2.1. (a) $T(x[n]) = g[n]x[n]$

- Stable: Let $|x[n]| \leq M$ then $|T[x[n]]| \leq |g[n]|M$. So, it is stable if $|g[n]|$ is bounded.
- Causal: $y_1[n] = g[n]x_1[n]$ and $y_2[n] = g[n]x_2[n]$, so if $x_1[n] = x_2[n]$ for all $n < n_0$, then $y_1[n] = y_2[n]$ for all $n < n_0$, and the system is causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= g[n](ax_1[n] + bx_2[n]) \\ &= ag[n]x_1[n] + bg[n]x_2[n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

So this is linear.

- Not time-invariant:

$$\begin{aligned} T(x[n-n_0]) &= g[n]x[n-n_0] \\ &\neq y[n-n_0] = g[n-n_0]x[n-n_0] \end{aligned}$$

which is not TI.

- Memoryless: $y[n] = T(x[n])$ depends only on the n^{th} value of x, so it is memoryless.

(b) $T(x[n]) = \sum_{k=n_0}^{n} x[k]$

- Not Stable: $|x[n]| \leq M \to |T(x[n])| \leq \sum_{k=n_0}^{n} |x[k]| \leq |n - n_0|M$. As $n \to \infty$, $T \to \infty$, so not stable.
- Not Causal: $T(x[n])$ depends on the future values of $x[n]$ when $n < n_0$, so this is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= \sum_{k=n_0}^{n} ax_1[k] + bx_2[k] \\ &= a\sum_{k=n_0}^{n} x_1[n] + b\sum_{k=n_0}^{n} x_2[n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

The system is linear.

- Not TI:

$$\begin{aligned} T(x[n-n_0]) &= \sum_{k=n_0}^{n} x[k-n_0] \\ &= \sum_{k=0}^{n-n_0} x[k] \\ &\neq y[n-n_0] = \sum_{k=n_0}^{n-n_0} x[k] \end{aligned}$$

The system is not TI.

- Not Memoryless: Values of $y[n]$ depend on past values for $n > n_0$, so this is not memoryless.

(c) $T(x[n]) \sum_{k=n-n_0}^{n+n_0} x[k]$

- Stable: $|T(x[n])| \leq \sum_{k=n-n_0}^{n+n_0} |x[k]| \leq \sum_{k=n-n_0}^{n+n_0} x[k]M \leq |2n_0 + 1|M$ for $|x[n]| \leq M$, so it is stable.
- Not Causal: $T(x[n])$ depends on future values of $x[n]$, so it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= \sum_{k=n-n_0}^{n+n_0} ax_1[k] + bx_2[k] \\ &= a\sum_{k=n-n_0}^{n+n_0} x_1[k] + b\sum_{k=n-n_0}^{n+n_0} x_2[k] = aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

This is linear.

- TI:

$$\begin{aligned} T(x[n-n_0] &= \sum_{k=n-n_0}^{n+n_0} x[k-n_0] \\ &= \sum_{k=n-n_0}^{n} x[k] \\ &= y[n-n_0] \end{aligned}$$

This is TI.

- Not memoryless: The values of $y[n]$ depend on $2n_0$ other values of x, not memoryless.

(d) $T(x[n]) = x[n-n_0]$

- Stable: $|T(x[n])| = |x[n-n_0]| \leq M$ if $|x[n] \leq M$, so stable.
- Causality: If $n_0 \geq 0$, this is causal, otherwise it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[n-n_0] + bx_2[n-n_0] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

This is linear.

- TI: $T(x[n-n_d] = x[n-n_0-n_d] = y[n-n_d]$. This is TI.
- Not memoryless: Unless $n_0 = 0$, this is not memoryless.

(e) $T(x[n]) = e^{x[n]}$

- Stable: $|x[n]| \leq M$, $|T(x[n])| = |e^{x[n]}| \leq e^{|x[n]|} \leq e^M$, this is stable.
- Causal: It doesn't use future values of $x[n]$, so it causal.
- Not linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= e^{ax_1[n]+bx_2[n]} \\ &= e^{ax_1[n]}e^{bx_2[n]} \\ &\neq aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

This is not linear.

- TI: $T(x[n-n_0]) = e^{x[n-n_0]} = y[n-n_0]$, so this is TI.
- Memoryless: $y[n]$ depends on the n^{th} value of x only, so it is memoryless.

(f) $T(x[n]) = ax[n] + b$

- Stable: $|T(x[n])| = |ax[n] + b| \leq a|M| + |b|$, which is stable for finite a and b.
- Causal: This doesn't use future values of $x[n]$, so it is causal.
- Not linear:

$$\begin{aligned} T(cx_1[n] + dx_2[n]) &= acx_1[n] + adx_2[n] + b \\ &\neq cT(x_1[n]) + dT(x_2[n]) \end{aligned}$$

This is not linear.

- TI: $T(x[n-n_0]) = ax[n-n_0] + b = y[n-n_0]$. It is TI.
- Memoryless: $y[n]$ depends on the n^{th} value of $x[n]$ only, so it is memoryless.

(g) $T(x[n]) = x[-n]$
- Stable: $|T(x[n])| \leq |x[-n]| \leq M$, so it is stable.
- Not causal: For $n < 0$, it depends on the future value of $x[n]$, so it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[-n] + bx_2x[-n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  This is linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0]) &= x[-n-n_0] \\ &\neq y[n-n_0] = x[-n+n_0] \end{aligned}$$

  This is not TI.
- Not memoryless: For $n \neq 0$, it depends on a value of x other than the n^{th} value, so it is not memoryless.

(h) $T(x[n]) = x[n] + u[n+1]$
- Stable: $|T(x[n])| \leq M + 3$ for $n \geq -1$ and $|T(x[n])| \leq M$ for $n < -1$, so it is stable.
- Causal: Since it doesn't use future values of $x[n]$, it is causal.
- Not linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[n] + bx_2[n] + 3u[n+1] \\ &\neq aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  This is not linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0] &= x[n-n_0] + 3u[n+1] \\ &= y[n-n_0] \\ &= x[n-n_0] + 3u[n-n_0+1] \end{aligned}$$

  This is not TI.
- Memoryless: $y[n]$ depends on the n^{th} value of x only, so this is memoryless.

2.2. For an LTI system, the output is obtained from the convolution of the input with the impulse response of the system:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

(a) Since $h[k] \neq 0$, for $(N_0 \leq n \leq N_1)$,

$$y[n] = \sum_{k=N_0}^{N_1} h[k]x[n-k]$$

The input, $x[n] \neq 0$, for $(N_2 \leq n \leq N_3)$, so

$$x[n-k] \neq 0, \text{ for } N_2 \leq (n-k) \leq N_3$$

Note that the minimum value of $(n-k)$ is N_2. Thus, the lower bound on n, which occurs for $k = N_0$ is

$$N_4 = N_0 + N_2.$$

Using a similar argument,

$$N_5 = N_1 + N_3.$$

Therefore, the output is nonzero for

$$(N_0 + N_2) \leq n \leq (N_1 + N_3).$$

(b) If $x[n] \neq 0$, for some $n_o \leq n \leq (n_o + N - 1)$, and $h[n] \neq 0$, for some $n_1 \leq n \leq (n_1 + M - 1)$, the results of part (a) imply that the output is nonzero for:

$$(n_o + n_1) \leq n \leq (n_o + n_1 + M + N - 2)$$

So the output sequence is $M + N - 1$ samples long. This is an important quality of the convolution for finite length sequences as we shall see in Chapter 8.

2.3. We desire the step response to a system whose impulse response is

$$h[n] = a^{-n}u[-n], \text{ for } 0 < a < 1.$$

The convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The step response results when the input is the unit step:

$$x[n] = u[n] = \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

Substitution into the convolution sum yields

$$y[n] = \sum_{k=-\infty}^{\infty} a^{-k}u[-k]u[n-k]$$

For $n \leq 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} a^{-k} \\ &= \sum_{k=-n}^{\infty} a^{k} \\ &= \frac{a^{-n}}{1-a} \end{aligned}$$

For $n > 0$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{0} a^{-k} \\ &= \sum_{k=0}^{\infty} a^{k} \\ &= \frac{1}{1-a} \end{aligned}$$

2.4. The difference equation:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1]$$

To solve, we take the Fourier transform of both sides.

$$Y(e^{j\omega}) - \frac{3}{4}Y(e^{j\omega})e^{-j\omega} + \frac{1}{8}Y(e^{j\omega})e^{-j2\omega} = 2 \cdot X(e^{j\omega})e^{-j\omega}$$

The system function is given by:

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{2e^{-j\omega}}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}} \end{aligned}$$

The impulse response (for $x[n] = \delta[n]$) is the inverse Fourier transform of $H(e^{j\omega})$.

$$H(e^{j\omega}) = \frac{-8}{1 + \frac{1}{4}e^{-j\omega}} + \frac{8}{1 - \frac{1}{2}e^{-j\omega}}$$

Thus,

$$h[n] = -8(\frac{1}{4})^n u[n] + 8(\frac{1}{2})^n u[n].$$

2.5. (a) The homogeneous difference equation:

$$y[n] - 5y[n-1] + 6y[n-2] = 0$$

Taking the Z-transform,

$$1 - 5z^{-1} + 6z^{-2} = 0$$
$$(1 - 2z^{-1})(1 - 3z^{-1}) = 0.$$

The homogeneous solution is of the form

$$y_h[n] = A_1(2)^n + A_2(3)^n.$$

(b) We take the z-transform of both sides:

$$Y(z)[1 - 5z^{-1} + 6z^{-2}] = 2z^{-1}X(z)$$

Thus, the system function is

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{2z^{-1}}{1 - 5z^{-1} + 6z^{-2}} \\ &= \frac{-2}{1 - 2z^{-1}} + \frac{2}{1 - 3z^{-1}}, \end{aligned}$$

where the region of convergence is outside the outermost pole, because the system is causal. Hence the ROC is $|z| > 3$. Taking the inverse z-transform, the impulse response is

$$h[n] = -2(2)^n u[n] + 2(3)^n u[n].$$

(c) Let $x[n] = u[n]$ (unit step), then

$$X(z) = \frac{1}{1 - z^{-1}}$$

and

$$\begin{aligned} Y(z) &= X(z) \cdot H(z) \\ &= \frac{2z^{-1}}{(1 - z^{-1})(1 - 2z^{-1})(1 - 3z^{-1})}. \end{aligned}$$

Partial fraction expansion yields

$$Y(z) = \frac{1}{1 - z^{-1}} - \frac{4}{1 - 2z^{-1}} + \frac{3}{1 - 3z^{-1}}.$$

The inverse transform yields:

$$y[n] = u[n] - 4(2)^n u[n] + 3(3)^n u[n].$$

2.6. (a) The difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

Taking the Fourier transform of both sides,

$$Y(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega}] = X(e^{j\omega})[1 + 2e^{-j\omega} + e^{-j2\omega}].$$

Hence, the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \end{aligned}$$

(b) A system with frequency response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}} \\ &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \end{aligned}$$

cross multiplying,

$$Y(e^{j\omega})[1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}] = X(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}],$$

and the inverse transform gives

$$y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3].$$

2.7. $x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

(a) $x[n]$ is periodic with period 12:

$$e^{j(\frac{\pi}{6}n)} = e^{j(\frac{\pi}{6})(n+N)} = e^{j(\frac{\pi}{6}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{\pi}{6}N, \text{for integers } k, N$$

Making $k = 1$ and $N = 12$ shows that $x[n]$ has period 12.

(b) $x[n]$ is periodic with period 8:

$$e^{j(\frac{3\pi}{4}n)} = e^{j(\frac{3\pi}{4})(n+N)} = e^{j(\frac{3\pi}{4}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{3\pi}{4}N, \text{for integers } k, N$$

$$\Longrightarrow N = \frac{8}{3}k, \text{for integers } k, N$$

The smallest k for which both k and N are integers are is 3, resulting in the period N being 8.

(c) $x[n] = [\sin(\pi n/5)]/(\pi n)$ is not periodic because the denominator term is linear in n.

(d) We will show that $x[n]$ is not periodic. Suppose that $x[n]$ is periodic for some period N:

$$e^{j(\frac{\pi}{\sqrt{2}}n)} = e^{j(\frac{\pi}{\sqrt{2}})(n+N)} = e^{j(\frac{\pi}{\sqrt{2}}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{\pi}{\sqrt{2}}N, \text{for integers } k, N$$

$$\Longrightarrow N = 2\sqrt{2}k, \text{for some integers } k, N$$

There is no integer k for which N is an integer. Hence $x[n]$ is not periodic.

2.8. We take the Fourier transform of both $h[n]$ and $x[n]$, and then use the fact that convolution in the time domain is the same as multiplication in the frequency domain.

$$\begin{aligned}
H(e^{j\omega}) &= \frac{5}{1+\frac{1}{2}e^{-j\omega}} \\
Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\
&= \frac{5}{1+\frac{1}{2}e^{-j\omega}} \cdot \frac{1}{1-\frac{1}{3}e^{-j\omega}} \\
&= \frac{3}{1+\frac{1}{2}e^{-j\omega}} + \frac{2}{1-\frac{1}{3}e^{-j\omega}} \\
y[n] &= 2(\frac{1}{3})^n u[n] + 3(-\frac{1}{2})^n u[n]
\end{aligned}$$

2.9. (a) First the frequency response:

$$Y(e^{j\omega}) - \frac{5}{6}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{6}e^{-2j\omega}Y(e^{j\omega}) = \frac{1}{3}e^{-2j\omega}X(e^{j\omega})$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{\frac{1}{3}e^{-2j\omega}}{1 - \frac{5}{6}e^{-j\omega} + \frac{1}{6}e^{-2j\omega}} \end{aligned}$$

Now we take the inverse Fourier transform to find the impulse response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{-2}{1 - \frac{1}{3}e^{-j\omega}} + \frac{2}{1 - \frac{1}{2}e^{-j\omega}} \\ h[n] &= -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n] \end{aligned}$$

For the step response $s[n]$:

$$\begin{aligned} s[n] &= \sum_{k=-\infty}^{\infty} h[k]u[n-k] \\ &= \sum_{k=-\infty}^{n} h[k] \\ &= -2\frac{1-(1/3)^{n+1}}{1-1/3}u[n] + 2\frac{1-(1/2)^{n+1}}{1-1/2}u[n] \\ &= (1 + (\frac{1}{3})^n - 2(\frac{1}{2})^n)u[n] \end{aligned}$$

(b) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 - \frac{5}{6}c + \frac{1}{6} = 0$$

So for $c = 1/2$ and $c = 1/3$, the general form for the homogeneous solution is:

$$y_h[n] = A_1(\frac{1}{2})^n + A_2(\frac{1}{3})^n$$

(c) The total solution is the sum of the homogeneous and particular solutions, with the particular solution being the impulse response found in part (a):

$$\begin{aligned} y[n] &= y_h[n] + y_p[n] \\ &= A_1(\frac{1}{2})^n + A_2(\frac{1}{3})^n + -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n] \end{aligned}$$

Now we use the constraint $y[0] = y[1] = 1$ to solve for A_1 and A_2:

$$\begin{aligned} y[0] &= A_1 + A_2 - 2 + 2 = 1 \\ y[1] &= A_1/2 + A_2/3 - 2/3 + 1 = 1 \\ A_1 + A_2 &= 1 \\ A_1/2 + A_2/3 &= 2/3 \end{aligned}$$

With $A_1 = 2$ and $A_2 = -1$ solving the simultaneous equations, we find that the impulse response is

$$y[n] = 2(\frac{1}{2})^n - (\frac{1}{3})^n + -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n]$$

2.10. (a)

$$\begin{aligned} y[n] &= h[n] * x[n] \\ &= \sum_{k=-\infty}^{\infty} a^k u[-k-1]u[n-k] \\ &= \begin{cases} \sum_{k=-\infty}^{n} a^k, & n \le -1 \\ \sum_{k=-\infty}^{-1} a^k, & n > -1 \end{cases} \\ &= \begin{cases} \frac{a^n}{1-1/a}, & n \le -1 \\ \frac{1/a}{1-1/a}, & n > -1 \end{cases} \end{aligned}$$

(b) First, let us define $v[n] = 2^n u[-n-1]$. Then, from part (a), we know that

$$w[n] = u[n] * v[n] = \begin{cases} 2^{n+1}, & n \le -1 \\ 1, & n > -1 \end{cases}$$

Now,

$$\begin{aligned} y[n] &= u[n-4] * v[n] \\ &= w[n-4] \\ &= \begin{cases} 2^{n-3}, & n \le 3 \\ 1, & n > 3 \end{cases} \end{aligned}$$

(c) Given the same definitions for $v[n]$ and $w[n]$ from part(b), we use the fact that $h[n] = 2^{n-1}u[-(n-1)-1] = v[n-1]$ to reduce our work:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= x[n] * v[n-1] \\ &= w[n-1] \\ &= \begin{cases} 2^n, & n \le 0 \\ 1, & n > 0 \end{cases} \end{aligned}$$

(d) Again, we use $v[n]$ and $w[n]$ to help us.

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= (u[n] - u[n-10]) * v[n] \\ &= w[n] - w[n-10] \\ &= (2^{n+1}u[-(n+1)] + u[n]) - (2^{n-9}u[-(n-9)] + u[n-10]) \\ &= \begin{cases} 2^{(n+1)} - 2^{(n-9)}, & n \le -2 \\ 1 - 2^{(n-9)}, & -1 \le n \le 8 \\ 0, & n \ge 9 \end{cases} \end{aligned}$$

2.11. First we re-write $x[n]$ as a sum of complex exponentials:

$$x[n] = \sin(\frac{\pi n}{4}) = \frac{e^{j\pi n/4} - e^{-j\pi n/4}}{2j}.$$

Since complex exponentials are eigenfunctions of LTI systems,

$$y[n] = \frac{H(e^{j\pi/4})e^{j\pi n/4} - H(e^{-j\pi/4})e^{-j\pi n/4}}{2j}$$

Evaluating the frequency response at $\omega = \pm\pi/4$:

$$\begin{aligned} H(e^{j\frac{\pi}{4}}) &= \frac{1 - e^{-j\pi/2}}{1 + 1/2e^{-j\pi}} = 2(1 - j) = 2\sqrt{2}e^{-j\pi/4} \\ H(e^{-j\frac{\pi}{4}}) &= \frac{1 - e^{j\pi/2}}{1 + 1/2e^{j\pi}} = 2(1 + j) = 2\sqrt{2}e^{j\pi/4} \end{aligned}$$

We get:

$$\begin{aligned} y[n] &= \frac{2\sqrt{2}e^{-j\pi/4}e^{j\pi n/4} - 2\sqrt{2}e^{j\pi/4}e^{-j\pi n/4}}{2j} \\ &= 2\sqrt{2}\sin(\pi n/4 - \pi/4). \end{aligned}$$

2.12. The difference equation:

$$y[n] = ny[n-1] + x[n]$$

Since the system is causal and satisfies initial-rest conditions, we may recursively find the response to any input.

