}e^{\mathbf{A}t}[\mathbf{I} - e^{-\mathbf{A}t}(\mathbf{I} + \mathbf{A}t)]\mathbf{B}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-2}[e^{\mathbf{A}t} - \mathbf{I} - \mathbf{A}t]\mathbf{B}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + [\mathbf{A}^{-2}(e^{\mathbf{A}t} - \mathbf{I}) - \mathbf{A}^{-1}t]\mathbf{B}$$

B-10-17. The equation of motion for the system is

$$m\ddot{y} = b(\dot{u} - \dot{y}) - ky$$

or

$$m\ddot{y} + b\dot{y} + ky = b\dot{u}$$

By substituting the given numerical values for m, b, and k into this last equation, we obtain

$$2\ddot{y} + 2\dot{y} + y = 2\dot{u}$$

or

$$\ddot{y} + \dot{y} + 0.5y = \dot{u} \tag{1}$$

where u is the input and y is the output.

By taking the Laplace transform of Equation (1), assuming the zero initial conditions, we have

$$(s^2 + s + 0.5)Y(s) = sU(s)$$

or

$$\frac{Y(s)}{U(s)} = \frac{s}{s^2 + s + 0.5} \tag{2}$$

Since the input u(t) is specified as a step displacement of 0.5 m, we have

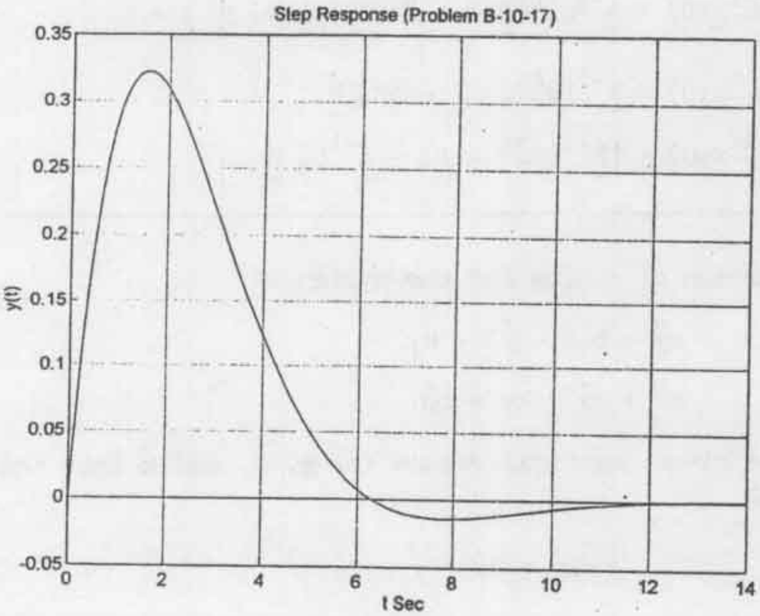
$$u = 0.5\ 1(t)$$

With this input, Equation (2) can be rewritten as

$$Y(s) = \frac{s}{s^2 + s + 0.5}\frac{0.5}{s} = \frac{0.5s}{s^2 + s + 0.5}\frac{1}{s}$$

The following MATLAB program will generate the desired response. The resulting response curve is shown below.

```
num = [0   0.5   0];
den = [1   1   0.5];
step(num,den)
grid
title('Step Response (Problem B-10-17)')
xlabel('t Sec')
ylabel('y(t)')
```

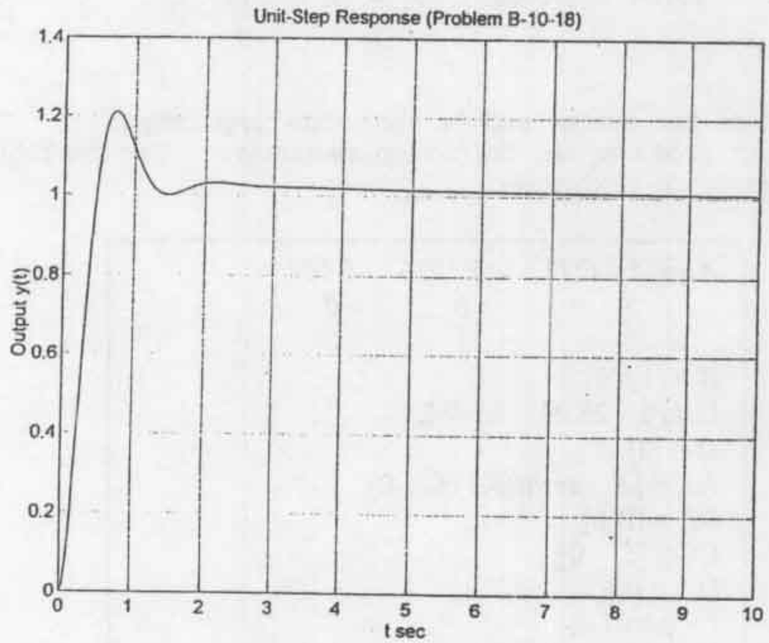


Note that the response reaches zero as t increases. This is because the transfer function Y(s)/U(s) given by Equation (2) involves s in the numerator.

B-10-18.

Unit-step response: The following MATLAB program yields the unit-step response of the given system. The resulting unit-step response curve is shown below.

```
A = [-5.03247   -25.1026   -5.008
       1          0          0
       0          1          0];
B = [1;0;0];
C = [0  25.04  5.008];
D = [0];
[y,x,t] = step(A,B,C,D);
plot(t,y)
grid
title('Unit-Step Response (Problem B-10-18)')
xlabel('t sec')
ylabel('Output y(t)')
```



Unit-ramp response: Referring to pages 596 - 599, we can obtain the unit-ramp response of the system by entering a MATLAB program similar to MATLAB Program 10-7 into the computer. Note that

$$AA = \begin{bmatrix} -5.03247 & -25.1026 & -5.008 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 25.04 & 5.008 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc|c} & & & 0 \\ & A & & 0 \\ & & & 0 \\ \hline 0 & 25.04 & 5.008 & 0 \end{array}\right] = [A \quad zeros(2,1);\ C \quad 0]$$

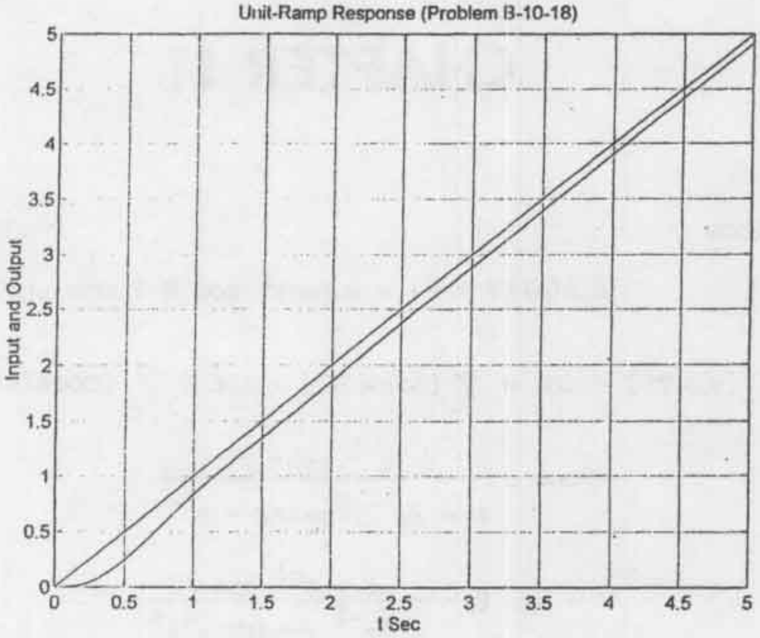
$$BB = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{c} B \\ \hline 0 \end{array}\right]$$

$$CC = [0 \quad 25.04 \quad 5.008 \quad 0] = [C \quad 0]$$

$$DD = [0]$$

x_4 is the output of the system and is the unit-ramp response. The following MATLAB program produces the unit-ramp response. The resulting response curve is shown on next page.

```
A = [-5.03247   -25.1026   -5.008
      1           0          0
      0           1          0];
B = [1;0;0];
C = [0   25.04   5.008];
D = [0];
AA = [A   zeros(3,1);C   0];
BB = [B;0];
CC = [C   0];
DD = [0];
t = 0:0.01:5;
[z,x,t] = step(AA,BB,CC,DD,1,t);
x4 = [0   0   0   1]*x';
plot(t,x4,t,t)
grid
title('Unit-Ramp Response (Problem B-10-18)')
xlabel('t Sec')
ylabel('Input and Output')
```


Unit-Ramp Response (Problem B-10-18)
Input and Output
t Sec
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

CHAPTER 11

B-11-1. Since

$$\sin(\omega kT + \theta) = \sin\omega kT \cos\theta + \cos\omega kT \sin\theta$$

we obtain

$$\mathcal{Z}[x(kT)] = \cos\theta\ \mathcal{Z}[\sin\omega kT] + \sin\theta\ \mathcal{Z}[\cos\omega kT]$$

$$= \cos\theta \frac{z^{-1}\sin\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

$$+ \sin\theta \frac{1 - z^{-1}\cos\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

$$= \frac{(\cos\theta)z^{-1}(\sin\omega T) + (\sin\theta)(1 - z^{-1}\cos\omega T)}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

$$= \frac{\sin\theta + z^{-1}\sin(\omega T - \theta)}{1 - 2z^{-1}\cos\omega T + z^{-2}}$$

B-11-2.

$$X_1(z)X_2(z) = \sum_{m=0}^{\infty} x_1(mT)z^{-m} \sum_{h=0}^{\infty} x_2(hT)z^{-h}$$

$$= \sum_{m=0}^{\infty} \sum_{h=0}^{\infty} x_1(mT)x_2(hT)z^{-(m+h)}$$

By defining $m = k - h$, we obtain

$$X_1(z)X_2(z) = \sum_{k=h}^{\infty} \sum_{h=0}^{\infty} x_1(kT - hT)x_2(hT)z^{-k}$$

Since $x_1(kT - hT) = 0$ for $k < h$, the limit of summation in $\sum_{k=h}^{\infty}$ can be changed to $\sum_{k=0}^{\infty}$. Thus

$$X_1(z)X_2(z) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} x_1(kT - hT)x_2(hT)z^{-k} \qquad (1)$$

Also,

$$X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k} \qquad (2)$$

Since $X(z) = X_1(z)X_2(z)$, by comparing Equations (1) and (2), we obtain

$$x(kT) = \sum_{h=0}^{\infty} x_1(kT - hT)x_2(hT) = \sum_{h=0}^{k} x_1(kT - hT)x_2(hT)$$

B-11-3.

$$\mathcal{Z}[x(k)] = X(z) = 0.2z^{-1} + 0.4z^{-2} + 0.6z^{-3} + 0.8z^{-4} + z^{-5} + z^{-6} + z^{-7} + \cdots$$

$$= 0.2z^{-1} + 0.4z^{-2} + 0.6z^{-3} + 0.8z^{-4} + z^{-5}(1 + z^{-1} + z^{-2} + \cdots)$$

$$= 0.2z^{-1} + 0.4z^{-2} + 0.6z^{-3} + 0.8z^{-4} + \frac{z^{-5}}{1 - z^{-1}}$$

$$= \frac{(z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4})(1 - z^{-1})}{5(1 - z^{-1})} + \frac{5z^{-5}}{5(1 - z^{-1})}$$

$$= \frac{z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}}{5(1 - z^{-1})}$$

$$= \frac{z^{-1}(1 - z^{-5})}{5(1 - z^{-1})^2}$$

B-11-4.

$$\mathcal{Z}[x(k)] = \mathcal{Z}[2^k] + \mathcal{Z}[3k] + \mathcal{Z}[1]$$

$$= \frac{1}{1 - 2z^{-1}} + \frac{3z^{-1}}{(1 - z^{-1})^2} + \frac{1}{1 - z^{-1}}$$

$$= \frac{2 - 2z^{-1} - 3z^{-2}}{(1 - 2z^{-1})(1 - z^{-1})^2}$$

B-11-5.