(a) Suppose $x[n] = \delta[n]$:

$$\begin{aligned} y[n] &= 0, \text{ for } n < 0 \\ y[0] &= 1 \\ y[1] &= 1 \\ y[2] &= 2 \\ y[3] &= 6 \\ y[4] &= 24 \end{aligned}$$

$$y[n] = h[n] = n!u[n]$$

(b) To determine if the system is linear, consider the input:

$$x[n] = a\delta[n] + b\delta[n]$$

performing the recursion,

$$\begin{aligned} y[n] &= 0, \text{ for } n < 0 \\ y[0] &= a + b \\ y[1] &= a + b \\ y[2] &= 2(a + b) \\ y[3] &= 6(a + b) \\ y[4] &= 24(a + b) \end{aligned}$$

Because the output of the superposition of two input signals is equivalent to the superposition of the individual outputs, the system is LINEAR.

(c) To determine if the system is time-invariant, consider the input:

$$x[n] = \delta[n-1]$$

the recursion yields

$$\begin{aligned} y[n] &= 0, \text{ for } n < 0 \\ y[0] &= 0 \\ y[1] &= 1 \\ y[2] &= 2 \\ y[3] &= 6 \\ y[4] &= 24 \end{aligned}$$

Using $h[n]$ from part (a),

$$h[n-1] = (n-1)!u[n-1] \neq y[n]|_{x[n]=\delta[n-1]}$$

Conclude: NOT TIME INVARIANT.

2.13. Eigenfunctions of LTI systems are of the form α^n, so functions (a), (b), and (e) are eigenfunctions. Notice that part (d), $\cos(\omega_0 n) = .5(e^{j\omega_0 n} + e^{-j\omega_0 n})$ is a sum of two α^n functions, and is therefore not an eigenfunction itself.

2.14. (a) The information given shows that the system satisfies the eigenfunction property of exponential sequences for LTI systems for one particular eigenfunction input. However, we do not know the system response for any other eigenfunction. Hence, we can say that the system may be LTI, but we cannot uniquely determine it. $\Longrightarrow$ (iv).

(b) If the system were LTI, the output should be in the form of $A(1/2)^n$, since $(1/2)^n$ would have been an eigenfunction of the system. Since this is not true, the system cannot be LTI. $\Longrightarrow$ (i).

(c) Given the information, the system *may* be LTI, but does not have to be. For example, for any input other than the given one, the system may output 0, making this system non-LTI. $\Longrightarrow$ (iii). If it were LTI, its system function can be found by using the DTFT:

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1}{1-\frac{1}{2}e^{-j\omega}} \\ h[n] &= (\frac{1}{2})^n u[n] \end{aligned}$$

2.15. (a) No. Consider the following input/outputs:

$$x_1[n] = \delta[n] \quad \Longrightarrow \quad y_1[n] = (\frac{1}{4})^n u[n]$$

$$x_2[n] = \delta[n-1] \quad \Longrightarrow \quad y_2[n] = (\frac{1}{4})^{n-1} u[n]$$

Even though $x_2[n] = x_1[n-1]$, $y_2[n] \neq y_1[n-1] = (\frac{1}{4})^{n-1}u[n-1]$

(b) No. Consider the input/output pair $x_2[n]$ and $y_2[n]$ above. $x_2[n] = 0$ for $n < 1$, but $y_2[0] \neq 0$.

(c) Yes. Since $h[n]$ is stable and multiplication with $u[n]$ will not cause any sequences to become unbounded, the entire system is stable.

2.16. (a) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 - \frac{1}{4}c + \frac{1}{8} = 0$$

So for $c = 1/2$ and $c = -1/4$, the general form for the homogeneous solution is:

$$y_h[n] = A_1(\frac{1}{2})^n + A_2(-\frac{1}{4})^n$$

(b) Taking the z-transform of both sides, we find that

$$Y(z)(1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}) = 3X(z)$$

and therefore

$$\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{3}{1 - 1/4z^{-1} - 1/8z^{-2}} \\
&= \frac{3}{(1 + 1/4z^{-1})(1 - 1/2z^{-1})} \\
&= \frac{1}{1 + 1/4z^{-1}} + \frac{2}{1 - 1/2z^{-1}}
\end{aligned}$$

The causal impulse response corresponds to assuming that the region of convergence extends outside the outermost pole, making

$$h_c[n] = ((-1/4)^n + 2(1/2)^n)u[n]$$

The anti-causal impulse response corresponds to assuming that the region of convergence is inside the innermost pole, making

$$h_{ac}[n] = -((-1/4)^n + 2(1/2)^n)u[-n-1]$$

(c) $h_c[n]$ is absolutely summable, while $h_{ac}[n]$ grows without bounds.

(d)

$$\begin{aligned}
Y(z) &= X(z)H(z) \\
&= \frac{1}{1 - \frac{1}{2}z^{-1}} \cdot \frac{1}{(1 + \frac{1}{4}z^{-1})(1 - \frac{1}{2}z^{-1})} \\
&= \frac{1/3}{1 + 1/4z^{-1}} + \frac{2}{1 - 1/2z^{-1}} + \frac{2/3}{1 - 1/2z^{-1}} \\
y[n] &= \frac{1}{3}(\frac{1}{4})^n u[n] + 4(n+1)(\frac{1}{2})^{n+1}u[n+1] + \frac{2}{3}(\frac{1}{2})^n u[n]
\end{aligned}$$

2.17. (a) We have

$$r[n] = \begin{cases} 1, & \text{for } 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

Taking the Fourier transform

$$\begin{aligned} R(e^{j\omega}) &= \sum_{n=0}^{M} e^{-j\omega n} \\ &= \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} \\ &= e^{-j\frac{M}{2}\omega} \left(\frac{e^{j\frac{M+1}{2}\omega} - e^{-j\frac{M+1}{2}\omega}}{e^{j\omega} - e^{-j\omega}} \right) \\ &= e^{-j\frac{M}{2}\omega} \left(\frac{\sin(\frac{M+1}{2}\omega)}{\sin(\omega/2)} \right) \end{aligned}$$

(b) We have

$$w[n] = \begin{cases} \frac{1}{2}(1 + \cos(\frac{2\pi n}{M}), & \text{for } 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

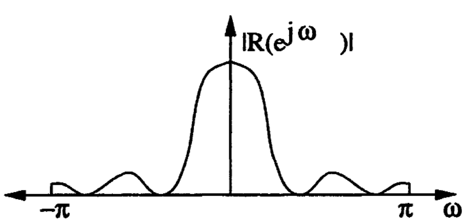
We note that,

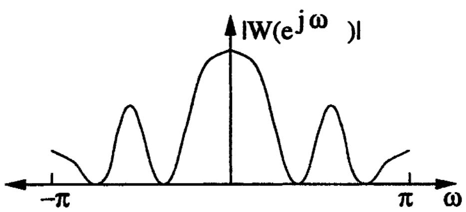
$$w[n] = r[n] \cdot \frac{1}{2}[1 + cos(\frac{2\pi n}{M})].$$

Thus,

$$\begin{aligned} W(e^{j\omega}) &= R(e^{j\omega}) * \sum_{n=-\infty}^{\infty} \frac{1}{2}(1 + cos(\frac{2\pi n}{M}))e^{-j\omega n} \\ &= R(e^{j\omega}) * \sum_{n=-\infty}^{\infty} \frac{1}{2}(1 + \frac{1}{2}e^{j\frac{2\pi n}{M}} + \frac{1}{2}e^{-j\frac{2\pi n}{M}})e^{-j\omega} \\ &= R(e^{j\omega}) * (\frac{1}{2}\delta(\omega) + \frac{1}{4}\delta(\omega + \frac{2\pi}{M}) + \frac{1}{4}\delta(\omega - \frac{2\pi}{M})) \end{aligned}$$

(c)





2.18. $h[n]$ is causal if $h[n] = 0$ for $n < 0$. Hence, (a) and (b) are causal, while (c), (d), and (e) are not.

2.19. $h[n]$ is stable if it is absolutely summable.

(a) Not stable because $h[n]$ goes to ∞ as n goes to ∞.

(b) Stable, because $h[n]$ is non-zero only for $0 \le n \le 9$.

(c) Stable.

$$\sum_n |h[n]| = \sum_{n=-\infty}^{-1} 3^n = \sum_{n=1}^{\infty} (1/3)^n = 1/2 < \infty$$

(d) Not stable. Notice that

$$\sum_{n=0}^{5} |\sin(\pi n/3)| = 2\sqrt{3}$$

and summing $|h[n]|$ over all positive n therefore grows to ∞.

(e) Stable. Notice that $|h[n]|$ is upperbounded by $(3/4)^{|n|}$, which is absolutely summable.

(f) Stable.

$$h[n] = \begin{cases} 2, & ,-5 \le n \le -1 \\ 1, & ,0 \le n \le 4 \\ 0, & ,\text{otherwise} \end{cases}$$

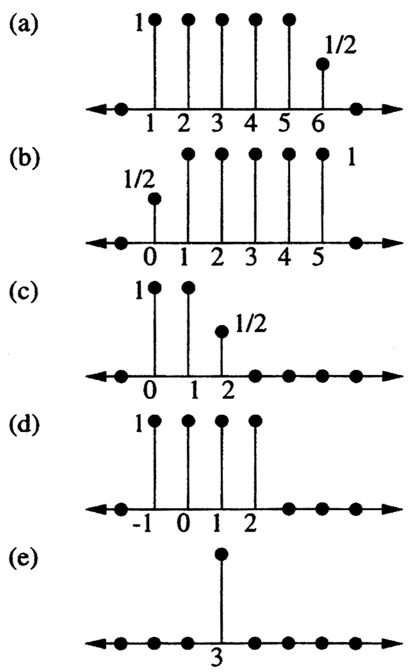
So $\sum |h[n]| = 15$.

2.20. (a) Taking the difference equation $y[n] = (1/a)y[n-1] + x[n-1]$ and assuming $h[0] = 0$ for $n < 0$:

$$\begin{aligned} h[0] &= 0 \\ h[1] &= 1 \\ h[2] &= 1/a \\ h[3] &= (1/a)^2 \\ &\vdots \\ h[n] &= (1/a)^{n-1}u[n-1] \end{aligned}$$

(b) $h[n]$ is absolutely summable if $|1/a| < 1$ or if $|a| > 1$

2.21.



2.22. For an LTI system, we use the convolution equation to obtain the output:

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Let $n = m + N$:

$$\begin{aligned} y[m+N] &= \sum_{k=-\infty}^{\infty} x[m+N-k]h[k] \\ &= \sum_{k=-\infty}^{\infty} x[(m-k)+N]h[k] \end{aligned}$$

Since $x[n]$ is periodic, $x[n] = x[n+rN]$ for any integer r. Hence,

$$\begin{aligned} y[m+N] &= \sum_{k=-\infty}^{\infty} x[m-k]h[k] \\ &= y[m] \end{aligned}$$

So, the output must also be periodic with period N.

2.23. (a) Since $\cos(\pi n)$ only takes on values of +1 or -1, this transformation outputs the current value of $x[n]$ multiplied by either ± 1. $T(x[n]) = (-1)^n x[n]$.

- Hence, it is stable, because it doesn't change the magnitude of $x[n]$ and hence satisfies bounded-in/bounded-out stability.
- It is causal, because each output depends only on the current value of $x[n]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = \cos(\pi n)x_1[n]$, and $y_2[n] = T(x_2[n]) = \cos(\pi n)x_2[n]$. Now

$$T(ax_1[n] + bx_2[n]) = \cos(\pi n)(ax_1[n] + bx_2[n]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = (-1)^n x[n]$, then $T(x[n-1]) = (-1)^n x[n-1] \neq y[n-1]$.

(b) This transformation simply "samples" $x[n]$ at location which can be expressed as k^2.

- The system is stable, since if $x[n]$ is bounded, $x[n^2]$ is also bounded.
- It is not causal. For example, $Tx[4] = x[16]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = x_1[n^2]$, and $y_2[n] = T(x_2[n]) = x_2[n^2]$. Now

$$T(ax_1[n] + bx_2[n]) = ax_1[n^2] + bx_2[n^2]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = x[n^2]$, then $T(x[n-1]) = x[n^2-1] \neq y[n-1]$.

(c) First notice that

$$\sum_{k=0}^{\infty} \delta[n-k] = u[n]$$

So $T(x[n]) = x[n]u[n]$. This transformation is therefore stable, causal, linear, but not time-invariant.

To see that it is not time invariant, notice that $T(\delta[n]) = \delta[n]$, but $T(\delta[n+1]) = 0$.

(d) $T(x[n]) = \sum_{k=n-1}^{\infty} x[k]$

- This is not stable. For example, $T(u[n]) = \infty$ for all n ¿ 1.
- It is not causal, since it sums *forward* in time.
- It is linear, since

$$\sum_{k=n-1}^{\infty} ax_1[k] + bx_2[k] = a \sum_{k=n-1}^{\infty} x_1[k] + b \sum_{k=n-1}^{\infty} x_2[k]$$

- It is time-invariant. Let

$$y[n] = T(x[n]) = \sum_{k=n-1}^{\infty} x[k],$$

then

$$T(x[n-n_0]) = \sum_{k=n-n_0-1}^{\infty} x[k] = y[n-n_0]$$

2.24. For an arbitrary linear system, we have

$$y[n] = T\{x[n]\},$$

Let $x[n] = 0$ for all n.

$$y[n] = T\{x[n]\}$$

For some arbitrary $x_1[n]$, we have

$$y_1[n] = T\{x_1[n]\}$$

Using the linearity of the system:

$$\begin{aligned} T\{x[n] + x_1[n]\} &= T\{x[n]\} + T\{x_1[n]\} \\ &= y[n] + y_1[n] \end{aligned}$$

Since $x[n]$ is zero for all n,

$$T\{x[n] + x_1[n]\} = T\{x_1[n]\} = y_1[n]$$

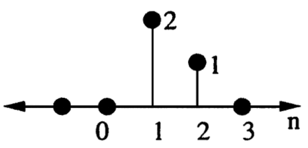
Hence, $y[n]$ must also be zero for all n.

2.25. We use the graphical approach to compute the convolution:

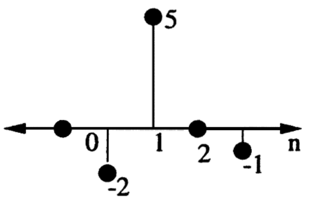
$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

(a) $y[n] = x[n] * h[n]$

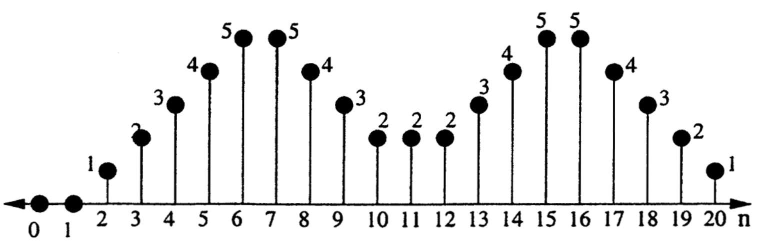
$$y[n] = \delta[n-1] * h[n] = h[n-1]$$



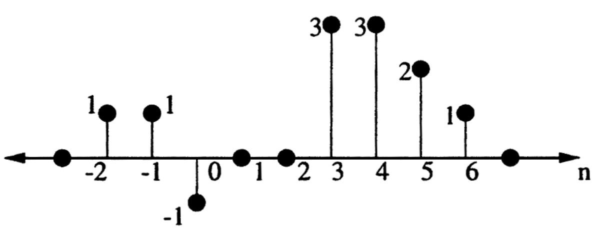
(b) $y[n] = x[n] * h[n]$



(c) $y[n] = x[n] * h[n]$



(d) $y[n] = x[n] * h[n]$



2.26. The response of the system to a delayed step:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} u[k-4]h[n-k] \end{aligned}$$

$$y[n] = \sum_{k=4}^{\infty} h[n-k]$$

Evaluating the above summation:

For $n < 4$: $y[n] = 0$
For $n = 4$: $y[n] = h[0] = 1$
For $n = 5$: $y[n] = h[1] + h[0] = 2$
For $n = 6$: $y[n] = h[2] + h[1] + h[0] = 3$
For $n = 7$: $y[n] = h[3] + h[2] + h[1] + h[0] = 4$
For $n = 8$: $y[n] = h[4] + h[3] + h[2] + h[1] + h[0] = 2$
For $n \geq 9$: $y[n] = h[5] + h[4] + h[3] + h[2] + h[1] + h[0] = 0$

2.27. The output is obtained from the convolution sum:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} x[k]u[n-k] \end{aligned}$$

The convolution may be broken into five regions over the range of n:

$$y[n] = 0, \text{for } n < 0$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{n} a^k \\ &= \frac{1-a^{(n+1)}}{1-a}, \text{for } 0 \leq n \leq N_1 \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k \\ &= \frac{1-a^{(N_1+1)}}{1-a}, \text{ for } N_1 < n < N_2 \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k + \sum_{k=N_2}^{n} a^{(k-N_2)} \\ &= \frac{1-a^{(N_1+1)}}{1-a} + \frac{1-a^{(n+1)}}{1-a} \\ &= \frac{2-a^{(N_1+1)}-a^{(n+1)}}{1-a}, \text{ for } N_2 \leq n \leq (N_1+N_2) \end{aligned}$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{N_1} a^k + \sum_{k=N_2}^{N_1+N_2} a^{(k-N_2)} \\ &= \sum_{k=0}^{N_1} a^k + \sum_{m=0} N_1 a^m \\ &= 2\sum_{k=0}^{N_1} a^k \\ &= 2 \cdot \left(\frac{1-a^{(N_1+1)}}{1-a}\right), \text{ for } n > (N_1+N_2) \end{aligned}$$

2.28. (a) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 + \frac{1}{15}c - \frac{2}{5} = 0$$

So for $c = 1/3$ and $c = -2/5$, the general form for the homogeneous solution is:

$$y_h[n] = A_1(\frac{1}{3})^n + A_2(-\frac{2}{5})^n$$

(b) We use the z-transform, and use different ROCs to generate the causal and anti-causal impulses responses:

$$H(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1+\frac{2}{5}z^{-1})} = \frac{5/11}{1-\frac{1}{3}z^{-1}} + \frac{6/11}{1+\frac{2}{5}z^{-1}}$$

$$h_c[n] = \frac{5}{11}(\frac{1}{3})^n u[n] + \frac{6}{11}(-\frac{2}{5})^n u[n]$$

$$h_{ac}[n] = -\frac{5}{11}(\frac{1}{3})^n u[-n-1] - \frac{6}{11}(-\frac{2}{5})^n u[-n-1]$$

(c) Since $h_c[n]$ is causal, and the two exponential bases in $h_c[n]$ are both less than 1, it is absolutely summable. $h_{ac}[n]$ grows without bounds as n approaches $-\infty$.

(d)

$$\begin{aligned}
Y(z) &= X(z)H(z) \\
&= \frac{1}{1-\frac{3}{5}z^{-1}} \cdot \frac{1}{(1-\frac{1}{3}z^{-1})(1+\frac{2}{5}z^{-1})} \\
&= \frac{-25/44}{1-1/3z^{-1}} + \frac{55/12}{1+2/5z^{-1}} + \frac{27/20}{1-3/5z^{-1}} \\
y[n] &= \frac{-25}{44}(\frac{1}{3})^n u[n] + \frac{55}{12}(-\frac{2}{5})^n u[n] + \frac{27}{20}(\frac{3}{5})^n u[n]
\end{aligned}$$

2.29. • System A:

$$x[n] = (\frac{1}{2})^n$$

This input is an eigenfunction of an LTI system. That is, if the system is linear, the output will be a replica of the input, scaled by a complex constant.
Since $y[n] = (\frac{1}{4})^n$, System A is NOT LTI.

- System B:

$$x[n] = e^{jn/8}u[n]$$

The Fourier transform of $x[n]$ is

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} e^{jn/8}u[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} e^{-j(\omega-\frac{1}{8})n} \\ &= \frac{1}{1-e^{-j(\omega-\frac{1}{8})}}. \end{aligned}$$

The output is $y[n] = 2x[n]$, thus

$$Y(e^{j\omega}) = \frac{2}{1-e^{-j(\omega-\frac{1}{8})}}.$$

Therefore, the frequency response of the system is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= 2. \end{aligned}$$

Hence, the system is a linear amplifier. We conclude that System B is LTI, and unique.

- System C: Since $x[n] = e^{jn/8}$ is an eigenfunction of an LTI system, we would expect the output to be given by

$$y[n] = \gamma e^{jn/8},$$

where γ is some complex constant, if System C were indeed LTI. The given output, $y[n] = 2e^{jn/8}$, indicates that this is so.
Hence, System C is LTI. However, it is not unique, since the only constraint is that

$$H(e^{j\omega})|_{\omega=1/8} = 2.$$

2.30

A. LTI systems are stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ (the summation should converge).

Then

$$S = \sum_{n=-\infty}^{\infty} |a|^n u[n]$$
$$= \sum_{n=0}^{\infty} |a|^n$$

S will converge only when $|a| < 1$ and $S = \frac{1}{1-|a|} < \infty$.

Therefore the system is stable for $|a| < 1$.

B. $y[n] = ay[n-1] + x[n] - a^N x[n-N]$. Therefore,

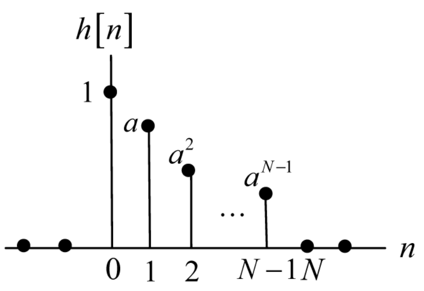
$h[n] = ah[n-1] + \delta[n] - a^N \delta[n-N]$.

Since the system is causal, $h[-1] = 0$. Then

$h[0] = 0 + 1 - 0 = 1$

$h[1] = a$, $h[2] = a^2$, $h[N] = a^N - a^N = 0$

$h[N+1] = a \times 0 + 0 - 0 = 0$.



C. We see that even though it is a recursive system (with feedback), its impulse respor finite in length. The length of $h[n]$ is N terms. Hence this system is FIR.

D. FIR systems are always stable as the sum $\sum_{n=-\infty}^{\infty} |h[n]|$ has at most a finite number of nonzero terms.

2.31. For $(-1 < a < 0)$, we have

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

(a) real part of $X(e^{j\omega})$:

$$\begin{aligned} X_R(e^{j\omega}) &= \frac{1}{2} \cdot [X(e^{j\omega}) + X^*(e^{j\omega})] \\ &= \frac{1 - a\cos(\omega)}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

(b) imaginary part:

$$\begin{aligned} X_I(e^{j\omega}) &= \frac{1}{2j} \cdot [X(e^{j\omega}) - X^*(e^{j\omega})] \\ &= \frac{-a\sin(\omega)}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

(c) magnitude:

$$\begin{aligned} |X(e^{j\omega})| &= [X(e^{j\omega})X^*(e^{j\omega})]^{\frac{1}{2}} \\ &= \left(\frac{1}{1 - 2a\cos(\omega) + a^2}\right)^{\frac{1}{2}} \end{aligned}$$

(d) phase:

$$\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin(\omega)}{1 - a\cos(\omega)}\right)$$

2.32

$$y[n] = -2x[n] + 4x[n-1] - 2x[n-2]$$

A. Impulse response:

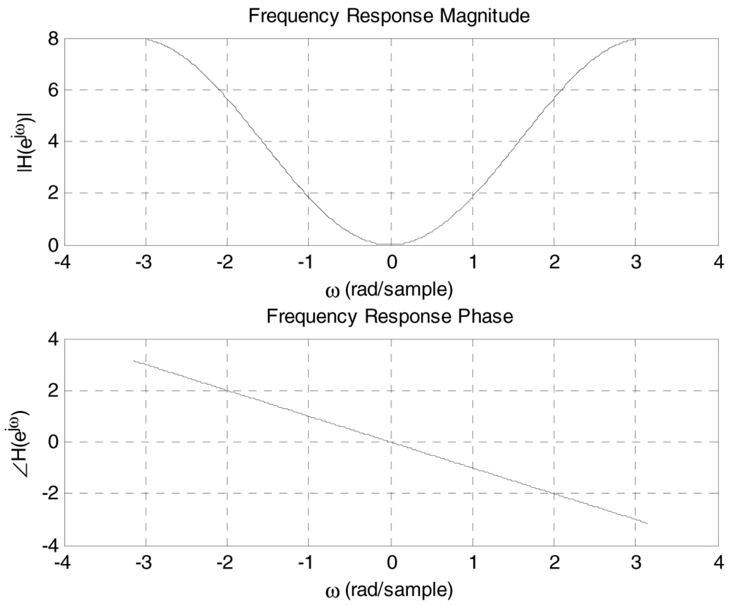
$$h[n] = -2\delta[n] + 4\delta[n-1] - 2\delta[n-2]$$

B.

$$\begin{aligned} H\left(e^{j\omega}\right) &= -2 + 4e^{-j\omega} - 2e^{-j2\omega} \\ &= -2e^{-j\omega}\left(e^{j\omega} + e^{-j\omega} - 2\right) \\ &= -2e^{-j\omega}\left(2\cos(\omega) - 2\right) \\ &= 4e^{-j\omega}\left(1 - \cos(\omega)\right) \\ &= 4e^{-j\omega}\left(2\sin^2(\omega/2)\right) \\ &= 8\sin^2(\omega/2)e^{-j\omega}. \end{aligned}$$

The delay is $n_d = 1$.

C.



D. If

$$\begin{aligned} x_1[n] &= 1 + e^{j0.5\pi n} \\ &= e^{j0n} + e^{j\frac{\pi}{2}n} \end{aligned}$$

then

$$\begin{aligned} y_1[n] &= H\left(e^{j0}\right)e^{j0n} + H\left(e^{j\frac{\pi}{2}n}\right)e^{j\frac{\pi}{2}n} \\ &= 0 \times e^{j0} + 8\sin^2(\pi/4)e^{-j\frac{\pi}{2}}e^{j\frac{\pi}{2}n} \\ &= 8 \times \frac{1}{2}e^{j\frac{\pi}{2}(n-1)}, \quad -\infty < n < \infty. \end{aligned}$$

E. Using the convolution sum,

$$\begin{aligned} y_2[n] &= \sum_{k=-\infty}^{\infty} h[k]x_2[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]\left(1 + e^{j\frac{\pi}{2}(n-k)}\right)u[n-k] \\ &= \sum_{k=-\infty}^{n} h[k]\left(1 + e^{j\frac{\pi}{2}(n-k)}\right) \end{aligned}$$

$$y_2[n] = \begin{cases} 0, & n < 0 \text{ (as the system is causal)} \\ \sum_{k=0}^{n} h[k]\left(1 + e^{j\frac{\pi}{2}(n-k)}\right), & n \geq 0 \end{cases}$$

Consider $n \geq 0$,

$$\begin{aligned} y_2[n] &= \left(\sum_{k=0}^{\infty} h[k]\left(1 + e^{j\frac{\pi}{2}(n-k)}\right)\right) - \left(\sum_{k=n+1}^{\infty} h[k]\left(1 + e^{j\frac{\pi}{2}(n-k)}\right)\right) \\ &= \sum_{k=0}^{\infty} h[k] + \left(\sum_{k=0}^{\infty} h[k]e^{-j\frac{\pi}{2}k}\right)e^{j\frac{\pi}{2}n} - \left(\sum_{k=n+1}^{\infty} h[k]\left(1 + e^{j\frac{\pi}{2}(n-k)}\right)\right) \\ &= H\left(e^{j0}\right) + H\left(e^{j\frac{\pi}{2}}\right)e^{j\frac{\pi}{2}n} - \left(\sum_{k=n+1}^{\infty} h[k]\left(1 + e^{j\frac{\pi}{2}(n-k)}\right)\right). \end{aligned}$$

Now $\left(\sum_{k=n+1}^{\infty} h[k]\left(1 + e^{j\frac{\pi}{2}(n-k)}\right)\right)$ becomes zero for $n \geq 2$ since $h[n] = 0$ for $n > 2$. Thus $y_2[n] = y_1[n]$ for all $n \geq 2$.

2.33. Recall that an eigenfunction of a system is an input signal which appears at the output of the system scaled by a complex constant.

(a) $x[n] = 5^n u[n]$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]5^{(n-k)}u[n-k] \\ &= 5^n \sum_{k=-\infty}^{n} h[k]5^{-k} \end{aligned}$$

Becuase the summation depends on n, $x[n]$ is NOT AN EIGENFUNCTION.

(b) $x[n] = e^{j2\omega n}$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]e^{j2\omega(n-k)} \\ &= e^{j2\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j2\omega k} \\ &= e^{j2\omega n} \cdot H(e^{j2\omega}) \end{aligned}$$

YES, EIGENFUNCTION.

(c) $e^{j\omega n} + e^{j2\omega n}$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} + \sum_{k=-\infty}^{\infty} h[k]e^{j2\omega(n-k)} \\ &= e^{j\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} + e^{j2\omega n} \sum_{k=-\infty}^{\infty} h[k]e^{-j2\omega k} \\ &= e^{j\omega n} \cdot H(e^{j\omega}) + e^{j2\omega n} \cdot H(e^{j2\omega}) \end{aligned}$$

Since the input cannot be extracted from the above expression, the sum of complex exponentials is NOT AN EIGENFUNCTION. (Although, separately the inputs are eigenfunctions. In general, complex exponential signals are always eigenfunctions of LTI systems.)

(d) $x[n] = 5^n$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]5^{(n-k)} \\ &= 5^n \sum_{k=-\infty}^{\infty} h[k]5^{-k} \end{aligned}$$

YES, EIGENFUNCTION.

(e) $x[n] = 5^n e^{j2\omega n}$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]5^{(n-k)}e^{j2\omega(n-k)} \\ &= 5^n e^{j2\omega n} \sum_{k=-\infty}^{\infty} h[k]5^{-k}e^{-j2\omega k} \end{aligned}$$

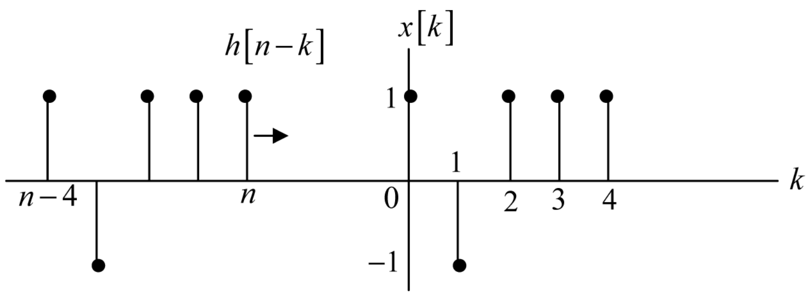
YES, EIGENFUNCTION.

2.34

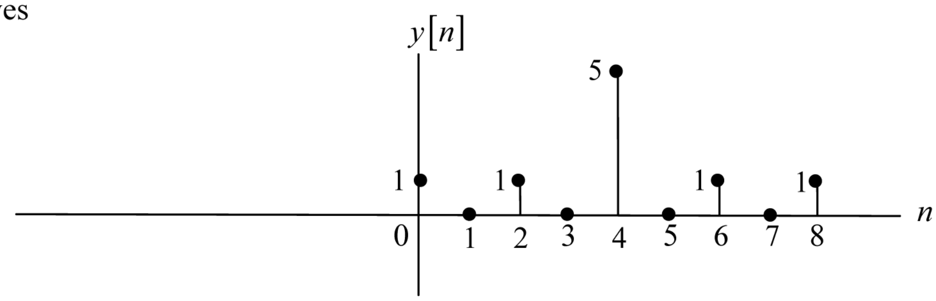
A. Note 1: The sequence $h[n]$ is one of the so-called "Barker codes."

Note 2: The impulse response of the filter satisfies $h[n] = x[4-n]$, that is, $h[n]$ is the "matched filter" for $x[n]$.

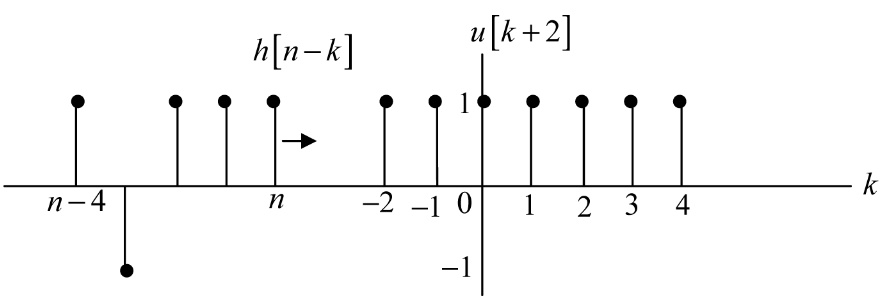
Using "flipping and shifting,"



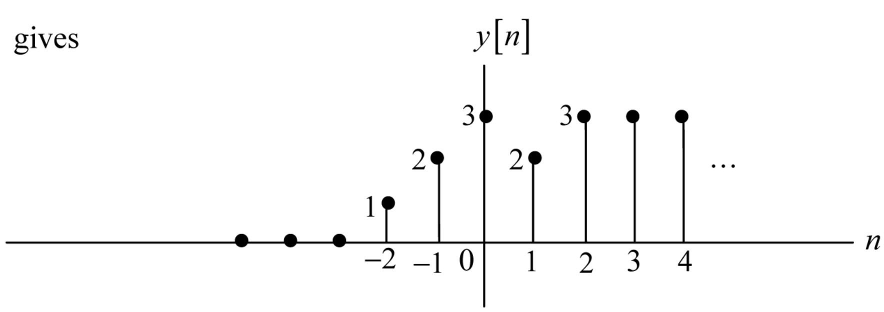
gives



B. Using "flipping and shifting,"



gives



2.35. We first re-write the system function $H(e^{j\omega})$:

$$\begin{aligned} H(e^{j\omega}) &= e^{j\pi/4} \cdot e^{-j\omega} \left(\frac{1 + e^{-j2\omega} + 4e^{-j4\omega}}{1 + \frac{1}{2}e^{-j2\omega}} \right) \\ &= e^{j\pi/4} G(e^{j\omega}) \end{aligned}$$

Let $y_1[n] = x[n] * g[n]$, then

$$\begin{aligned} x[n] &= \cos(\frac{\pi n}{2}) = \frac{e^{j\pi n/2} + e^{-j\pi n/2}}{2} \\ y_1[n] &= \frac{G(e^{j\pi/2})e^{j\pi n/2} + G(e^{-j\pi/2})e^{-j\pi n/2}}{2} \end{aligned}$$

Evaluating the frequency response at $\omega = \pm\pi/2$:

$$\begin{aligned} G(e^{j\frac{\pi}{2}}) &= e^{-j\frac{\pi}{2}} \left(\frac{1 + e^{-j\pi} + 4e^{-j2\pi}}{1 + \frac{1}{2}e^{-j\pi}} \right) = 8e^{-j\pi/2} \\ G(e^{-j\frac{\pi}{2}}) &= 8e^{j\pi/2} \end{aligned}$$

Therefore,

$$y_1[n] = (8e^{j(\pi n/2 - \pi/2)} + 8e^{j(-\pi n/2 + \pi/2)})/2 = 8\cos(\frac{\pi}{2}n - \frac{\pi}{2})$$

and

$$y[n] = e^{j\pi/4} y_1[n] = 8e^{j\pi/4} \cos(\frac{\pi}{2}n - \frac{\pi}{2})$$

2.36

A.

$$H\left(e^{j\omega}\right)=\frac{1}{1-0.8e^{-j\omega}}+\frac{e^{-j2\omega}}{1-0.8e^{-j\omega}}$$

$$h[n]=(0.8)^n u[n]+(0.8)^{n-2} u[n-2]$$

B.

$$H\left(e^{j\omega}\right)=\frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}=\frac{1+e^{-j2\omega}}{1-0.8e^{-j\omega}}$$

$$Y\left(e^{j\omega}\right)-0.8e^{-j\omega}Y\left(e^{j\omega}\right)=X\left(e^{j\omega}\right)+e^{-j2\omega}X\left(e^{j\omega}\right)$$

$$y[n]-0.8y[n-1]=x[n]+x[n-2]$$

$$y[n]=0.8y[n-1]+x[n]+x[n-2]$$

C. Using the frequency response we can write the output as

$$y[n]=H\left(e^{j0}\right)4+2\left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0 n+\angle H\left(e^{j\omega_0}\right)\right).$$

To get $y[n]=$ constant we need $\left|H\left(e^{j\omega_0}\right)\right|=0$, which means $1+e^{-j2\omega_0}=0$, or $\omega_0=\pi/2$.

Then $y[n]=4\frac{1+1}{1-0.8}=40$.