(a)

$$\mathcal{Z}[e^{-at}x(t)] = \mathcal{Z}[e^{-akT}x(kT)] = \sum_{k=0}^{\infty} e^{-akT}x(kT)z^{-k}$$

$$= \sum_{k=0}^{\infty} x(kT)(ze^{aT})^{-k} = X(ze^{aT})$$

(b) Consider

$$X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

Differentiating X(z) with respect to z, we obtain

$$\frac{d}{dz} X(z) = \sum_{k=0}^{\infty} (-k)x(kT)z^{-k-1}$$

Multiplying both sides of the last equation by -Tz gives

$$-Tz \frac{d}{dz} X(z) = \sum_{k=0}^{\infty} (kT)x(kT)z^{-k}$$

Thus we have

$$\sum_{k=0}^{\infty} (kT)x(kT)z^{-k} = \mathcal{Z}[kT\ x(kT)] = \mathcal{Z}[t\ x(t)]$$

$$= -Tz \frac{d}{dz} X(z)$$

B-11-6.

$$\frac{X(z)}{z} = \frac{(1 - e^{-aT})z}{(z - 1)(z - e^{-aT})} = \frac{1}{z - 1} - \frac{e^{-aT}}{z - e^{-aT}}$$

Hence

$$X(z) = \frac{z}{z - 1} - \frac{e^{-aT} z}{z - e^{-aT}}$$

$$= \frac{1}{1 - z^{-1}} - \frac{e^{-aT}}{1 - e^{-aT}z^{-1}}$$

The inverse z transform of X(z) gives

$$x(k) = 1 - e^{-aT}e^{-akT}$$

$$= 1 - e^{-aT(1 + k)}$$

B-11-7. X(z) can be rewritten as

$$X(z) = \frac{z}{(z - 1)^2(z - 2)}$$

Then

$$\frac{X(z)}{z} = \frac{1}{(z-1)^2(z-2)} = -\frac{1}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{z-2}$$

or

$$X(z) = -\frac{z^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-2z^{-1}}$$

Hence

$$x(k) = -k - 1 + 2^k \qquad k = 0, 1, 2, \ldots$$

B-11-8. Since

$$X(z) = \frac{(1-e^{-T})z}{(z-1)(z-e^{-T})}$$

we have

$$\frac{X(z)}{z} = \frac{1-e^{-T}}{(z-1)(z-e^{-T})} = \frac{1}{z-1} - \frac{1}{z-e^{-T}}$$

or

$$X(z) = \frac{z}{z-1} - \frac{z}{z-e^{-T}} = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T}z^{-1}}$$

Thus

$$x(k) = 1 - e^{-kT} \qquad k = 0, 1, 2, \ldots$$

B-11-9. Referring to Table 11-1, we find

$$\mathcal{Z}[k^2a^{k-1}] = \frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$$

By writing $e^{-1} = a$, we obtain

$$X(z) = \frac{e^{-3}z^{-1}}{(1-e^{-1}z^{-1})^3} = \frac{a^3z^{-1}}{(1-az^{-1})^3} = \frac{a^3z^{-1}(1+az^{-1}-az^{-1})}{(1-az^{-1})^3}$$

$$= \frac{a^3z^{-1}(1+az^{-1})}{(1-az^{-1})^3} + \frac{a^3z^{-1}(-az^{-1}+1-1)}{(1-az^{-1})^3}$$

$$= \frac{a^3z^{-1}(1+az^{-1})}{(1-az^{-1})^3} + \frac{a^3z^{-1}}{(1-az^{-1})^2} - \frac{a^3z^{-1}}{(1-az^{-1})^3}$$

Hence we have

$$\frac{2a^3z^{-1}}{(1-az^{-1})^3} = \frac{a^3z^{-1}(1+az^{-1})}{(1-az^{-1})^3} + \frac{a^3z^{-1}}{(1-az^{-1})^2}$$

Taking the inverse z transform of each term of this last equation, we obtain

$$2x(k) = a^3 k^2 a^{k-1} + a^3 k a^{k-1}$$

Hence

$$x(k) = \frac{a^3}{2} (k^2 a^{k-1} + k a^{k-1}) = \frac{a^{k+2}}{2} k(k + 1)$$

Since $a = e^{-1}$, we have

$$x(k) = \frac{e^{-k-2}}{2} k(k + 1) \qquad k = 0, 1, 2, \ldots$$

B-11-10. The z transform of the given difference equation becomes

$$[z^2X(z) - z^2x(0) - zx(1)] + 2[zX(z) - zx(0)] + X(z) = \frac{z^{-1}}{(1 - z^{-1})^2}$$

Substituting the initial data into this last equation, we get

$$(z^2 + 2z + 1)X(z) = \frac{z}{(z - 1)^2}$$

or

$$X(z) = \frac{z}{(z + 1)^2(z - 1)^2}$$

Hence

$$\frac{X(z)}{z} = \frac{1}{(z + 1)^2(z - 1)^2} = \frac{0.25}{(z + 1)^2} + \frac{0.25}{z + 1} + \frac{0.25}{(z - 1)^2} - \frac{0.25}{z - 1}$$

or

$$X(z) = \frac{0.25z^{-1}}{(1 + z^{-1})^2} + \frac{0.25}{1 + z^{-1}} + \frac{0.25z^{-1}}{(1 - z^{-1})^2} - \frac{0.25}{1 - z^{-1}}$$