2.37

A. The impulse response $h_2[n] = 2\dfrac{\sin(0.5\pi n)}{\pi n}$ corresponds to a frequency response of

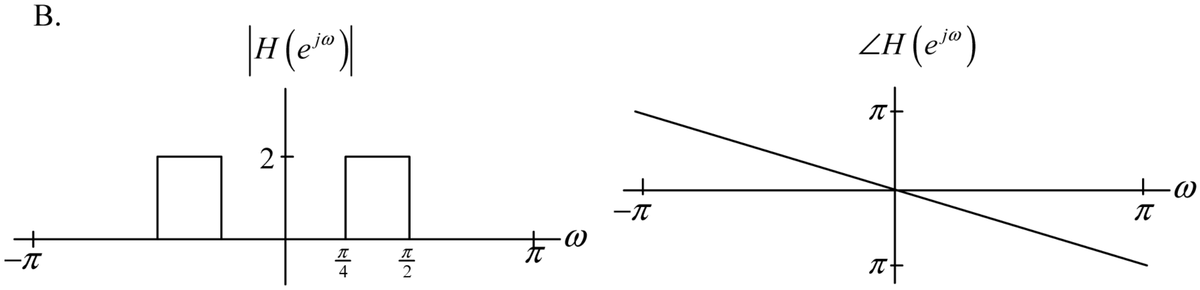
$$H_2(e^{j\omega}) = \begin{cases} 2, & \omega < \pi/2 \\ 0, & \pi/2 < \omega \le \pi \end{cases}$$

Then

$$H(e^{j\omega}) = H_1(e^{j\omega})H_2(e^{j\omega})$$

$$= e^{-j\omega} \times \begin{cases} 0, & |\omega| < \pi/4 \\ 2, & \pi/4 < |\omega| < \pi/2 \\ 0, & \pi/2 < |\omega| \le \pi. \end{cases}$$

B.



C. Method 1 (Easiest):
The overall cascade system can be viewed as the difference of two lowpass filters with a one-sample delay.

$$h[n] = 2\frac{\sin\left(\frac{\pi}{2}(n-1)\right)}{\pi(n-1)} - 2\frac{\sin\left(\frac{\pi}{4}(n-1)\right)}{\pi(n-1)}$$

Method 2 (Harder):
The overall cascade system can be viewed as having a lowpass response modulated up to frequency $3\pi/8$.

$$h[n] = 4\frac{\sin\left(\frac{\pi}{8}(n-1)\right)}{\pi(n-1)}\cos\left(\tfrac{3\pi}{8}(n-1)\right)$$

Method 3 (Direct):
Just evaluate

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega$$

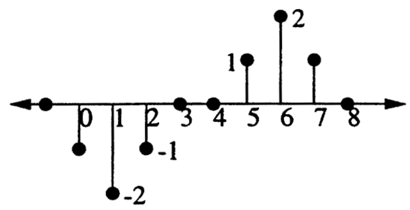
2.38. (a) Notice that

$$x[n] = x_0[n-2] + 2x_0[n-4] + x_0[n-6]$$

Since the system is LTI,

$$y[n] = y_0[n-2] + 2y_0[n-4] + y_0[n-6],$$

and we get sequence shown here:



(b) Since

$$y_0[n] = -1x_0[n+1] + x_0[n-1] = x_0[n] * (-\delta[n+1] + \delta[n-1]),$$
$$h[n] = -\delta[n+1] + \delta[n-1]$$

2.39. The ideal delay system:

$$y[n] = T\{x[n]\} = x[n - n_o]$$

Using the definition of linearity:

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= ax_1[n - n_o] + bx_2[n - n_o] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

So, the ideal delay system is LINEAR.

The moving average system:

$$y[n] = Tx[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

by linearity:

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} (ax_1[n] + bx_2[n]) \\ &= \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} ax_1[n] + \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} bx_2[n] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

Conclude, the moving average is LINEAR.

2.40. $x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

(a) $x[n]$ is periodic with period 5:

$$e^{j(\frac{2\pi}{5}n)} = e^{j(\frac{2\pi}{5})(n+N)} = e^{j(\frac{2\pi}{5}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{2\pi}{5}N, \text{for integers } k, N$$

Making $k = 1$ and $N = 5$ shows that $x[n]$ has period 5.

(b) $x[n]$ is periodic with period 38. Since the sin function has period of 2π:

$$x[n+38] = \sin(\pi(n+38)/19) = \sin(\pi n/19 + 2\pi) = x[n]$$

(c) This is not periodic because the linear term n is not periodic.

(d) This is again not periodic. $e^{j\omega}$ is periodic over period 2π, so we have to find k, N such that

$$x[n+N] = e^{j(n+N)} = e^{j(n+2\pi k)}$$

Since we can make k and N integers at the same time, $x[n]$ is not periodic.

2.41. Since $H(e^{-j\omega}) = H^*(e^{j\omega})$, we can apply the results of Example 2.13 from the text,

$$y[n] = |H(e^{j\frac{3\pi}{2}})|\cos(\frac{3\pi}{2}n + \frac{\pi}{4} + \angle H(e^{j\frac{3\pi}{2}}))$$

To find $H(e^{j\frac{3\pi}{2}})$, we use the fact that $H(e^{j\omega})$ is periodic over 2π, so

$$H(e^{j\frac{3\pi}{2}}) = H(e^{-j\frac{\pi}{2}}) = e^{j\frac{2\pi}{3}}$$

Therefore,

$$y[n] = \cos(\frac{3\pi}{2}n + \frac{\pi}{4} + \frac{2\pi}{3}) = \cos(\frac{3\pi}{2}n + \frac{11\pi}{12})$$

2.42

The autocorrelation function of $s[n]$ is

$$\phi_{ss}[m] = E\{s[n]s[n+m]\}.$$

Substituting $s[n] = x[n]w[n]$ gives

$$\phi_{ss}[m] = E\{x[n]w[n]x[n+m]w[n+m]\}.$$

Since $x[n]$ and $w[n]$ are statistically independent we have

$$\begin{aligned}\phi_{ss}[m] &= E\{x[n]x[n+m]\}E\{w[n]w[n+m]\} \\ &= E\{x[n]x[n+m]\}\sigma_w^2\delta[m].\end{aligned}$$

Then $\phi_{ss}[m] = 0$ for $m \neq 0$, so $s[n]$ is white.

We can find the mean of $s[n]$ by

$$E\{s[n]\} = E\{x[n]w[n]\} = E\{x[n]\}E\{w[n]\} = 0,$$

where we have used the statistical independence of $x[n]$ and $w[n]$ together with the fact that $x[n]$ and $w[n]$ each have zero mean. The variance of $s[n]$ is then given by

$$\begin{aligned}\sigma_s^2 &= E\{s^2[n]\} - E^2\{s[n]\} \\ &= E\{s^2[n]\} \\ &= \phi_{ss}[0] \\ &= E\{x^2[n]\}\sigma_w^2.\end{aligned}$$

Since $x[n]$ has zero mean, $E\{x^2[n]\} = \sigma_x^2$. Then $\sigma_s^2 = \sigma_x^2\sigma_w^2$ as was to have been shown.

2.43

1. The Fourier transform of $x[n]$ is given by $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. Then

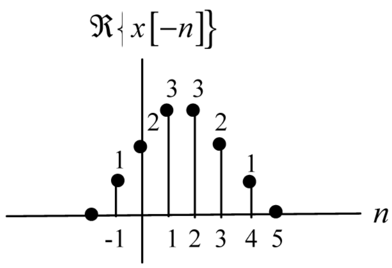
$$X(e^{j\omega})\Big|_{\omega=0} = \sum_{n=-\infty}^{\infty} x[n] = 12.$$

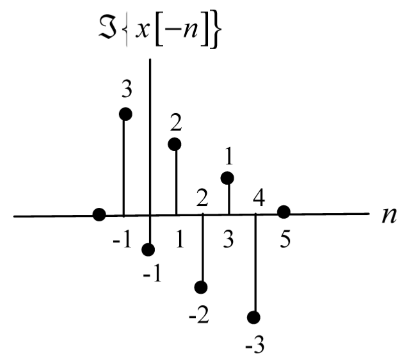
2. $X(e^{j\omega})\Big|_{\omega=\pi} = \sum_{n=-\infty} x[n]e^{-j\pi n} = \sum_{n=-\infty} (-1)^n x[n] = -j12$.

3. The inverse Fourier transform is given by $\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = x[n]$. Then

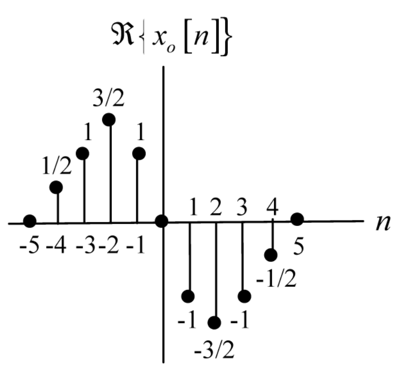
$$\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega = 2\pi x[n]$$
$$\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega 0}d\omega = 2\pi x[0]$$
$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi(2-j).$$

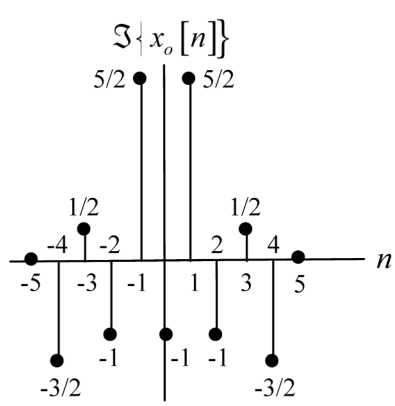
4. If $x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})$, then $x[-n] \overset{F}{\longleftrightarrow} X(e^{-j\omega})$.





5. If $x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})$, then $x_o[n] \overset{F}{\longleftrightarrow} j\Im\{X(e^{j\omega})\}$, where $x_o[n] = \frac{1}{2}(x[n] - x^*[-n])$.





2.44. First $x[n]$ goes through a lowpass filter with cutoff frequency 0.5π. Since the cosine has a frequency of 0.6π, it will be filtered out. The delayed impulse will be filtered to a delayed sinc and the constant will remain unchanged. We thus get:

$$w[n] = 3\frac{sin(0.5\pi(n-5))}{\pi(n-5)} + 2.$$

$y[n]$ is then given by:

$$y[n] = 3\frac{sin(0.5\pi(n-5))}{\pi(n-5)} - 3\frac{sin(0.5\pi(n-6))}{\pi(n-6)}.$$

2.45

$$x[n] = -b^n u[-n-1] = \begin{cases} -b^n, & n \le -1 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{-1} -b^n e^{-j\omega n}.$$

Let $k = -n$. Then

$$X\left(e^{j\omega}\right) = \sum_{k=1}^{\infty} -b^{-k} e^{j\omega k}$$
$$= -\left\{ \left(\sum_{k=0}^{\infty} b^{-k} e^{j\omega k} \right) - 1 \right\}$$
$$= 1 - \sum_{k=0}^{\infty} \left(b^{-1} e^{j\omega} \right)^k$$
$$= 1 - \frac{1}{1 - \frac{e^{j\omega}}{b}},$$

where the last step is true only for $\left|b^{-1} e^{j\omega}\right| < 1$, or $\left|b^{-1}\right| < 1$, or $|b| > 1$. Now we have

$$X\left(e^{j\omega}\right) = \frac{1 - \frac{e^{-j\omega}}{b} - 1}{1 - \frac{e^{-j\omega}}{b}}$$
$$= \frac{-be^{-j\omega}\left(-\frac{1}{b} e^{j\omega} \right)}{-be^{-j\omega}\left(1 - \frac{1}{b} e^{j\omega} \right)}$$
$$X\left(e^{j\omega}\right) = \frac{1}{1 - be^{-j\omega}}$$

only when $|b| > 1$.

Now suppose

$$Y\left(e^{j\omega}\right) = \frac{2e^{-j\omega}}{1 + 2e^{-j\omega}} = 2 \frac{1}{1 - (-2)e^{-j\omega}} e^{-j\omega}$$

Using the above transform pair and then shifting to the right by one,

$$y[n] = 2\left[-(-2)^{n-1} u[-(n-1)-1] \right] = -2(-2)^{n-1} u[-n]$$
$$= (-2)^n u[-n].$$

2.46

$$x[n] = w[n]\cos(\omega_0 n)$$

A. Fourier transforming gives

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{2\pi} W(e^{j\omega}) * \{\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\} \\ &= \frac{1}{2\pi}\left\{\pi W\left(e^{j(\omega-\omega_0)}\right) + \pi W\left(e^{j(\omega+\omega_0)}\right)\right\} \\ &= \tfrac{1}{2} W\left(e^{j(\omega-\omega_0)}\right) + \tfrac{1}{2} W\left(e^{j(\omega+\omega_0)}\right), \end{aligned}$$

for $-\pi < \omega \le \pi$.

B. We know from tables that if

$$y[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise,} \end{cases}$$

then the DTFT $Y(e^{j\omega})$ is

$$Y(e^{j\omega}) = \frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)} e^{-j\omega M/2}$$

Let $M = 2L$. Then we have

$$y[n] = \begin{cases} 1, & 0 \le n \le 2L \\ 0, & \text{otherwise,} \end{cases}$$

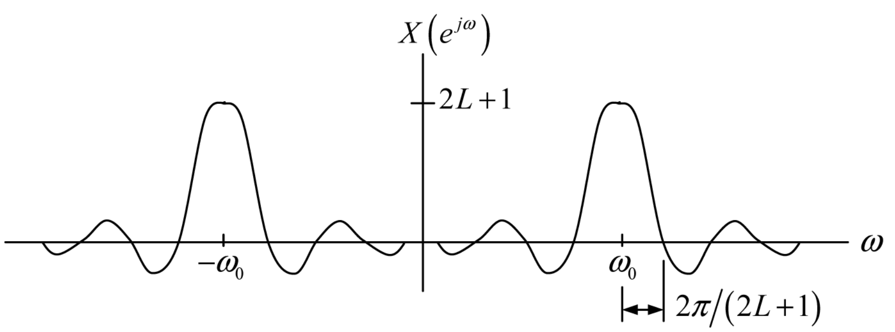
with DTFT

$$Y(e^{j\omega}) = \frac{\sin(\omega(2L+1)/2)}{\sin(\omega/2)} e^{-j\omega L}$$

Now $w[n] = y[n+L]$, which implies $W(e^{j\omega}) = Y(e^{j\omega})e^{j\omega L}$. That is,

$$W(e^{j\omega}) = \frac{\sin(\omega(2L+1)/2)}{\sin(\omega/2)}$$

C. $X(e^{j\omega}) = \tfrac{1}{2} W\left(e^{j(\omega-\omega_0)}\right) + \tfrac{1}{2} W\left(e^{j(\omega+\omega_0)}\right)$



As ω_0 gets closer to $\omega = 0$, the two peaks merge into a single peak. We will have two distinct peaks if $\omega_0 \ge \dfrac{2\pi}{2L+1}$.

2.47. (a) Notice that $x_1[n] = x_2[n] + x_3[n+4]$, so if $T\{\cdot\}$ is linear,

$$\begin{aligned} T\{x_1[n]\} &= T\{x_2[n]\} + T\{x_3[n+4]\} \\ &= y_2[n] + y_3[n+4] \end{aligned}$$

From Fig P2.4, the above equality is not true. Hence, the system is NOT LINEAR.
(b) To find the impulse response of the system, we note that

$$\delta[n] = x_3[n+4]$$

Therefore,

$$\begin{aligned} T\{\delta[n]\} &= y_3[n+4] \\ &= 3\delta[n+6] + 2\delta[n+5] \end{aligned}$$

(c) Since the system is known to be time-invariant and not linear, we cannot use choices such as:

$$\delta[n] = x_1[n] - x_2[n]$$

and

$$\delta[n] = \frac{1}{2}x_2[n+1]$$

to determine the impulse response. With the given information, we can only use shifted inputs.

2.48. (a) Suppose we form the impulse:

$$\delta[n] = \frac{1}{2}x_1[n] - \frac{1}{2}x_2[n] + x_3[n]$$

Since the system is linear,

$$L\{\delta[n]\} = \frac{1}{2}y_1[n] - \frac{1}{2}y_2[n] + y_3[n]$$

A shifted impulse results when:

$$\delta[n-1] = -\frac{1}{2}x_1[n] + \frac{1}{2}x_2[n]$$

The response to the shifted impulse

$$L\{\delta[n-1]\} = -\frac{1}{2}y_1[n] + \frac{1}{2}y_2[n]$$

Since,

$$L\{\delta[n]\} \neq L\{\delta[n-1]\}$$

The system is NOT TIME INVARIANT.

(b) An impulse may be formed:

$$\delta[n] = \frac{1}{2}x_1[n] - \frac{1}{2}x_2[n] + x_3[n]$$

since the system is linear,

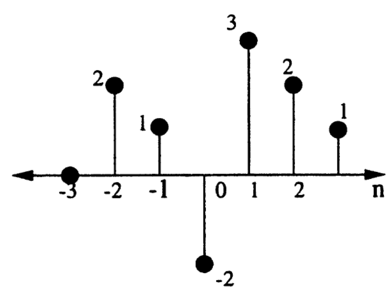
$$\begin{aligned} L\{\delta[n]\} &= \frac{1}{2}y_1[n] - \frac{1}{2}y_2[n] + y_3[n] \\ &= h[n] \end{aligned}$$

from the figure,

$$y_1[n] = -\delta[n+1] + 3\delta[n] + 3\delta[n-1] + \delta[n-3]$$
$$y_2[n] = -\delta[n+1] + \delta[n] - 3\delta[n-1] - \delta[n-3]$$
$$y_3[n] = 2\delta[n+2] + \delta[n+1] - 3\delta[n] + 2\delta[n-2]$$

Combining:

$$\begin{aligned} h[n] &= 2\delta[n+2] + \delta[n+1] - 2\delta[n] + 3\delta[n-1] \\ &\quad + 2\delta[n-2] + \delta[n-3] \end{aligned}$$



2.49. (a) The homogeneous solution to the second order difference equation,

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1],$$

is obtained by setting the input (forcing term) to zero.

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 0$$

Solving,

$$1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} = 0,$$

$$(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1}) = 0,$$

and the homogeneous solution takes the form

$$y_h[n] = A_1(\frac{1}{2})^n + A_2(\frac{1}{4})^n,$$

for the constants A_1 and A_2.

(b) Substituting the intial conditions,

$$y_h[-1] = A_1(\frac{1}{2})^{-1} + A_2(\frac{1}{4})^{-1} = 1,$$

and

$$y_h[0] = A_1 + A_2 = 0.$$

We have

$$2A_1 + 4A_2 = 1$$
$$A_1 + A_2 = 0$$

Solving,

$$A_1 = -1/2$$

and

$$A_2 = 1/2.$$

(c) Homogeneous equation:

$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = 0$$

Solving,

$$1 - z^{-1} + \frac{1}{4}z^{-2} = 0,$$

$$(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z^{-1}) = 0,$$

and the homogeneous solution takes the form

$$y_h[n] = A_1(\frac{1}{2})^n.$$

Invoking the intial conditions, we have

$$y_h[-1] = 2A_1 = 1$$
$$y_h[0] = A_1 = 0$$

Evident from the above contradiction, the initial conditions cannot be met.