The inverse z transform of X(z) gives

$$x(k) = 0.25\, k(-1)^{k-1} + 0.25(-1)^k + 0.25\, k - 0.25$$

$$= 0.25\, [k(-1)^{k-1} + (-1)^k + k - 1] \qquad k = 0, 1, 2, \ldots$$

B-11-11. If $a \neq 0$, then

$$x(1) = x(0) + 1 = 1$$

$$x(2) = x(1) + a = 1 + a$$

$$x(3) = x(2) + a^2 = 1 + a + a^2$$

$$x(4) = x(3) + a^3 = 1 + a + a^2 + a^3$$

$$\vdots$$

$$x(k) = 1 + a + a^2 + \cdots + a^{k-1} = \frac{1 - a^k}{1 - a} \qquad (a \neq 1,\ a \neq 0)$$

$$= k \qquad (a = 1)$$

If a = 0, then

$$x(k) = 0 \qquad k = 0, 1, 2, \ldots \qquad (a = 0)$$

B-11-12. The pulse transfer function for $G_p(s)$ including the zero-order hold is obtained as follows:

$$\frac{Y(z)}{U(z)} = G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{s + 3}{(s + 1)(s + 2)}\right]$$

$$= (1 - z^{-1})\ \mathcal{Z}\left[\frac{s + 3}{s(s + 1)(s + 2)}\right]$$

$$= (1 - z^{-1})\ \mathcal{Z}\left[\frac{1.5}{s} - \frac{2}{s + 1} + \frac{0.5}{s + 2}\right]$$

$$= (1 - z^{-1})\left[\frac{1.5}{1 - z^{-1}} - \frac{2}{1 - e^{-T}z^{-1}} + \frac{0.5}{1 - e^{-2T}z^{-1}}\right]$$

$$= \frac{(1.5 - 2e^{-T} + 0.5e^{-2T})z + e^{-T}(1.5e^{-2T} - 2e^{-T} + 0.5)}{(z - e^{-T})(z - e^{-2T})}$$

B-11-13. The pulse transfer function for the system can be rewritten as

$$\frac{Y(z)}{U(z)} = \frac{z + 2}{z^2 + z + 0.16}$$

or

$$(z^2 + z + 0.16)Y(z) = (z + 2)U(z)$$

The inverse z transform of this last equation gives

$$y(k + 2) + y(k + 1) + 0.16\, y(k) = u(k + 1) + 2\, u(k)$$

Comparing this difference equation with the standard second-order difference equation:

$$y(k+2) + a_1 y(k+1) + a_2 y(k) = b_0 u(k+2) + b_1 u(k+1) + b_2 u(k)$$

we find

$$a_1 = 1, \qquad a_2 = 0.16, \qquad b_0 = 0, \qquad b_1 = 1, \qquad b_2 = 2$$

Define state variables as follows:

$$x_1(k) = y(k) - h_0 u(k)$$

$$x_2(k) = x_1(k+1) - h_1 u(k)$$

where

$$h_0 = b_0 = 0$$

$$h_1 = b_1 - a_1 h_0 = 1$$

Also, define

$$h_2 = b_2 - a_1 h_1 - a_2 h_0 = 2 - 1 = 1$$

Then the state variables become

$$x_1(k) = y(k)$$

$$x_2(k) = x_1(k+1) - u(k)$$

Referring to Section 11-6, the state equation for the system can be given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} u(k)$$

or

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

The output equation is

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Note that a number of different state space representations are possible for the system.

B-11-14. From the given transfer function we obtain

$$\dot{y} = -ay + u$$

Define a state variable x as

$$x = y$$

(The input variable is u and the output variable is y.) Then we have the following continuous-time state equation and output equation:

$$\dot{x} = -ax + u$$

$$y = x$$

The discretized state equation is obtained as

$$x((k+1)T) = G(T)x(kT) + H(T)u(kT)$$

where

$$G(T) = e^{AT} = e^{-aT}$$

$$H(T) = \int_0^T e^{At}\,dt\,B = \int_0^T e^{-at}\,dt$$

$$= -\frac{e^{-at}}{a}\Bigg|_0^T = \frac{1}{a}(1 - e^{-aT})$$

Hence the discrete-time state space representation for the system is given by

$$x((k+1)T) = e^{-aT}x(kT) + \frac{1}{a}(1 - e^{-aT})u(kT)$$

$$y(kT) = x(kT)$$

The pulse transfer function for the discretized system is obtained from Equation (11-58) as follows:

$$\frac{Y(z)}{U(z)} = F(z) = C(zI - G)^{-1}H + D$$

$$= (1)(z - e^{-aT})^{-1}\,\frac{1}{a}\,(1 - e^{-aT}) + 0$$

$$= \frac{1 - e^{-aT}}{a(z - e^{-aT})} = \frac{(1 - e^{-aT})z^{-1}}{a(1 - e^{-aT}z^{-1})}$$

B-11-15. Referring to Equation (11-58), the pulse transfer function F(z) is given by

$$\frac{Y(z)}{U(z)} = F(z) = \underset{\sim}{C}(z\underset{\sim}{I} - \underset{\sim}{G})^{-1}\underset{\sim}{H} + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z - 1 & -1 + e^{-T} \\ 0 & z - e^{-T} \end{bmatrix}^{-1} \begin{bmatrix} T + e^{-T} - 1 \\ 1 - e^{-T} \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(z-1)(z-e^{-T})} \begin{bmatrix} z - e^{-T} & 1 - e^{-T} \\ 0 & z - 1 \end{bmatrix}$$

$$\times \begin{bmatrix} T + e^{-T} - 1 \\ 1 - e^{-T} \end{bmatrix}$$

$$= \frac{1}{(z-1)(z-e^{-T})} \begin{bmatrix} z - e^{-T} & 1 - e^{-T} \end{bmatrix} \begin{bmatrix} T + e^{-T} - 1 \\ 1 - e^{-T} \end{bmatrix}$$

$$= \frac{(z - e^{-T})(T + e^{-T} - 1) + (1 - e^{-T})^2}{(z-1)(z-e^{-T})}$$

$$= \frac{(T + e^{-T} - 1)z + 1 - e^{-T}(1 + T)}{(z-1)(z-e^{-T})}$$

B-11-16. The discretized state equation is

$$\underset{\sim}{x}((k+1)T) = \underset{\sim}{G}(T)\underset{\sim}{x}(kT) + \underset{\sim}{H}(T)u(kT)$$

where

$$\underset{\sim}{G}(T) = e^{\underset{\sim}{A}T}, \qquad \underset{\sim}{H}(T) = \left(\int_0^T e^{\underset{\sim}{A}T}\, dt\right)\underset{\sim}{B}$$

$\underset{\sim}{G}(T)$ and $\underset{\sim}{H}(T)$ can be obtained as follows:

$$e^{\underset{\sim}{A}t} = \mathcal{L}^{-1}[(s\underset{\sim}{I} - \underset{\sim}{A})^{-1}] = \mathcal{L}^{-1}\left\{\begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1}\right\}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Hence

$$\underset{\sim}{G}(T) = \underset{\sim}{G}(0.1) = e^{0.1\underset{\sim}{A}} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}$$

Since

$$\int_0^{0.1} e^{\underset{\sim}{A}t}\, dt = \int_0^{0.1} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} dt = \begin{bmatrix} t & \frac{t^2}{2} \\ 0 & t \end{bmatrix}_0^{0.1}$$

$$= \begin{bmatrix} 0.1 & 0.005 \\ 0 & 0.1 \end{bmatrix}$$

we have

$$\underset{\sim}{H}(T) = \underset{\sim}{H}(0.1) = \left(\int_0^{0.1} e^{At}\, dt\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.005 \\ 0 & 0.1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix}$$

Thus, the discrete-time state equation is given by

$$\begin{bmatrix} x_1((k+1)T) \\ x_2((k+1)T) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix} + \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix} u(kT)$$

where $T = 0.1$ s. The output equation is

$$y(kT) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix}$$

Referring to Equation (11-58), the pulse transfer function for this discretized system is given by

$$F(z) = \underset{\sim}{C}(z\underset{\sim}{I} - \underset{\sim}{G})^{-1}\underset{\sim}{H} + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z - 1 & -0.1 \\ 0 & z - 1 \end{bmatrix}^{-1} \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{z - 1} & \frac{0.1}{(z - 1)^2} \\ 0 & \frac{1}{z - 1} \end{bmatrix} \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix}$$

$$= \frac{0.005(z + 1)}{(z - 1)^2} = \frac{0.005(1 + z^{-1})z^{-1}}{(1 - z^{-1})^2}$$

B-11-17. The discretized state equation is

$$\mathbf{x}((k + 1)T) = \mathbf{G}(T)\mathbf{x}(kT) + \mathbf{H}(T)u(kT)$$

where

$$\mathbf{G}(T) = e^{\mathbf{A}T}$$

$$\mathbf{H}(T) = \left(\int_0^T e^{\mathbf{A}t}\, dt \right) \mathbf{B}$$

$\mathbf{G}(T)$ and $\mathbf{H}(T)$ can be obtained as follows: Since

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \mathcal{L}^{-1}\left\{ \begin{bmatrix} s & -1 \\ 3 & s + 4 \end{bmatrix}^{-1} \right\}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} \frac{s + 4}{(s + 1)(s + 3)} & \frac{1}{(s + 1)(s + 3)} \\ \frac{-3}{(s + 1)(s + 3)} & \frac{s}{(s + 1)(s + 3)} \end{bmatrix}$$

$$= \mathcal{L}^{-1} \begin{bmatrix} \frac{1.5}{s + 1} - \frac{0.5}{s + 3} & \frac{0.5}{s + 1} - \frac{0.5}{s + 3} \\ \frac{-1.5}{s + 1} + \frac{1.5}{s + 3} & \frac{-0.5}{s + 1} + \frac{1.5}{s + 3} \end{bmatrix}$$

$$= \begin{bmatrix} 1.5\, e^{-t} - 0.5\, e^{-3t} & 0.5\, e^{-t} - 0.5\, e^{-3t} \\ -1.5\, e^{-t} + 1.5\, e^{-3t} & -0.5\, e^{-t} + 1.5\, e^{-3t} \end{bmatrix}$$

we have

$$\underset{\sim}{G}(T) = \underset{\sim}{G}(0.2) = e^{0.2\underset{\sim}{A}}$$

$$= \begin{bmatrix} 1.5\ e^{-0.2} - 0.5\ e^{-0.6} & 0.5\ e^{-0.2} - 0.5\ e^{-0.6} \\ -1.5\ e^{-0.2} + 1.5\ e^{-0.6} & -0.5\ e^{-0.2} + 1.5\ e^{-0.6} \end{bmatrix}$$

$$= \begin{bmatrix} 0.9537 & 0.1350 \\ -0.4049 & 0.4139 \end{bmatrix}$$

Also,

$$\int_0^{0.2} e^{\underset{\sim}{A}t}\, dt = \int_0^{0.2} \begin{bmatrix} 1.5\ e^{-t} - 0.5\ e^{-3t} & 0.5\ e^{-t} - 0.5\ e^{-3t} \\ -1.5\ e^{-t} + 1.5\ e^{-3t} & -0.5\ e^{-t} + 1.5\ e^{-3t} \end{bmatrix} dt$$

$$= \begin{bmatrix} -1.5(e^{-0.2} - 1) + \frac{0.5}{3}(e^{-0.6} - 1) & -0.5(e^{-0.2} - 1) + \frac{0.5}{3}(e^{-0.6} - 1) \\ 1.5(e^{-0.2} - 1) - \frac{1.5}{3}(e^{-0.6} - 1) & 0.5(e^{-0.2} - 1) - \frac{1.5}{3}(e^{-0.6} - 1) \end{bmatrix}$$

$$= \begin{bmatrix} 0.1967 & 0.01544 \\ -0.04631 & 0.1350 \end{bmatrix}$$

Hence

$$\underset{\sim}{H}(T) = \underset{\sim}{H}(0.2) = \left(\int_0^{0.2} e^{\underset{\sim}{A}t}\, dt\right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1967 & 0.01544 \\ -0.04631 & 0.1350 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.01544 \\ 0.1350 \end{bmatrix}$$