(d) The homogeneous difference equation:

$$y[n] - y[n-1] + \frac{1}{4}y[n-2] = 0$$

Suppose the homogeneous solution is of the form

$$y_h[n] = A_1(\frac{1}{2})^n + nB_1(\frac{1}{2})^n,$$

substituting into the difference equation:

$$y_h[n] - y_h[n-1] + \frac{1}{4}y_h[n-2] = 0$$

$$\begin{aligned} A_1(\frac{1}{2})^n + nB_1(\frac{1}{2})^n - A_1(\frac{1}{2})^{n-1} - (n-1)B_1(\frac{1}{2})^{n-1} \\ \frac{1}{4}A_1(\frac{1}{2})^{n-2} + \frac{1}{4}(n-2)B_1(\frac{1}{2})^{n-2} &= 0. \end{aligned}$$

(e) Using the solution from part (d):

$$y_h[n] = A_1(\frac{1}{2})^n + nB_1(\frac{1}{2})^n,$$

and the initial conditions

$$y_h[-1] = 1$$

and

$$y_h[0] = 0,$$

we solve for A_1 and B_1:

$$A_1 = 0$$
$$B_1 = -1/2.$$

2.50. (a) For $x_1[n] = \delta[n]$,

$$\begin{aligned} y_1[0] &= 1 \\ y_1[1] &= ay[0] = a \end{aligned}$$

For $x_2[n] = \delta[n-1]$,

$$\begin{aligned} y_2[0] &= 1 \\ y_2[1] &= ay[0] + x_2[1] = a + 1 \neq y_1[0] \end{aligned}$$

Even though $x_2[n] = x_1[n-1]$, $y_2[n] \neq y_2[n-1]$. Hence the system is NOT TIME INVARIANT.

(b) A linear system has the property that

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}\}$$

Hence, if the input is doubled, the output must also double at each value of n.
Because $y[0] = 1$, always, the system is NOT LINEAR.

(c) Let $x_3 = \alpha x_1[n] + \beta x_2[n]$.
For $n \geq 0$:

$$\begin{aligned} y_3[n] &= x_3[n] + ay_3[n-1] \\ &= \alpha x_1[n] + \beta x_2[n] + a(x_3[n-1] + y_3[n-2]) \\ &= \alpha \sum_{k=0}^{n-1} a^k x_1[n-k] + \beta \sum_{k=0}^{n-1} a^k x_2[n-k] \\ &= \alpha(h[n] * x_1[n]) + \beta(h[n] * x_2[n]) \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

For $n < 0$:

$$\begin{aligned} y_3[n] &= a^{-1}(y_3[n+1] - x_3[n]) \\ &= -\alpha \sum_{k=-1}^{n} a^k x_1[n-k] - \beta \sum_{k=-1}^{n} a^k x_2[n-k] \\ &= \alpha y_1[n] + \beta y_2[n]. \end{aligned}$$

For $n = 0$:

$$y_3[n] = y_1[n] = y_2[n] = 0.$$

Conclude,

$$y_3[n] = \alpha y_1[n] + \beta y_2[n], \text{ for all } n.$$

Therefore, the system is LINEAR. The system is still NOT TIME INVARIANT.

2.51. For $(-1 < a < 0)$, we have

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

(a) real part of $X(e^{j\omega})$:

$$\begin{aligned} X_R(e^{j\omega}) &= \frac{1}{2} \cdot [X(e^{j\omega}) + X^*(e^{j\omega})] \\ &= \frac{1 - a\cos(\omega)}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

(b) imaginary part:

$$\begin{aligned} X_I(e^{j\omega}) &= \frac{1}{2j} \cdot [X(e^{j\omega}) - X^*(e^{j\omega})] \\ &= \frac{-a\sin(\omega)}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

(c) magnitude:

$$\begin{aligned} |X(e^{j\omega})| &= [X(e^{j\omega})X^*(e^{j\omega})]^{\frac{1}{2}} \\ &= \left(\frac{1}{1 - 2a\cos(\omega) + a^2}\right)^{\frac{1}{2}} \end{aligned}$$

(d) phase:

$$\angle X(e^{j\omega}) = \arctan\left(\frac{-a\sin(\omega)}{1 - a\cos(\omega)}\right)$$

2.52. For the input

$$\begin{aligned} x[n] &= \cos(\pi n)u[n] \\ &= (-1)^n u[n], \end{aligned}$$

the output is

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} (j/2)^k u[k](-1)^{(n-k)} u[n-k] \\ &= (-1)^n \sum_{k=0}^{n} (j/2)^k (-1)^{-k} \\ &= (-1)^n \sum_{k=0}^{n} (-j/2)^k \\ &= (-1)^n \left(\frac{1-(-j/2)^{(n+1)}}{1+j/2} \right) \end{aligned}$$

For large n, $(-j/2)^{(n+1)} \to 0$. Thus, the steady-state response becomes

$$\begin{aligned} y[n] &= \frac{(-1)^n}{1+j/2} \\ &= \frac{\cos(\pi n)}{1+j/2}. \end{aligned}$$

2.53. The input sequence,

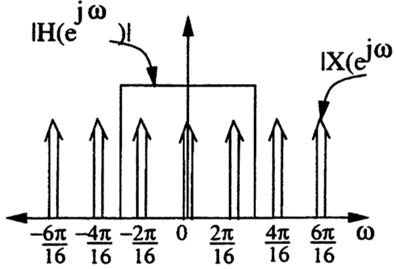
$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n+16k],$$

has the Fourier representation

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \delta[n+16k]e^{-j\omega n} \\ &= \frac{1}{16}\sum_{k=-\infty}^{\infty} \delta(\omega + \frac{2\pi k}{16}). \end{aligned}$$

Therefore, the frequency representation of the input is also a periodic impulse train. There are *l*6 frequency impulses in the range $-\pi \leq \omega \leq \pi$.

We sketch the magnitudes of $X(e^{j\omega})$ and $H(e^{j\omega})$:



From the sketch, we observe that the LTI system is a lowpass filter which removes all but three of the frequency impulses. To these, it multiplies a phase factor $e^{-j3\omega}$.

The Fourier transform of the output is

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{16}\delta(\omega) + \frac{1}{16}e^{-j\frac{6\pi}{16}}\delta(\omega - \frac{2\pi}{16}) \\ &\quad + \frac{1}{16}e^{j\frac{6\pi}{16}}\delta(\omega + \frac{2\pi}{16}) \end{aligned}$$

Thus the output sequence is

$$y[n] = \frac{1}{16} + \frac{1}{8}\cos(\frac{2\pi n}{16} + \frac{3\pi}{8}).$$

2.54. (a) From the figure,

$$\begin{aligned} y[n] &= (x[n] + x[n] * h_1[n]) * h_2[n] \\ &= (x[n] * (\delta[n] + h_1[n])) * h_2[n]. \end{aligned}$$

Let $h[n]$ be the impulse response of the overall system,

$$y[n] = x[n] * h[n].$$

Comparing with the above expression,

$$\begin{aligned} h[n] &= (\delta[n] + h_1[n]) * h_2[n] \\ &= h_2[n] + h_1[n] * h_2[n] \\ &= \alpha^n u[n] + \beta^{(n-1)} u[n-1]. \end{aligned}$$

(b) Taking the Fourier transform of $h[n]$ from part (a),

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\omega n} + \beta \sum_{n=-\infty}^{\infty} \alpha^{(n-1)} u[n-1] e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} + \beta \sum_{\ell=0}^{\infty} \alpha^{(\ell-1)} e^{-j\omega \ell}, \end{aligned}$$

where we have used $\ell = (n-1)$ in the second sum.

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{1-\alpha e^{-j\omega}} + \frac{\beta e^{-j\omega}}{1-\alpha e^{-j\omega}} \\ &= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \text{ for } |\alpha| < 1. \end{aligned}$$

Note that the Fourier transform of $\alpha^n u[n]$ is well known, and the second term of $h[n]$ (see part (a)) is just a scaled and shifted version of $\alpha^n u[n]$. So, we could have used the properties of the Fourier transform to reduce the algebra.

(c) We have

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}, \end{aligned}$$

cross multiplying,

$$Y(e^{j\omega})[1-\alpha e^{-j\omega}] = X(e^{j\omega})[1+\beta e^{-j\omega}]$$

taking the inverse Fourier transform, we have

$$y[n] - \alpha y[n-1] = x[n] + \beta x[n-1].$$

(d) From part (a):

$$h[n] = 0, \text{ for } n < 0.$$

This implies that the system is CAUSAL.

If the system is stable, its Fourier transform exists. Therefore, the condition for stability is the same as the condition imposed on the frequency response of part (b). That is, STABLE, if $|\alpha| < 1$.

2.55. (a)

$$\begin{aligned} X(e^{j\omega})|_{\omega=0} &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}|_{\omega=0} \\ &= \sum_{n=-\infty}^{\infty} x[n] \\ &= 6 \end{aligned}$$

(b)

$$\begin{aligned} X(e^{j\omega})|_{\omega=\pi} &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n} \\ &= \sum_{n=-\infty}^{\infty} x[n](-1)^n \\ &= 2 \end{aligned}$$

(c) Because $x[n]$ is symmetric about $n = 2$ this signal has linear phase.

$$X(e^{j\omega}) = A(\omega)e^{-j2\omega}$$

$A(\omega)$ is a zero phase (real) function of ω. Hence,

$$\angle X(e^{j\omega}) = -2\omega, \qquad -\pi \leq \omega \leq \pi$$

(d)

$$\int_{-\pi}^{\pi} X(e^{j\omega})e^{-j\omega n}d\omega = 2\pi x[n]$$

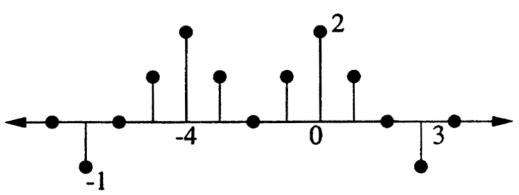
for $n = 0$:

$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0] = 4\pi$$

(e) Let $y[n]$ be the unknown sequence. Then

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{-j\omega}) \\ &= \sum_n x[n]e^{j\omega n} \\ &= \sum_n x[-n]e^{-j\omega n} \\ &= \sum_n y[n]e^{-j\omega n} \end{aligned}$$

Hence $y[n] = x[-n]$.



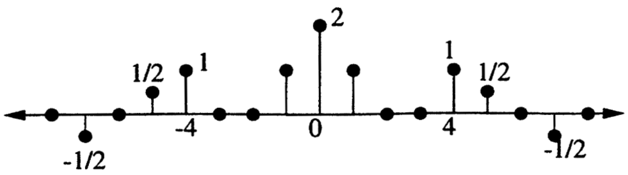
(f) We have determined that:

$$X(e^{j\omega}) = A(\omega)e^{-j2\omega}$$

$$\begin{aligned} X_R(e^{j\omega}) &= \mathcal{R}e\{X(e^{j\omega})\} \\ &= A(\omega)\cos(2\omega) \\ &= \frac{1}{2}A(\omega)\left(e^{j2\omega} + e^{-j2\omega}\right) \end{aligned}$$

Taking the inverse transform, we have

$$\frac{1}{2}a[n+2] + \frac{1}{2}a[n-2] = \frac{1}{2}x[n+4] + \frac{1}{2}x[n]$$



2.56. Let $x[n] = \delta[n]$, then

$$X(e^{j\omega}) = 1$$

The output of the ideal lowpass filter:

$$W(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = H(e^{j\omega})$$

The multiplier:

$$(-1)^n w[n] = e^{-j\pi n} w[n]$$

causes a shift in the frequency domain:

$$W(e^{j(\omega-\pi)}) = H(e^{j(\omega-\pi)})$$

The overall output:

$$y[n] = e^{-j\pi n} w[n] + w[n]$$

$$Y(e^{j\omega}) = H(e^{j(\omega-\pi)}) + H(e^{j\omega})$$

Noting that:

$$H(e^{j(\omega-\pi)}) = \begin{cases} 1, & \frac{\pi}{2} \leq |\omega| \leq \pi \\ 0, & |\omega| < \frac{\pi}{2} \end{cases}$$

$Y(e^{j\omega}) = 1$, thus $y[n] = \delta[n]$.

2.57. (a) We first perform a partial-fraction expansion of $X(e^{j\omega})$:

$$
\begin{aligned}
X(e^{j\omega}) &= \frac{1-a^2}{(1-ae^{-j\omega})(1-ae^{j\omega})} \\
&= \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}} \\
x[n] &= a^n u[n] + a^{-n} u[-n-1] \\
&= a^{|n|}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\cos(\omega)d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\frac{e^{j\omega}+e^{-j\omega}}{2}d\omega \\
&= \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega}d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{-j\omega}d\omega\right) \\
&= \frac{1}{2}(x[n-1]+x[n+1]) \\
&= \frac{1}{2}(a^{|n-1|}+a^{|n+1|})
\end{aligned}
$$

2.58. (a)

$$\begin{aligned} y[n] &= x[n] + 2x[n-1] + x[n-2] \\ &= x[n] * h[n] \\ &= x[n] * (\delta[n] + 2\delta[n-1] + \delta[n-2]) \\ h[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \end{aligned}$$

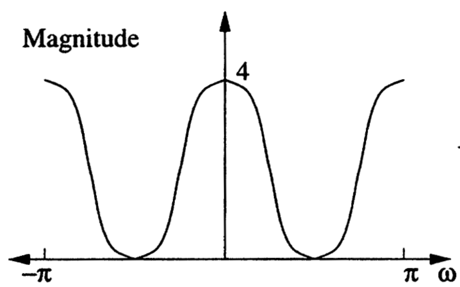
(b) Yes. $h[n]$ is finite-length and absolutely summable.

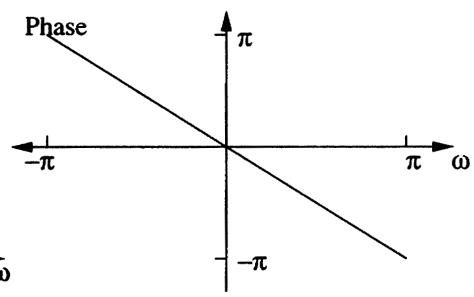
(c)

$$\begin{aligned} H(e^{j\omega}) &= 1 + 2e^{-j\omega} + e^{-2j\omega} \\ &= 2e^{-j\omega}(\frac{1}{2}e^{j\omega} + 1 + \frac{1}{2}e^{-j\omega}) \\ &= 2e^{-j\omega}(\cos(\omega) + 1) \end{aligned}$$

(d)

$$\begin{aligned} |H(e^{j\omega})| &= 2(\cos(\omega) + 1) \\ \angle H(e^{j\omega}) &= -\omega \end{aligned}$$





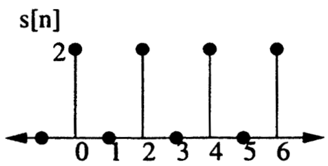
(e)

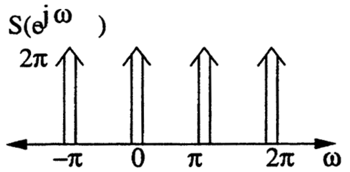
$$\begin{aligned} h_1[n] &= \frac{1}{2\pi}\int_{<2\pi>} H_1(e^{j\omega})e^{j\omega}d\omega \\ &= \frac{1}{2\pi}\int_{<2\pi>} H(e^{j(\omega+\pi)})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{<2\pi>} H(e^{j(\omega)})e^{j(\omega-\pi)n}d\omega \\ &= e^{-j\pi n}\frac{1}{2\pi}\int_{<2\pi>} H(e^{j(\omega)})e^{j\omega n}d\omega \\ &= -1^n h[n] \\ &= \delta[n] - 2\delta[n-1] + \delta[n-2] \end{aligned}$$

2.59. (a) Notice that

$$s[n] = 1 + \cos(\pi n) = 1 + (-1)^n$$

$$S(e^{j\omega}) = 2\pi \sum_k \delta(\omega - k\pi)$$





(b) Since $y[n] = x[n]s[n]$,

$$\begin{aligned} Y(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\theta})X(e^{j(\omega-\theta)})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S(e^{j\theta})X(e^{j(\omega-\theta)})d\omega \\ &= X(e^{j\omega}) + X(e^{j(\omega-\pi)}) \end{aligned}$$

$Y(e^{j\omega})$ contains copies of $X(e^{j\omega})$ replicated at intervals of π.

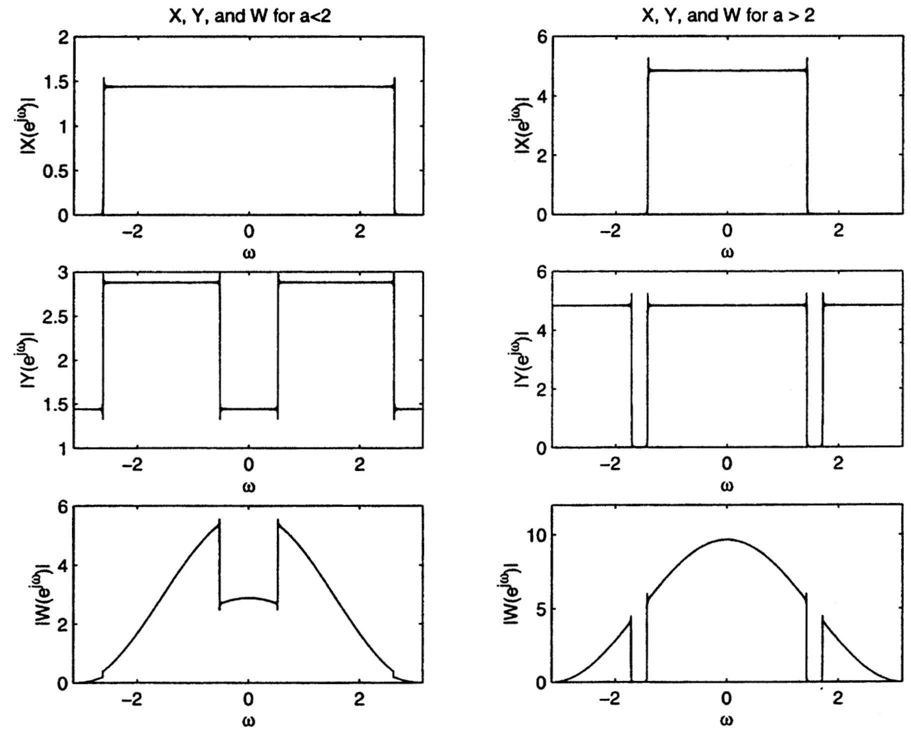
(c) Since $w[n] = y[n] + (1/2)(y[n+1] + y[n-1])$,

$$\begin{aligned} W(e^{j\omega}) &= Y(e^{j\omega}) + \frac{1}{2}\left(e^{j\omega}Y(e^{j\omega}) + e^{-j\omega}Y(e^{j\omega})\right) \\ &= Y(e^{j\omega})(1 + \cos(\omega)) \end{aligned}$$

(d) The following figure shows $X(e^{j\omega})$, $Y(e^{j\omega})$, and $W(e^{j\omega})$ for $a < 2$ and $a > 2$. Notice that

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \pi/a, \\ \cdot 0, & \pi/a \le |\omega| \ge \pi \end{cases}$$

So, for $a > 2$, $Y(e^{j\omega})$ contains two non-overlapping replications of $X(e^{j\omega})$, whereas for $a < 2$, "aliasing" occurs. When there is aliasing, $W(e^{j\omega})$ is not at all close to $X(e^{j\omega})$. Hence, a must be greater than 2 for $w[n]$ to be "close" to $x[n]$.



2.60. (a) We start by interpreting each clue.

(i) The system is causal implies

$$h[n] = 0 \text{ for } n \leq 0.$$

(ii) The Fourier transform is conjugate symmetric implies $h[n]$ is real.

(iii) The DTFT of the sequence $h[n+1]$ is real implies $h[n+1]$ is even.

From the above observations, we deduce that $h[n]$ has length 3, therefore it has finite duration.

(b) From part (a) we know that $h[n]$ is length 3 with even symmetry around $h[1]$. Let $h[0] = h[2] = a$ and $h[1] = b$, from (iv) and using Parseval's theorem, we have

$$2a^2 + b^2 = 2.$$

From (v), we also have

$$2a - b = 0.$$

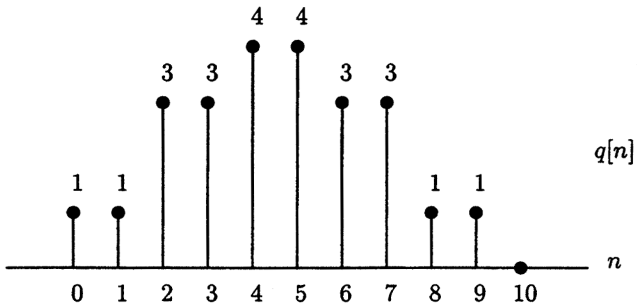
Solving the above equations, we get

$$\begin{aligned} h[0] &= \frac{1}{\sqrt{3}} \\ h[1] &= \frac{2}{\sqrt{3}} \\ h[2] &= \frac{1}{\sqrt{3}} \end{aligned}$$

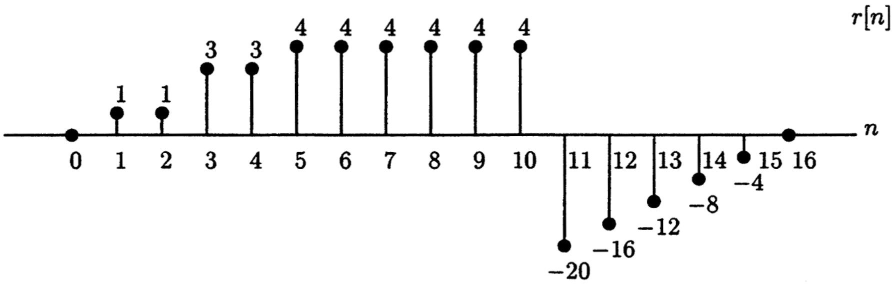
or

$$\begin{aligned} h[0] &= -\frac{1}{\sqrt{3}} \\ h[1] &= -\frac{2}{\sqrt{3}} \\ h[2] &= -\frac{1}{\sqrt{3}}. \end{aligned}$$

2.61. (a) Carrying out the convolution sum, we get the following sequence $q[n]$:



(b) Again carrying out the convolution sum, we get the following sequence $r[n]$:



(c) Let $a[n] = v[-n]$ and $b[n] = w[-n]$, then:

$$
\begin{aligned}
a[n] * b[n] &= \sum_{k=-\infty}^{+\infty} a[k]b[n-k] \\
&= \sum_{k=-\infty}^{+\infty} v[-k]w[k-n] \\
&= \sum_{r=-\infty}^{+\infty} v[r]w[-n-r] \text{ where } r = -k \\
&= q[-n].
\end{aligned}
$$

We thus conclude that $q[-n] = v[-n] * w[-n]$.

2.62.

$$\begin{aligned}
x[n] &= \cos(\frac{15\pi n}{4} - \frac{\pi}{3}) \\
&= \cos(-\frac{\pi n}{4} - \frac{\pi}{3}) \\
&= \cos(\frac{\pi n}{4} + \frac{\pi}{3}) \\
&= \frac{e^{j\frac{\pi}{3}} e^{j\frac{\pi n}{4}}}{2} + \frac{e^{-j\frac{\pi}{3}} e^{-j\frac{\pi n}{4}}}{2}.
\end{aligned}$$

Using the fact that complex exponentials are eigenfunctions of LTI systems, we get:

$$\begin{aligned}
y[n] &= e^{-j\frac{3\pi}{8}} \frac{e^{j\frac{\pi}{3}} e^{j\frac{\pi n}{4}}}{2} + e^{-j\frac{\pi}{8}} \frac{e^{-j\frac{\pi}{3}} e^{-j\frac{\pi n}{4}}}{2} \\
&= \frac{e^{-j\frac{\pi}{24}} e^{j\frac{\pi n}{4}}}{2} + \frac{e^{-j\frac{11\pi}{24}} e^{-j\frac{\pi n}{4}}}{2} \\
&= e^{-j\frac{\pi}{4}} (\frac{e^{j\frac{5\pi}{24}} e^{j\frac{\pi n}{4}}}{2} + \frac{e^{-j\frac{5\pi}{24}} e^{-j\frac{\pi n}{4}}}{2}) \\
&= e^{-j\frac{\pi}{4}} \cos(\frac{\pi n}{4} + \frac{5\pi}{24}).
\end{aligned}$$

2.63. (a)

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n} x[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k} x[k] h[n-k]. \end{aligned}$$

Let $x[n] = a x_1[n] + b x_2[n]$, then:

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n}(a x_1[n] + b x_2[n])) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k}(a x_1[k] + b x_2[k]) h[n-k] \\ &= a \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k} x_1[k] h[n-k] + b \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 k} x_2[k] h[n-k] \\ &= a y_1[n] + b y_2[n] \end{aligned}$$

where $y_1[n]$ and $y_2[n]$ are the responses to $x_1[n]$ and $x_2[n]$ respectively. We thus conclude that system S is linear.

(b) Let $x_2[n] = x[n - n_0]$, then:

$$\begin{aligned} y_2[n] &= h[n] * (e^{-j\omega_0 n} x_2[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)} x_2[n-k] h[k] \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)} x[n - n_0 - k] h[k] \\ &\neq y[n - n_0]. \end{aligned}$$

We thus conclude that system S is not time invariant.

(c) Since the magnitude of $e^{-j\omega_0 n}$ is always bounded by 1 and $h[n]$ is stable, a bounded input $x[n]$ will always produce a bounded input to the stable LTI system and therefore the output $y[n]$ will be bounded. We thus conclude that system S is stable.

(d) We can rewrite $y[n]$ as:

$$\begin{aligned} y[n] &= h[n] * (e^{-j\omega_0 n} x[n]) \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 (n-k)} x[n-k] h[k] \\ &= \sum_{k=-\infty}^{+\infty} e^{-j\omega_0 n} e^{j\omega_0 k} x[n-k] h[k] \\ &= e^{-j\omega_0 n} \sum_{k=-\infty}^{+\infty} e^{j\omega_0 k} x[n-k] h[k]. \end{aligned}$$

System C should therefore be a multiplication by $e^{-j\omega_0 n}$.

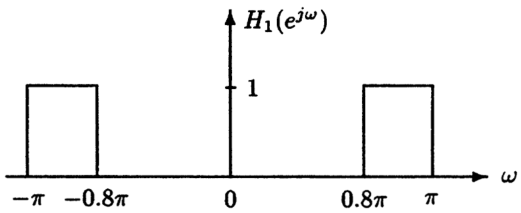
2.64. (a) $H_1(e^{j\omega})$ corresponds to a frequency shifted version of $H(e^{j\omega})$, specifically:

$$H_1(e^{j\omega}) = H(e^{j(\omega-\pi)}).$$

We thus have:

$$H_1(e^{j\omega}) = \begin{cases} 0 & , \quad |\omega| < 0.8\pi \\ 1 & , \quad 0.8\pi \le |\omega| \le \pi. \end{cases}$$

This is a highpass filter.



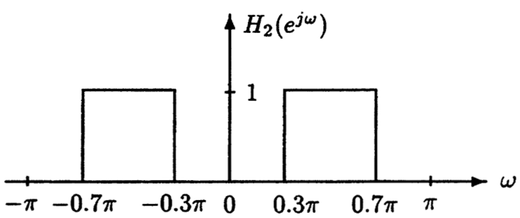
(b) $H_2(e^{j\omega})$ corresponds to a frequency modulated version of $H(e^{j\omega})$, specifically:

$$H_2(e^{j\omega}) = H(e^{j\omega}) * (\delta(\omega - 0.5\pi) + \delta(\omega + 0.5\pi)) \qquad \text{where } |\omega| \le \pi.$$

We thus have:

$$H_2(e^{j\omega}) = \begin{cases} 0 & , \quad |\omega| < 0.3\pi \\ 1 & , \quad 0.3\pi \le |\omega| \le 0.7\pi \\ 0 & , \quad 0.7\pi < |\omega| \le \pi. \end{cases}$$

This is a bandpass filter.



(c) $H_3(e^{j\omega})$ corresponds to a periodic convolution of $H_{lp}(e^{j\omega})$ with another lowpass filter, specifically:

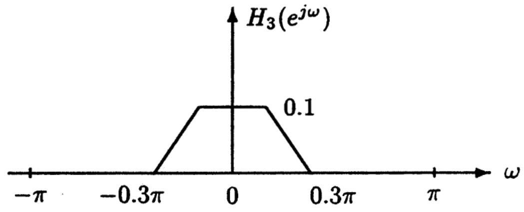
$$H_3(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\theta})H_{lp}(e^{j\omega-\theta})\,d\theta$$

where $H(e^{j\omega})$ is given by:

$$H(e^{j\omega}) = \begin{cases} 1 & , \quad |\omega| < 0.1\pi \\ 0 & , \quad 0.1\pi \le |\omega| \le \pi \end{cases}$$

Carrying out the convolution, we get:

$$H_3(e^{j\omega}) = \begin{cases} 0.1 & , \quad |\omega| < 0.1\pi \\ -\frac{|\omega|}{2\pi} + 0.15 & , \quad 0.1\pi \le |\omega| \le 0.3\pi \\ 0 & , \quad 0.3\pi < |\omega| \le \pi. \end{cases}$$



2.65. Note that $X(e^{j\omega})$ is real, and $Y(e^{j\omega})$ is given by:

$$Y(e^{j\omega}) = \begin{cases} -jX(e^{j\omega}) & , \quad 0 < \omega < \pi \\ +jX(e^{j\omega}) & , \quad -\pi < \omega < 0. \end{cases}$$

$w[n] = x[n] + jy[n]$, therefore:

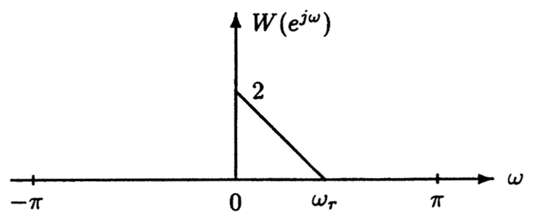
$$W(e^{j\omega}) = X(e^{j\omega}) + jY(e^{j\omega}).$$

Using the above, we get:

$$jY(e^{j\omega}) = \begin{cases} X(e^{j\omega}) & , \quad 0 < \omega < \pi \\ -X(e^{j\omega}) & , \quad -\pi < \omega < 0. \end{cases}$$

We thus conclude:

$$W(e^{j\omega}) = \begin{cases} 2X(e^{j\omega}) & , \quad 0 < \omega < \pi \\ 0 & , \quad -\pi < \omega < 0. \end{cases}$$



2.66. (a) Using the change of variable: $r = -k$, we can rewrite $R_x[n]$ as:

$$R_x[n] = \sum_{r=-\infty}^{\infty} x^*[-r]x[n-r] = x^*[-n] * x[n].$$

We therefore have:

$$g[n] = x^*[-n].$$

(b) The Fourier transform of $x^*[-n]$ is $X^*(e^{j\omega})$, therefore:

$$R_x(e^{j\omega}) = X^*(e^{j\omega})X(e^{j\omega}) = |X(e^{j\omega})|^2.$$

2.67. **(a)** Note that $x_2[n] = -\sum_{k=0}^{k=4} x[n-k]$. Since the system is LTI, we have:

$$y_2[n] = -\sum_{k=0}^{k=4} y[n-k].$$

(b) By carrying out the convolution, we get:

$$h[n] = \begin{cases} -1 & , \quad n = 0, n = 2 \\ -2 & , \quad n = 1 \\ 0 & , \quad \text{o.w.} \end{cases}$$

2.68. The system is not stable, any bounded input that excites the zero input response will result in an unbounded output.

The solution to the difference equation is given by:

$$y[n] = y_{zir}[n] + y_{zsr}[n]$$

where $y_{zir}[n]$ is the zero input response and $y_{zsr}[n]$ is the zero state response, the response to zero initial conditions:

$$\begin{aligned} y_{zir}[n] &= a(\frac{1}{2})^n \qquad \text{where } a \text{ is a constant determined by the initial condition.} \\ y_{zsr}[n] &= (\frac{1}{2})^n u[n] * x[n]. \end{aligned}$$

An example of a bounded input that results in an unbounded output is:

$$x[n] = \delta[n+1].$$

The output is unbounded and given by:

$$y[n] = (\frac{1}{2})^{n+1} u[n+1] - \frac{1}{2}(\frac{1}{2})^n.$$

3.1. (a)

$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n u[n]\right] = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{2z}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

(b)

$$\begin{aligned}\mathcal{Z}\left[-\left(\frac{1}{2}\right)^n u(-n-1)\right] &= -\sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^n z^{-n} = -\sum_{n=1}^{\infty}(2z)^n \\ &= -\frac{2z}{1-2z} = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| < \frac{1}{2}\end{aligned}$$

(c)

$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n u[-n]\right] = \sum_{n=-\infty}^{0}(2z)^n = \frac{1}{1-2z} \qquad |z| < \frac{1}{2}$$

(d)

$$\mathcal{Z}[\delta[n]] = z^0 = 1 \qquad \text{all } z$$

(e)

$$\mathcal{Z}[\delta[n-1]] = z^{-1} \qquad |z| > 0$$

(f)

$$\mathcal{Z}[\delta[n+1]] = z^{+1} \qquad 0 \le |z| < \infty$$

(g)

$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n (u[n]-u[n-10])\right] = \sum_{n=0}^{9}\left(\frac{1}{2z}\right)^n = \frac{1-(2z)^{-10}}{1-(2z)^{-1}} \qquad |z| > 0$$

3.2.

$$x[n] = \begin{cases} n, & 0 \le n \le N-1 \\ N, & N \le n \end{cases} = n\, u[n] - (n-N)u[n-N]$$

$$n\, x[n] \quad \Leftrightarrow \quad -z\frac{d}{dz}X(z) \Rightarrow n\, u[n] \Leftrightarrow -z\frac{d}{dz}\frac{1}{1-z^{-1}} \quad |z| > 1$$

$$n\, u[n] \quad \Leftrightarrow \quad \frac{z^{-1}}{(1-z^{-1})^2} \quad |z| > 1$$

$$x[n-n_0] \quad \Leftrightarrow \quad X(z)\cdot z^{-n_0} \Rightarrow (n-N)u[n-N] \Leftrightarrow \frac{z^{-N-1}}{(1-z^{-1})^2} \quad |z| > 1$$

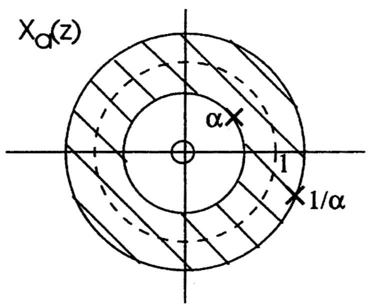
therefore

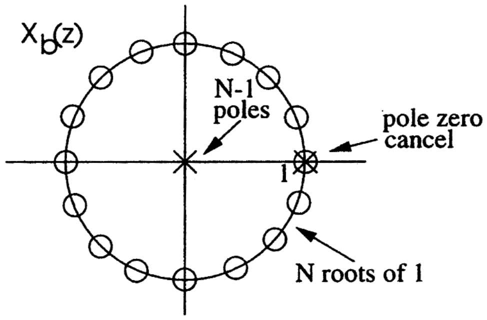
$$X(z) = \frac{z^{-1} - z^{-N-1}}{(1-z^{-1})^2} = \frac{z^{-1}(1-z^{-N})}{(1-z^{-1})^2}$$

3.3. **(a)**

$$x_a[n] = \alpha^{|n|} \qquad 0 < |\alpha| < 1$$

$$\begin{aligned} X_a(z) &= \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \sum_{n=1}^{\infty} \alpha^n z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\ &= \frac{\alpha z}{1-\alpha z} + \frac{1}{1-\alpha z^{-1}} = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}, \qquad |\alpha| < |z| < \frac{1}{|\alpha|} \end{aligned}$$





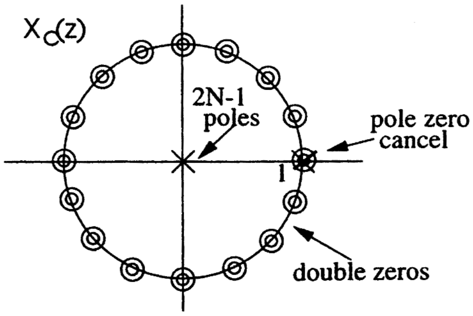
(b)

$$x_b = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & N \le n \\ 0, & n < 0 \end{cases} \Rightarrow X_b(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)} \quad z \neq 0$$

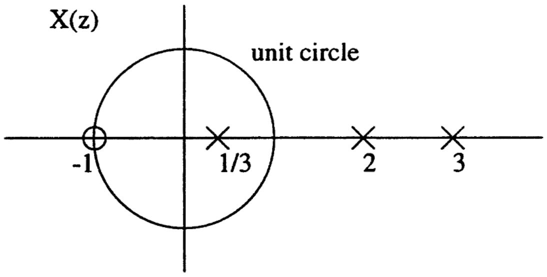
(c)

$$x_c[n] = x_b[n-1] * x_b[n] \Leftrightarrow X_c(z) = z^{-1} X_b(z) \cdot X_b(z)$$

$$X_c(z) = z^{-1}\left(\frac{z^N - 1}{z^{N-1}(z-1)}\right)^2 = \frac{1}{z^{2N-1}}\left(\frac{z^N-1}{z-1}\right)^2 \qquad z \neq 0, 1$$



3.4. The pole-zero plot of $X(z)$ appears below.



(a) For the Fourier transform of $x[n]$ to exist, the z-transform of $x[n]$ must have an ROC which includes the unit circle, therefore, $|\frac{1}{3}| < |z| < |2|$.
Since this ROC lies outside $\frac{1}{3}$, this pole contributes a right-sided sequence. Since the ROC lies inside 2 and 3, these poles contribute left-sided sequences. The overall $x[n]$ is therefore two-sided.

(b) Two-sided sequences have ROC's which look like washers. There are two possibilities. The ROC's corresponding to these are: $|\frac{1}{3}| < |z| < |2|$ and $|2| < |z| < |3|$.