Thus, the discretized state space representation of the system becomes as follows:

$$\begin{bmatrix} x_1((k + 1)T) \\ x_2((k + 1)T) \end{bmatrix} = \begin{bmatrix} 0.9537 & 0.1350 \\ -0.4049 & 0.4139 \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix} + \begin{bmatrix} 0.01544 \\ 0.1350 \end{bmatrix} u(kT)$$

$$y(kT) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix}$$

where T = 0.2 s.

Referring to Equation (11-58), the pulse transfer function is obtained as follows:

$$\frac{Y(z)}{U(z)} = F(z) = \underset{\sim}{C}(z\underset{\sim}{I} - \underset{\sim}{G})^{-1}\underset{\sim}{H} + D$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z - 0.9537 & -0.1350 \\ 0.4049 & z - 0.4139 \end{bmatrix}^{-1} \begin{bmatrix} 0.01544 \\ 0.1350 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{z^2 - 1.3675z + 0.4493} \begin{bmatrix} z - 0.4139 & 0.1350 \\ -0.4049 & z - 0.9537 \end{bmatrix} \begin{bmatrix} 0.01544 \\ 0.1350 \end{bmatrix}$$

$$= \frac{0.01544z + 0.01183}{z^2 - 1.3675z + 0.4493}$$

$$= \frac{0.01544z^{-1} + 0.01183\,z^{-2}}{1 - 1.3675\,z^{-1} + 0.4493\,z^{-2}}$$

B-11-18 The given system has no input function, but is subjected to initial conditions. It is similar to free vibrations of a mechanical system consisting of a mass, damper, and spring.

Consider first the following system:

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_2 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

a state space representation of the system is

$$x(k + 1) = Gx(k) + Hu(k)$$

$$y(k) = Cx(k) + Du(k)$$

where

$$G = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [b_1 - a_1 b_0 \;\vdots\; b_2 - a_2 b_0], \quad D = b_0$$

Notice that the coefficients b_0, b_1, and b_2 affect only matrices C and D.

Next, consider the case where the system has no input (u = 0). For such a case, MATLAB produces a pulse transfer function of the following form:

$$\frac{Y(z)}{U(z)} = \frac{0 + 0z^{-1} + 0z^{-2}}{1 + a_1z^{-1} + a_2z^{-2}}$$

If the original system had an input function u such that

$$\ddot{y} + 2\dot{y} + 10y = u$$

MATLAB produces the pulse transfer function

$$\frac{Y(z)}{U(z)} = \frac{1}{1 + a_1z^{-1} + a_2z^{-2}}$$

The following MATLAB program will yield a discrete-time state-space representation when the sampling period T is 0.1 s. This program will also yield the pulse transfer function.

```
num = [0   0   0];
den = [1   2   10];
[A,B,C,D] = tf2ss(num,den);
[G,H] = c2d(A,B,0.1)

G =

   0.7753  -0.8913
   0.0891   0.9536


H =

   0.0891
   0.0046
[numz,denz] = ss2tf(G,H,C,D)

numz =

    0   0   0


denz =

   1.0000  -1.7288   0.8187
```

Based on the MATLAB output, the discrete-time state equation is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.7753 & -0.8913 \\ 0.0891 & 0.9536 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The pulse transfer function obtained from the MATLAB output is

$$\frac{X(z)}{U(z)} = \frac{0 + 0z^{-1} + 0z^{-2}}{1 - 1.7288z^{-1} + 0.8187z^{-2}} \qquad (1)$$

If the original system had an input function u, the numerator becomes nonzero. The difference equation corresponding to Equation (1) is

$$y(k + 2) - 1.7288\, y(k + 1) + 0.8187\, y(k) = 0 \qquad (2)$$

The initial data y(0) and y(1) are given by

$$y(0) = 1, \qquad y(1) = 0.9536$$

Note that the original differential equation system given by

$$\ddot{y} + 2\dot{y} + 10y = 0, \qquad y(0) = 1, \quad \dot{y}(0) = 0$$

has the response curve (free vibration curve) as shown in Figure (a).

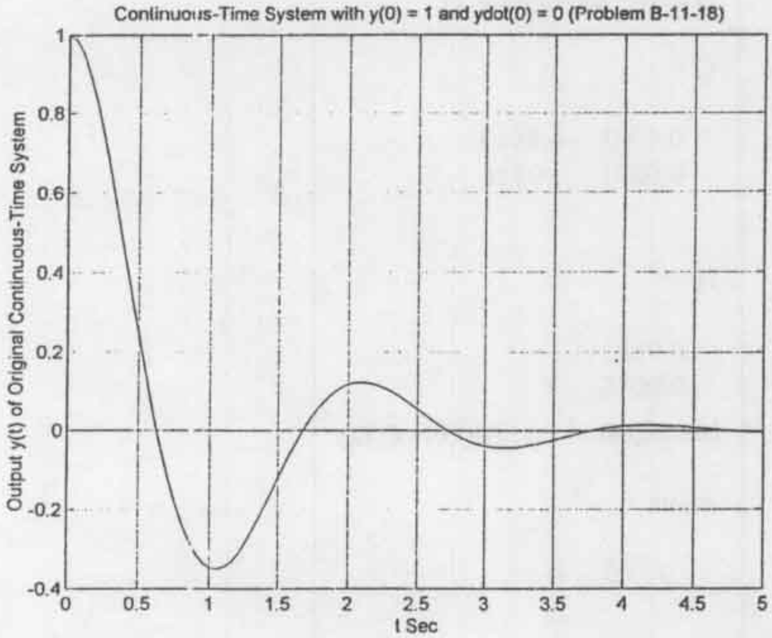


Figure (a)