(c) The ROC must be a connected region. For stability, the ROC must contain the unit circle. For causality the ROC must be outside the outermost pole. These conditions cannot be met by any of the possible ROC's of this pole-zero plot.

3.5.

$$
\begin{aligned}
X(z) &= (1+2z)(1+3z^{-1})(1-z^{-1}) \\
&= 2z + 5 - 4z^{-1} - 3z^{-2} \\
&= \sum_{n=-\infty}^{\infty} x[n]z^{-n}
\end{aligned}
$$

Therefore,

$$x[n] = 2\delta[n+1] + 5\delta[n] - 4\delta[n-1] - 3\delta[n-2]$$

(d)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$\begin{aligned} X(z) &= \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{1}{1 + \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \\ x[n] &= \left(-\frac{1}{2}\right)^n u[n] \end{aligned}$$

Long division: see part (i) above.

(e)

$$X(z) = \frac{1 - az^{-1}}{z^{-1} - a} \quad |z| > |a^{-1}|$$

Partial Fractions:

$$X(z) = -a - \frac{a^{-1}(1 - a^2)}{1 - a^{-1}z^{-1}} \quad |z| > |a^{-1}|$$

$$x[n] = -a\delta[n] - (1 - a^2)a^{-(n+1)}u[n]$$

Long division:

$$\begin{array}{r|llll} & -\frac{1}{a} & -\left(\frac{a^{-1}-a}{a}\right)z^{-1} & -\left(\frac{a^{-1}-a}{a^2}\right)z^{-2} & + \dots \\ \hline -a + z^{-1} & 1 & -\, az^{-1} & & \\ & 1 & -\, az^{-1} & & \\ \cline{2-3} & & (a^{-1} - a)z^{-1} & & \dots \end{array}$$

$$\Longrightarrow \qquad x[n] = -a\delta[n] - (1 - a^2)a^{-(n+1)}u[n]$$

(d)

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} \quad |z| > \frac{1}{2}$$

Partial Fractions:

$$\begin{aligned} X(z) &= \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{1}{1 + \frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \\ x[n] &= \left(-\frac{1}{2}\right)^n u[n] \end{aligned}$$

Long division: see part (i) above.

(e)

$$X(z) = \frac{1 - az^{-1}}{z^{-1} - a} \quad |z| > |a^{-1}|$$

Partial Fractions:

$$X(z) = -a - \frac{a^{-1}(1 - a^2)}{1 - a^{-1}z^{-1}} \qquad |z| > |a^{-1}|$$

$$x[n] = -a\delta[n] - (1 - a^2)a^{-(n+1)}u[n]$$

Long division:

$$\begin{array}{r|llll} & -\frac{1}{a} & -\left(\frac{a^{-1}-a}{a}\right)z^{-1} & -\left(\frac{a^{-1}-a}{a^2}\right)z^{-2} & + \ldots \\ \hline -a + z^{-1} & 1 & -\,az^{-1} & & \\ & 1 & -\,az^{-1} & & \\ \cline{2-3} & & (a^{-1} - a)z^{-1} & \ldots & \end{array}$$

$$\Longrightarrow \qquad x[n] = -a\delta[n] - (1 - a^2)a^{-(n+1)}u[n]$$

3.7. (a)

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$$

$$\Rightarrow \qquad X(z) = \frac{-1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \frac{1}{2} < |z| < 1$$

Now to find $H(z)$ we simply use $H(z) = Y(z)/X(z)$; i.e.,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} \cdot \frac{(1-z^{-1})(1-\frac{1}{2}z^{-1})}{-\frac{1}{2}z^{-1}} = \frac{1-z^{-1}}{1+z^{-1}}$$

$H(z)$ causal $\Rightarrow$ ROC $|z| > 1$.

(b) Since one of the poles of $X(z)$, which limited the ROC of $X(z)$ to be less than 1, is cancelled by the zero of $H(z)$, the ROC of $Y(z)$ is the region in the z-plane that satisfies the remaining two constraints $|z| > \frac{1}{2}$ and $|z| > 1$. Hence $Y(z)$ converges on $|z| > 1$.

(c)

$$Y(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1+z^{-1}} \qquad |z| > 1$$

Therefore,

$$y[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}(-1)^n u[n]$$

3.8. The causal system has system function

$$H(z) = \frac{1 - z^{-1}}{1 + \frac{3}{4}z^{-1}}$$

and the input is $x[n] = \left(\frac{1}{3}\right)^n u[n] + u[-n-1]$. Therefore the z-transform of the input is

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 - z^{-1}} = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - z^{-1})} \qquad \frac{1}{3} < |z| < 1$$

(a) $h[n]$ causal $\Rightarrow$

$$h[n] = \left(-\frac{3}{4}\right)^n u[n] - \left(-\frac{3}{4}\right)^{n-1} u[n-1]$$

(b)

$$\begin{aligned} Y(z) &= X(z)H(z) = \frac{-\frac{2}{3}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 + \frac{3}{4}z^{-1})} \qquad \frac{3}{4} < |z| \\ &= \frac{-\frac{8}{13}}{1 - \frac{1}{3}z^{-1}} + \frac{\frac{8}{13}}{1 + \frac{3}{4}z^{-1}} \end{aligned}$$

Therefore the output is

$$y[n] = -\frac{8}{13}\left(\frac{1}{3}\right)^n u[n] + \frac{8}{13}\left(-\frac{3}{4}\right)^n u[n]$$

(c) For $h[n]$ to be causal the ROC of $H(z)$ must be $\frac{3}{4} < |z|$ which includes the unit circle. Therefore, $h[n]$ absolutely summable.

3.9.

$$H(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{2}{(1-\frac{1}{2}z^{-1})} - \frac{1}{(1+\frac{1}{4}z^{-1})}$$

(a) $h[n]$ causal $\Rightarrow$ ROC outside $|z| = \frac{1}{2} \Rightarrow |z| > \frac{1}{2}$.

(b) ROC includes $|z| = 1 \Rightarrow$ stable.

(c)

$$\begin{aligned}
y[n] &= -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1] \\
Y(z) &= \frac{-\frac{1}{3}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} \\
&= \frac{1+z^{-1}}{(1+\frac{1}{4}z^{-1})(1-2z^{-1})} \qquad \frac{1}{4} < |z| < 2 \\
X(z) &= \frac{Y(z)}{H(z)} = \frac{(1-\frac{1}{2}z^{-1})}{(1-2z^{-1})} \qquad |z| < 2 \\
x[n] &= -(2)^n u[-n-1] + \frac{1}{2}(2)^{n-1}u[-n]
\end{aligned}$$

(d)

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]$$

3.10. (a)

$$\begin{aligned} x[n] &= \left(\frac{1}{2}\right)^n u[n-10] + \left(\frac{3}{4}\right)^n u[n-10] \\ &= \left(\frac{1}{2}\right)^n u[n] + \left(\frac{3}{4}\right)^n u[n] \\ &\quad - \left[\left(\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right)(u[n] - u[n-11])\right] \end{aligned}$$

The last term is finite length and converges everywhere except at $z = 0$. Therefore, ROC outside largest pole $\frac{3}{4} < |z|$.

(b)

$$x[n] = \begin{cases} 1, & -10 \le n \le 10 \\ 0, & \text{otherwise} \end{cases}$$

Finite length but has positive and negative powers at z in its $X(z)$. Therefore the ROC is $0 < |z| < \infty$.

(c)

$$\begin{aligned} x[n] &= 2^n u[-n] = \left(\frac{1}{2}\right)^{-n} u[-n] \\ x[-n] &\leftrightarrow X(1/z) \\ \left(\frac{1}{2}\right)^n u[n] &\Rightarrow \text{ROC is} \quad |z| > \frac{1}{2} \\ \left(\frac{1}{2}\right)^{-n} u[-n] &\Rightarrow \text{ROC is} \quad |z| < 2 \end{aligned}$$

(d)

$$x[n] = \left[\left(\frac{1}{4}\right)^{n+4} - (e^{j\pi/3})^n\right] u[n-1]$$

$x[n]$ is right-sided, so its ROC extends outward from the outermost pole $e^{j\pi/3}$. But since it is non-zero at $n = -1$, the ROC does not include ∞. So the ROC is $1 < |z| < \infty$.

(e)

$$\begin{aligned} x[n] &= u[n+10] - u[n+5] \\ &= \begin{cases} 1, & -10 \le n \le -6 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$x[n]$ is finite-length and has only positive powers of z in its $X(z)$. So the ROC is $|z| < \infty$.

(f)

$$x[n] = \left(\frac{1}{2}\right)^{n-1} u[n] + (2+3j)^{n-2} u[-n-1]$$

$x[n]$ is two-sided, with two poles. Its ROC is the ring between the two poles: $\frac{1}{2} < |z| < \left|\frac{1}{2+3j}\right|$, or $\frac{1}{2} < |z| < \frac{1}{\sqrt{13}}$.

3.11.

$$x[n] \text{ causal } \Rightarrow X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

which means this summation will include *no* positive powers of z. This means that the closed form of $X(z)$ must converge at $z = \infty$, i.e., $z = \infty$ must be in the ROC of $X(z)$, or $\lim_{z\to\infty} X(z) \neq \infty$.

(a)

$$\lim_{z\to\infty} \frac{(1 - z^{-1})^2}{(1 - \frac{1}{2}z^{-1})} = 1 \qquad \text{could be causal}$$

(b)

$$\lim_{z\to\infty} \frac{(z - 1)^2}{(z - \frac{1}{2})} = \infty \qquad \text{could not be causal}$$

(c)

$$\lim_{z\to\infty} \frac{(z - \frac{1}{4})^5}{(z - \frac{1}{2})^6} = 0 \qquad \text{could be causal}$$

(d)

$$\lim_{z\to\infty} \frac{(z - \frac{1}{4})^6}{(z - \frac{1}{2})^5} = \infty \qquad \text{could not be causal}$$

3.12. (a)

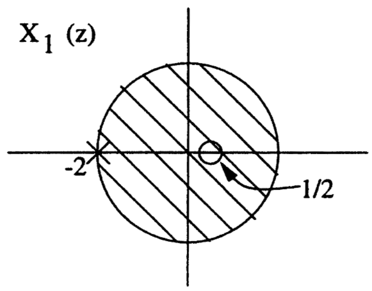
$$X_1(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + 2z^{-1}}$$

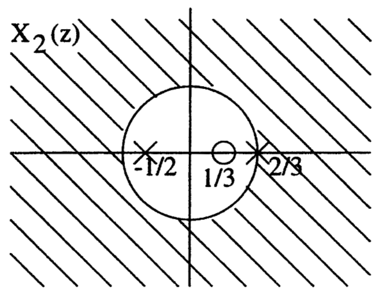
The pole is at -2, and the zero is at 1/2.

(b)

$$X_2(z) = \frac{1 - \frac{1}{3}z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 - \frac{2}{3}z^{-1})}$$

The poles are at -1/2 and 2/3, and the zero is at 1/3. Since $x_2[n]$ is causal, the ROC is extends from the outermost pole: $|z| > 2/3$.

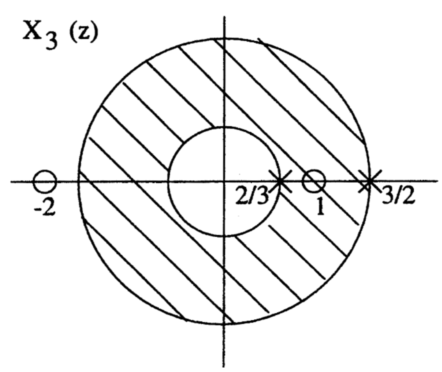




(c)

$$X_3(z) = \frac{1 + z^{-1} - 2z^{-2}}{1 - \frac{13}{6}z^{-1} + z^{-2}}$$

The poles are at 3/2 and 2/3, and the zeros are at 1 and -2. Since $x_3[n]$ is absolutely summable, the ROC must include the unit circle: $2/3 < |z| < 3/2$.



3.13.

$$\begin{aligned} G(z) &= \sin(z^{-1})(1+3z^{-2}+2z^{-4}) \\ &= (z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!})(1+3z^{-2}+2z^{-4}) \\ &= \sum_n g[n]z^{-n} \end{aligned}$$

$g[11]$ is simply the coefficient in front of z^{-11} in this power series expansion of $G(z)$:

$$g[11] = -\frac{1}{11!} + \frac{3}{9!} - \frac{2}{11!}$$

3.14.

$$\begin{aligned} H(z) &= \frac{1}{1-\frac{1}{4}z^{-2}} \\ &= \frac{1}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})} \\ &= \frac{0.5}{1-\frac{1}{2}z^{-1}} + \frac{0.5}{1+\frac{1}{2}z^{-1}} \end{aligned}$$

Taking the inverse z-Transform:

$$h[n] = \frac{1}{2}(\frac{1}{2})^n u[n] + \frac{1}{2}(-\frac{1}{2})^n u[n]$$

So,

$$A_1 = \frac{1}{2}; \quad \alpha_1 = \frac{1}{2}; \quad A_2 = \frac{1}{2}; \quad \alpha_2 = -\frac{1}{2};$$

3.15. Using long division, we get

$$\begin{aligned} H(z) &= \frac{1 - \frac{1}{1024}z^{-10}}{1 - \frac{1}{2}z^{-1}} \\ &= \sum_{n=0}^{n=9} (\frac{1}{2})^n z^{-n} \end{aligned}$$

Taking the inverse z-transform,

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n = 0, 1, 2, \ldots, 9 \\ 0, & \text{otherwise} \end{cases}$$

Since $h[n]$ is 0 for $n < 0$, the system is causal.

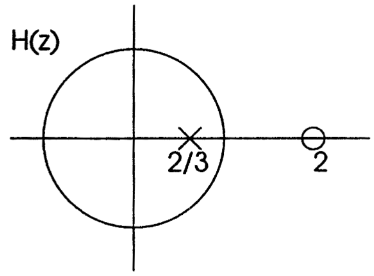
3.16. (a) To determine $H(z)$, we first find $X(z)$ and $Y(z)$:

$$\begin{aligned} X(z) &= \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1-2z^{-1}} \\ &= \frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}, \qquad \frac{1}{3} < |z| < 2 \\ Y(z) &= \frac{5}{1-\frac{1}{3}z^{-1}} - \frac{5}{1-\frac{2}{3}z^{-1}} \\ &= \frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{2}{3}z^{-1})}, \qquad |z| > \frac{2}{3} \end{aligned}$$

Now

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}} \qquad |z| > \frac{2}{3} \end{aligned}$$

The pole-zero plot of $H(z)$ is plotted below.



(b) Taking the inverse z-transform of $H(z)$, we get

$$\begin{aligned} h[n] &= (\frac{2}{3})^n u[n] - 2(\frac{2}{3})^{n-1}u[n-1] \\ &= (\frac{2}{3})^n (u[n] - 3u[n-1]) \end{aligned}$$

(c) Since

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{2}{3}z^{-1}},$$

we can write

$$Y(z)(1-\frac{2}{3}z^{-1}) = X(z)(1-2z^{-1}),$$

whose inverse z-transform leads to

$$y[n] - \frac{2}{3}y[n-1] = x[n] - 2x[n-1]$$

(d) The system is stable because the ROC includes the unit circle. It is also causal since the impulse response $h[n] = 0$ for $n < 0$.

3.17. We solve this problem by finding the system function $H(z)$ of the system, and then looking at the different impulse responses which can result from our choice of the ROC.

Taking the z-transform of the difference equation, we get

$$Y(z)(1-\frac{5}{2}z^{-1}+z^{-2}) = X(z)(1-z^{-1}),$$

and thus

$$\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}} \\
&= \frac{1-z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} \\
&= \frac{2/3}{1-2z^{-1}} + \frac{1/3}{1-\frac{1}{2}z^{-1}}
\end{aligned}$$

If the ROC is

(a) $|z| < \frac{1}{2}$:

$$h[n] = -\frac{2}{3}2^n u[-n-1] - \frac{1}{3}(\frac{1}{2})^n u[-n-1]$$
$$\Longrightarrow \quad h[0] = 0.$$

(b) $\frac{1}{2} < |z| < 2$:

$$h[n] = -\frac{2}{3}2^n u[-n-1] + \frac{1}{3}(\frac{1}{2})^n u[n]$$
$$\Longrightarrow \quad h[0] = \frac{1}{3}.$$

(c) $|z| > 2$:

$$h[n] = \frac{2}{3}2^n u[n] + \frac{1}{3}(\frac{1}{2})^n u[n]$$
$$\Longrightarrow \quad h[0] = 1.$$

(d) $|z| > 2$ or $|z| < \frac{1}{2}$:

$$h[n] = \frac{2}{3}2^n u[n] - \frac{1}{3}(\frac{1}{2})^n u[n-1]$$
$$\Longrightarrow \quad h[0] = \frac{2}{3}.$$

3.18. (a)

$$\begin{aligned} H(z) &= \frac{1+2^{z}-1+z^{-2}}{(1+\frac{1}{2}z^{-1})(1-z^{-1})} \\ &= -2+\frac{\frac{1}{3}}{1+\frac{1}{2}z^{-1}}+\frac{\frac{8}{3}}{1-z^{-1}} \end{aligned}$$

Taking the inverse z-transform:

$$h[n] = -2\delta[n]+\frac{1}{3}(-\frac{1}{2})^{n}u[n]+\frac{8}{3}u[n].$$

(b)

Given

$$H(z)=\frac{1+2z^{-1}+z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)},$$

$z=2$ is inside the ROC. Therefore,

$$\begin{aligned} y[n] &= H(z)\big|_{z=2}2^{n} \\ &= \frac{18}{5}2^{n}. \end{aligned}$$

3.19. The ROC($Y(z)$) includes the intersection of ROC($H(z)$) and ROC($X(z)$).

(a)

$$Y(z) = \frac{1}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

The intersection of ROCs of $H(z)$ and $X(z)$ is $|z| > \frac{1}{2}$. So the ROC of $Y(z)$ is $|z| > \frac{1}{2}$.

(b) The ROC of $Y(z)$ is exactly the intersection of ROCs of $H(z)$ and $X(z)$: $\frac{1}{3} < |z| < 2$.

(c)

$$Y(z) = \frac{1}{(1 - \frac{1}{5}z^{-1})(1 + \frac{1}{3}z^{-1})}$$

The ROC is $|z| > \frac{1}{3}$.

3.20. In both cases, the ROC of $H(z)$ has to be chosen such that ROC($Y(z)$) includes the intersection of ROC($H(z)$) and ROC($X(z)$).

(a)

$$H(z) = \frac{1 - \frac{3}{4}z^{-1}}{1 + \frac{2}{3}z^{-1}}$$

The ROC is $|z| > \frac{2}{3}$.

(b)

$$H(z) = \frac{1}{1 - \frac{1}{6}z^{-1}}$$

The ROC is $|z| > \frac{1}{6}$.