The MATLAB program used to obtain this curve is given below.

```
num = [1   2   0];
den = [1   2   10];
step(num,den)
grid
title('Continuous-Time System with y(0) = 1 and ydot(0) = 0 (Problem B-11-18)')
xlabel('t Sec')
ylabel('Output y(t) of Original Continuous-Time System')
```

The response curve (free vibration curve) of the discretized version of the system given by Equation (2) is shown in Figure (b).

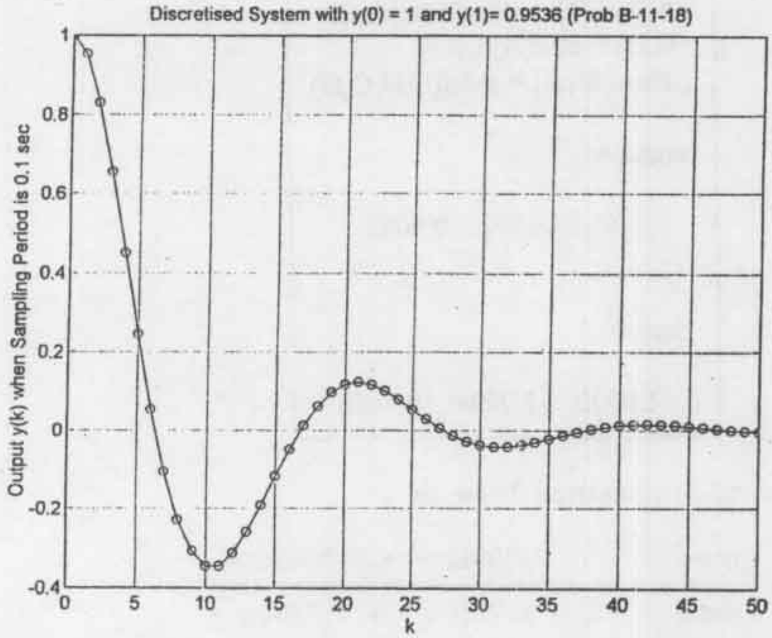


Figure (b)

The MATLAB program used to obtain this curve is given below.

```
num1 = [1  -0.7752  0];
den1 = [1  -1.7288  0.8187];
u = [1  zeros(1,50)];
k = 0:50;
y = filter(num1,den1,u);
plot(k,y,'o',k,y,'-')
grid
title('Discretised System with y(0) = 1 and y(1)= 0.9536 (Prob B-11-18)')
xlabel('k')
ylabel('Output y(k) when Sampling Period is 0.1 sec')
```

B-11-19.

$$G(s) = \frac{2}{(s + 1)(s + 2)} = \frac{2}{s^2 + 3s + 2}$$

The pulse transfer function for this system may be obtained by use of the MATLAB program shown on next page.

```
num = [0   0   2];
den = [1   3   2];
[A,B,C,D] = tf2ss(num,den);
[G,H] = c2d(A,B,0.1);
[numz,denz] = ss2tf(G,H,C,D)

numz =

       0   0.0091   0.0082


denz =

    1.0000  -1.7236   0.7408
```

The pulse transfer function obtained here is

$$G(z) = \frac{\text{numz}}{\text{denz}} = \frac{0.0091z^{-1} + 0.0082z^{-2}}{1 - 1.7236z^{-1} + 0.7408z^{-2}}$$

$$= \frac{0.0091z^{-1} + 0.0082z^{-2}}{(1 - 0.9048z^{-1})(1 - 0.8187z^{-1})}$$

This result is identical with the pulse transfer function obtained in Problem A-11-12.

<u>B-11-20</u>.

$$x(k + 2) = x(k + 1) + x(k) \qquad x(0) = 0,\ x(1) = 1$$

Taking the z transform of this difference equation, we obtain

$$z^2X(z) - z^2x(0) - zx(1) = zX(z) - zx(0) + X(z)$$

Hence

$$(z^2 - z - 1)X(z) = z$$

or

$$X(z) = \frac{z}{z^2 - z - 1} \qquad (1)$$

The Fibonacci series can be generated as the response of X(z) to the Kronecker delta input, where X(z) is given by Equation (1) and the Kronecker delta input u(k) is defined by

$$u(0) = 1$$

$$u(k) = 0 \qquad k = 1, 2, 3, \ldots$$

The following MATLAB program will generate the Fibonacci series.

```
num = [0   1   0];
den = [1   -1   -1];
u = [1   zeros(1,30)];
x = filter(num,den,u)

x =

 Columns 1 through 6

     0       1       1       2       3       5

 Columns 7 through 12

     8      13      21      34      55      89

 Columns 13 through 18

   144     233     377     610     987    1597

 Columns 19 through 24

  2584    4181    6765   10946   17711   28657

 Columns 25 through 30

 46368   75025  121393  196418  317811  514229

 Column 31

 832040
```

Note that column 1 corresponds to k = 0 and column 31 corresponds to k = 30. Thus, the Fibonacci series is given by

$$x(0) = 0$$

$$x(1) = 1$$

$$x(2) = 1$$

$$x(3) = 2$$

$$\vdots$$

$$x(29) = 514229$$

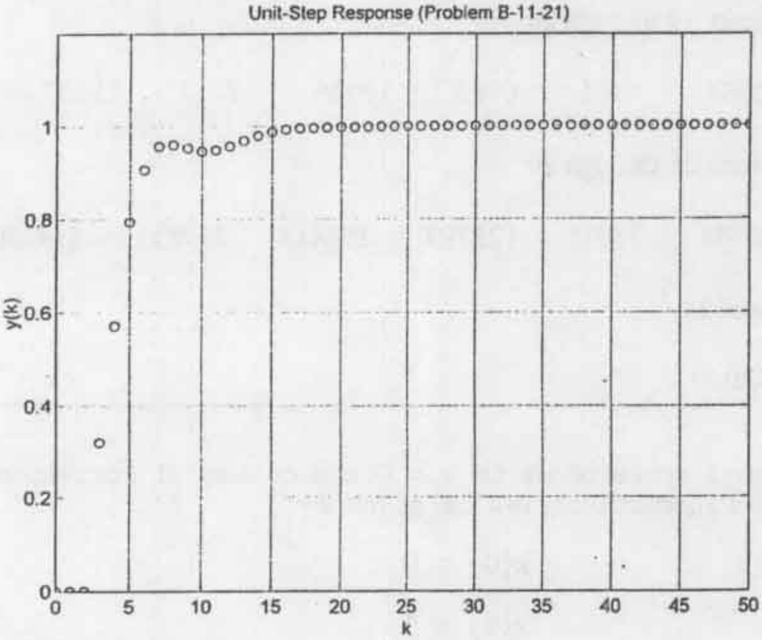
$$x(30) = 832040$$

B-11-21.

$$\frac{Y(z)}{U(z)} = \frac{0.3205z^{-3} - 0.1885z^{-4}}{1 - 1.3679z^{-1} + 0.3679z^{-2} + 0.3205z^{-3} - 0.1885z^{-4}}$$

The following MATLAB program will generate a plot of the unit-step response up to k = 50. The resulting unit-step response is shown below.

```
num = [0   0   0   0.3205   -0.1885];
den = [1  -1.3679   0.3679   0.3205   -0.1885];
u = ones(1,51);
k = 0:50;
y = filter(num,den,u);
plot(k,y,'o')
v = [0   50   0   1.2]; axis(v)
grid
title('Unit-Step Response (Problem B-11-21)')
xlabel('k')
ylabel('y(k)')
```

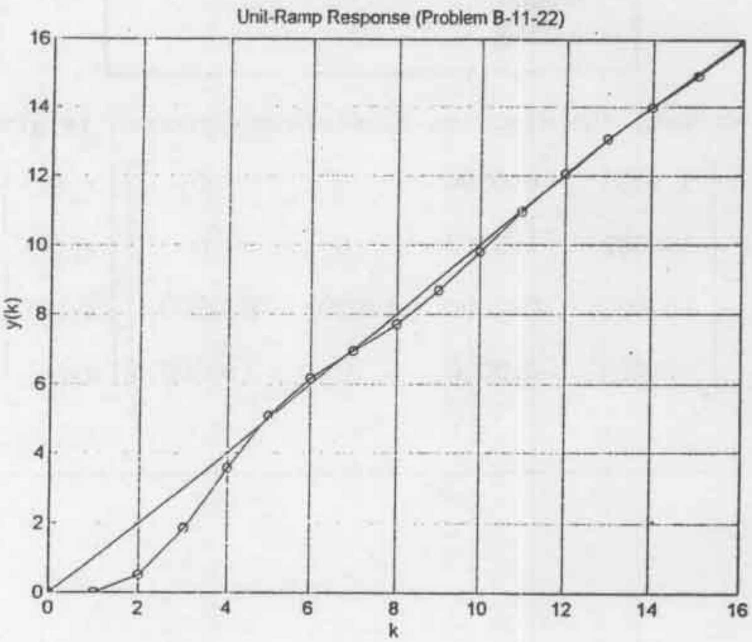


B-11-22.

$$\frac{Y(z)}{U(z)} = \frac{0.5151z^{-1} - 0.1452z^{-2} - 0.2963z^{-3} + 0.0528z^{-4}}{1 - 1.8528z^{-1} + 1.5906z^{-2} - 0.6642z^{-3} + 0.0528z^{-4}}$$

The following MATLAB program will generate a plot of the unit-ramp response of the system up to k = 16. (The sampling period is 1 s.) The resulting plot is shown on next page.

```
num = [0   0.5151   -0.1452   -0.2963   0.0528];
den = [1   -1.8528   1.5906   -0.6642   0.0528];
k = 0:16;
x = [k];
y = filter(num,den,x);
plot(k,y,'o',k,y,'-',k,k,'--')
grid
title('Unit-Ramp Response (Problem B-11-22)')
xlabel('k')
ylabel('y(k)')
```



B-11-23. A MATLAB program to discretize the given equation is presented below. (The sampling period T is 0.05 s.)

```
A = [ 0          1  0  0
      20.601     0  0  0
      0          0  0  1
      -0.4905    0  0  0];
B = [0;-1;0;0.5];
[G,H] = c2d(A,B,0.05)
```

This MATLAB program produces the output as shown on next page.

```
G =

   1.0259   0.0504        0        0
   1.0389   1.0259        0        0
  -0.0006   0.0000   1.0000   0.0500
  -0.0247  -0.0006        0   1.0000

H =

  -0.0013
  -0.0504
   0.0006
   0.0250
```

From the MATLAB output, the discrete-time state equation is given below.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 1.0259 & 0.0504 & 0 & 0 \\ 1.0389 & 1.0259 & 0 & 0 \\ -0.0006 & 0.0000 & 1.0000 & 0.0500 \\ -0.0247 & -0.0006 & 0 & 1.0000 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} -0.0013 \\ -0.0504 \\ 0.0006 \\ 0.0250 \end{bmatrix} u$$

The END