3.21.

$$H(z)=\frac{4+0.25z^{-1}-0.5z^{-2}}{\left(1-0.25z^{-1}\right)\left(1+0.5z^{-1}\right)}$$

A. Poles are located at $z=0.25$ and $z=-0.5$. Since the system is causal, the ROC is the disk that extends outwards from the outermost pole. Hence, $|z|>0.5$.

B. The unit circle has to be contained in the ROC to ensure stability. The above ROC contains the ROC, and hence the system is stable.

C.

$$H(z)=\frac{Y(z)}{X(z)}=\frac{4+0.25z^{-1}-0.5z^{-2}}{\left(1-0.25z^{-1}\right)\left(1+0.5z^{-1}\right)}$$

$$\left(1+0.5z^{-1}-0.25z^{-1}-0.125z^{-2}\right)Y(z)=\left(4+0.25z^{-1}-0.5z^{-2}\right)X(z)$$

$$Y(z)+0.25z^{-1}Y(z)-0.125z^{-2}Y(z)=4X(z)+0.25z^{-1}X(z)-0.5z^{-2}X(z)$$

Taking the inverse z-transform of both sides gives

$$y[n]+0.25y[n-1]-0.125y[n-2]=4x[n]+0.25x[n-1]-0.5x[n-2]$$

$$y[n]=-0.25y[n-1]+0.125y[n-2]+4x[n]+0.25x[n-1]-0.5x[n-2].$$

D. Since the degree of the numerator is equal to the degree of the denominator, we need to do long division. The result is

$$H(z)=4-\frac{0.75z^{-1}}{1+0.25z^{-1}-0.125z^{-2}}.$$

Then

$$\frac{0.75z^{-1}}{\left(1-0.25z^{-1}\right)\left(1+0.5z^{-1}\right)}=\frac{A}{\left(1-0.25z^{-1}\right)}+\frac{B}{\left(1+0.5z^{-1}\right)}=P(z).$$

We have

$$A=\left(1-0.25z^{-1}\right)P(z)\Big|_{z=0.25}=\frac{0.75(4)}{1+0.5(4)}=1$$

and

$$B=\left(1+0.5z^{-1}\right)P(z)\Big|_{z=-0.5}=\frac{0.75(-2)}{1-0.25(-2)}=-1$$

so that

$$H(z)=4-\frac{1}{\left(1-0.25z^{-1}\right)}+\frac{1}{\left(1+0.5z^{-1}\right)}.$$

The inverse z-transform is

$$h[n]=4\delta[n]-(0.25)^n u[n]+(-0.5)^n u[n].$$

E. $x[n] = u[-n-1]$. Note that this is a left-sided sequence. The z-transform is

$$X(z) = \frac{-1}{1-z^{-1}}, \quad \text{ROC } |z| < 1.$$

Then

$$\begin{aligned}
Y(z) &= H(z)X(z), \quad \text{ROC } R_H \cap R_X \\
&= \frac{4+0.25z^{-1}-0.5z^{-2}}{\left(1-0.25z^{-1}\right)\left(1+0.5z^{-1}\right)} \frac{-1}{\left(1-z^{-1}\right)}, \quad \text{ROC } \left[|z|>0.5\right] \cap \left[|z|<1\right] \\
&= \frac{-4-0.25z^{-1}+0.5z^{-2}}{\left(1-0.25z^{-1}\right)\left(1+0.5z^{-1}\right)\left(1-z^{-1}\right)}, \quad \text{ROC } 0.5 < |z| < 1.
\end{aligned}$$

F. Notice that the degree of the numerator is less than the degree of the denominator. Thus there is no need for long division and we can proceed directly to the partial fraction expansion.

$$Y(z) = \frac{A}{\left(1-0.25z^{-1}\right)} + \frac{B}{\left(1+0.5z^{-1}\right)} + \frac{C}{\left(1-z^{-1}\right)}.$$

We have

$$A = \left(1-0.25z^{-1}\right)Y(z)\Big|_{z=0.25} = \frac{-4-0.25(4)+0.5(4)^2}{\left(1+0.5(4)\right)(1-4)} = \frac{-1}{3}$$

$$B = \left(1+0.5z^{-1}\right)Y(z)\Big|_{z=-0.5} = \frac{-4-0.25(-2)+0.5(-2)^2}{\left(1+0.5(-2)\right)\left(1-(-2)\right)} = \frac{-1}{3}$$

$$C = \left(1-z^{-1}\right)Y(z)\Big|_{z=1} = \frac{-4-0.25+0.5}{(1-0.25)(1+0.5)} = \frac{-10}{3}.$$

Then

$$Y(z) = \frac{-\frac{1}{3}}{\left(1-0.25z^{-1}\right)} + \frac{-\frac{1}{3}}{\left(1+0.5z^{-1}\right)} + \frac{-\frac{10}{3}}{\left(1-z^{-1}\right)}.$$

The inverse z-transform is

$$y[z] = -\frac{1}{3}(0.25)^n u[n] - \frac{1}{3}(-0.5)^n u[n] + \frac{10}{3} u[-n-1].$$

3.22. A. The system is linear, so we can find the response to each term in the input express add the responses together.

For input $2\cos\left(\frac{\pi}{2}n\right)$, we can evaluate $H(z)$ at $z = e^{j\frac{\pi}{2}}$. The steady-state respons then $\left|H\left(e^{j\frac{\pi}{2}}\right)\right| 2\cos\left(\frac{\pi}{2}n + \measuredangle H\left(e^{j\frac{\pi}{2}}\right)\right)$.

For input $u[n]$, the steady-state response is equal to the DC gain; that is, $H\left(e^{j0}\right)$.

B. Given $H(z) = \dfrac{1-4z^2}{1+0.5z^{-1}}$, we have

$$H\left(e^{j\frac{\pi}{2}}\right) = \frac{1-4e^{-j\pi}}{1+0.5e^{-j\frac{\pi}{2}}} = \frac{5}{1-j0.5} = 4.47e^{j0.464}.$$

Then $y_1[n] = 8.94\cos\left(\frac{\pi}{2}n + 0.464\right)$.

Next, $H\left(e^{j0}\right) = \dfrac{1-4}{1+0.5} = -2.00$, so that $y_2[n] = -2.00 \times 1 = -2.00$.

As n gets large the response becomes

$$y[n] = y_1[n] + y_2[n] = -2.00 + 8.94\cos\left(\tfrac{\pi}{2}n + 0.464\right).$$

3.23. (a)

$$\begin{aligned}
y[n] &= 0 \qquad n<0 \\
y[n] &= \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} a^{n-k} = a^n \frac{1-a^{-(n+1)}}{1-a^{-1}} = \frac{1-a^{n+1}}{1-a} \qquad 0 \le n < N-1 \\
y[n] &= \sum_{k=0}^{N-1} x[k]h[n-k] = \sum_{k=0}^{N-1} a^{n-k} = a^n \frac{1-a^{-N}}{1-a^{-1}} = a^{n+1}\frac{1-a^{-N}}{a-1}, \qquad n \ge N
\end{aligned}$$

(b)

$$\begin{aligned}
H(z) &= \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-az^{-1}} \qquad |z|>|a| \\
X(z) &= \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} \qquad |z|>0
\end{aligned}$$

Therefore,

$$Y(z) = \frac{1-z^{-N}}{(1-az^{-1})(1-z^{-1})} = \frac{1}{(1-az^{-1})(1-z^{-1})} - \frac{z^{-N}}{(1-az^{-1})(1-z^{-1})} \qquad |z|>|a|$$

Now,

$$\frac{1}{(1-az^{-1})(1-z^{-1})} = \frac{\frac{1}{1-a^{-1}}}{1-az^{-1}} + \frac{\frac{1}{1-a}}{1-z^{-1}} = \left(\frac{1}{1-a}\right)\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right)$$

So

$$\begin{aligned}
y[n] &= \left(\frac{1}{1-a}\right)\left[u[n] - a^{n+1}u[n] - u[n-N] - a^{n-N+1}u[n-N]\right] \\
&= \frac{1-a^{n+1}}{1-a}u[n] - \frac{1-a^{n-N+1}}{1-a}u[n-N] \\
y[n] &= \begin{cases} 0 & n<0 \\ \frac{1-a^{n+1}}{1-a} & 0 \le n \le N-1 \\ a^{n+1}\left(\frac{1-a^{-N}}{a-1}\right) & n \ge N \end{cases}
\end{aligned}$$

3.24. (a)

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=-\infty}^{\infty} \left(3\left(-\frac{1}{3}\right)^k u[k]\right) u[n-k] \\ &= \sum_{k=0}^{n} 3\left(-\frac{1}{3}\right)^k \\ &= \begin{cases} \frac{9}{4}\left(1-\left(-\frac{1}{3}\right)^{n+1}\right), & n \geq 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \frac{3}{1+\frac{1}{3}z^{-1}} \frac{1}{1-z^{-1}} \\ &= \frac{\frac{3}{4}}{1+\frac{1}{3}z^{-1}} + \frac{\frac{9}{4}}{1-z^{-1}} \\ y[n] &= \frac{3}{4}\left(-\frac{1}{3}\right)^n u[n] + \frac{9}{4}u[n] \\ &= \frac{9}{4}\left(1+\frac{1}{3}\left(-\frac{1}{3}\right)^n\right) u[n] \\ &= \frac{9}{4}\left(1-\left(-\frac{1}{3}\right)^{n+1}\right) u[n] \end{aligned}$$

3.25. **(a)**

$$\begin{aligned}
H(z) &= \frac{1-\frac{1}{2}z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} \\
&= -4+\frac{5+\frac{7}{2}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} \\
&= -4-\frac{2}{1-\frac{1}{2}z^{-1}}+\frac{7}{1-\frac{1}{4}z^{-1}} \\
h[n] &= -4\delta[n]-2\left(\frac{1}{2}\right)^n u[n]+7\left(\frac{1}{4}\right)^n u[n]
\end{aligned}$$

(b)

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=x[n]-\frac{1}{2}x[n-2]$$

3.26. The plots of the sequences are shown below.

(a) Let

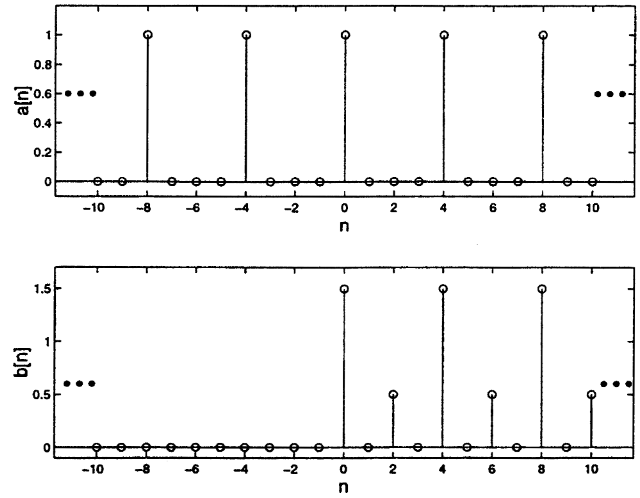
$$a[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k],$$

Then

$$A(z) = \sum_{k=-\infty}^{\infty} z^{-4n}$$

(b)

$$\begin{aligned}
b[n] &= \frac{1}{2}\left[e^{j\pi n} + \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{2} + 2\pi n\right)\right] u[n] \\
&= \frac{1}{2}\left[(-1)^n + \cos\left(\frac{\pi}{2}n\right) + 1\right] u[n] \\
&= \begin{cases} \frac{3}{2}, & n = 4k,\ k \geq 0 \\ \frac{1}{2}, & n = 4k+2,\ k \geq 0 \\ 0, & \text{otherwise} \end{cases} \\
B(z) &= \sum_{n=0}^{\infty} \frac{3}{2} z^{-4n} + \sum_{n=0}^{\infty} \frac{1}{2} z^{-(4n+2)} \\
&= \frac{3/2 + 1/2z^{-2}}{1 - z^{-4}}, \qquad |z| > 1
\end{aligned}$$



3.27.

$$X(z) = \frac{z^2}{(z-a)(z-b)} = \frac{z^2}{z^2-(a+b)z+ab}$$

Obtain a proper fraction:

$$\begin{array}{r|l} & \quad 1 \\ \hline z^2-(a+b)z+ab & z^2 \\ & \underline{z^2 \quad -(a+b)z \quad +ab} \\ & \qquad (a+b)z \quad -ab \end{array}$$

$$\begin{aligned} X(z) &= 1 + \frac{(a+b)z-ab}{(z-a)(z-b)} = 1 + \frac{\frac{(a+b)a-ab}{a-b}}{z-a} + \frac{\frac{(a+b)b-ab}{b-a}}{z-b} \\ &= 1 + \frac{\frac{a^2}{a-b}}{z-a} - \frac{\frac{b^2}{a-b}}{z-b} = 1 + \frac{1}{a-b}\left(\frac{a^2z^{-1}}{1-az^{-1}} - \frac{b^2z^{-1}}{1-bz^{-1}}\right) \\ x[n] &= \delta[n] + \frac{a^2}{a-b}a^{n-1}u[n-1] - \frac{b^2}{a-b}b^{n-1}u[n-1] \\ &= \delta[n] + \left(\frac{1}{a-b}\right)(a^{n+1}-b^{n+1})u[n-1] \end{aligned}$$

3.31

$$H(z)=\frac{1-z^{-1}}{1-0.25z^{-2}}=\frac{1-z^{-1}}{\left(1-0.5z^{-1}\right)\left(1+0.5z^{-1}\right)}$$

A. Given $x[n]=u[n]$, we have $X(z)=\frac{1}{1-z^{-1}},\quad 1<|z|$. Then

$$\begin{aligned}Y(z)&=H(z)X(z)\\&=\frac{1-z^{-1}}{\left(1-0.5z^{-1}\right)\left(1+0.5z^{-1}\right)}\frac{1}{1-z^{-1}}\\&=\frac{1}{\left(1-0.5z^{-1}\right)\left(1+0.5z^{-1}\right)}\\&=\frac{\frac{1}{2}}{\left(1-0.5z^{-1}\right)}+\frac{\frac{1}{2}}{\left(1+0.5z^{-1}\right)},\quad 0.5<|z|.\end{aligned}$$

(The ROC for $Y(z)$ includes the intersection of the ROC of $H(z)$ with the ROC of $X(z)$.)

Inverse z-transforming gives

$$y[n]=\tfrac{1}{2}(0.5)^n u[n]+\tfrac{1}{2}(-0.5)^n u[n].$$

B. If $y[n]=\delta[n]-\delta[n-1]$, then $Y(z)=1-z^{-1},\quad 0<|z|$. We have

$$\begin{aligned}X(z)&=\frac{Y(z)}{H(z)}\\&=\frac{1-z^{-1}}{\left(\frac{1-z^{-1}}{1-0.25z^{-2}}\right)}\\&=1-0.25z^{-2},\quad 0<|z|.\end{aligned}$$

Inverse z-transforming gives

$$x[n]=\delta[n]-0.25\delta[n-2].$$

C. Now $x[n]=\cos(0.5\pi n),\quad -\infty<n<\infty$. At $\omega=0.5\pi$ we have

$$\begin{aligned}H\left(e^{j0.5\pi}\right)&=\frac{1-e^{-j0.5\pi}}{1-0.25e^{-j\pi}}\\&=1.13e^{j\frac{\pi}{4}}.\end{aligned}$$

Then

$$y[n]=1.13\cos\left(0.5\pi n+\tfrac{\pi}{4}\right).$$

3.32. (a) $x[n]$ is right-sided and

$$X(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}$$

Long division:

$$\begin{array}{r|llll} & 1 & -\frac{2}{3}z^{-1} & +\frac{2}{9}z^{-2} & + \dots \\ \hline 1+\frac{1}{3}z^{-1} & 1 & -\frac{1}{3}z^{-1} & & \\ & 1 & -\frac{1}{3}z^{-1} & & \\ \hline & & -\frac{2}{3}z^{-1} & & \\ & & -\frac{2}{3}z^{-1} & -\frac{2}{9}z^{-2} & \\ \hline & & & +\frac{2}{9}z^{-2} & \end{array}$$

Therefore, $x[n] = 2(-\frac{1}{3})^n u[n] - \delta[n]$

(b)

$$X(z) = \frac{3}{z - \frac{1}{4} - \frac{1}{8}z^{-1}} = \frac{3z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{4}z^{-1})} = \frac{4}{1 - \frac{1}{2}z^{-1}} - \frac{4}{1 - \frac{1}{4}z^{-1}}$$

Poles at $\frac{1}{2}$, and $-\frac{1}{4}$. $x[n]$ stable, $\Rightarrow |z| > \frac{1}{2} \Rightarrow$ causal.
Therefore,

$$x[n] = 4\left(\frac{1}{2}\right)^n u[n] - 4\left(-\frac{1}{4}\right)^n u[n]$$

(c)

$$\begin{aligned} X(z) &= \ln(1 - 4z) \qquad |z| < \frac{1}{4} \\ &= -\sum_{i=1}^{\infty} \frac{(4z)^i}{i} = -\sum_{\ell=-\infty}^{-1} \frac{1}{\ell}(4z)^{-\ell} \end{aligned}$$

Therefore,

$$x[n] = \frac{1}{n}(4)^{-n} u[-n-1]$$

(d)

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-3}} \qquad |z| > (3)^{-\frac{1}{3}} \Rightarrow \text{causal}$$

By long division:

$$\begin{array}{r|llll} & 1 & +\frac{1}{3}z^{-3} & +\frac{1}{9}z^{-6} & + \dots \\ \hline 1-\frac{1}{3}z^{-3} & 1 & & & \\ & 1 & -\frac{1}{3}z^{-3} & & \\ \hline & & +\frac{1}{3}z^{-3} & & \\ & & +\frac{1}{3}z^{-3} & -\frac{1}{9}z^{-6} & \\ \hline & & & +\frac{1}{9}z^{-6} & \end{array}$$

$$\Longrightarrow \qquad x[n] = \begin{cases} (\frac{1}{3})^{\frac{n}{3}}, & n = 0, 3, 6, \dots \\ 0, & \text{otherwise} \end{cases}$$

3.33. (a)

$$\begin{aligned} X(z) &= \frac{1}{(1+\frac{1}{2}z^{-1})^2(1-2z^{-1})(1-3z^{-1})} \qquad \frac{1}{2} < |z| < 2 \\ &= \frac{\frac{1}{35}}{(1+\frac{1}{2}z^{-2})^2} + \frac{\frac{88}{1225}}{(1+\frac{1}{2}z^{-1})} - \frac{\frac{1568}{1225}}{(1-2z^{-1})} + \frac{\frac{2700}{1225}}{(1-3z^{-1})} \end{aligned}$$

Therefore,

$$x[n] = \frac{1}{35}(n+1)\left(\frac{-1}{2}\right)^{n+1} u[n+1] + \frac{58}{(35)^2}\left(\frac{-1}{2}\right)^{n} u[n] + \frac{1568}{(35)^2}(2)^n u[-n-1] - \frac{2700}{(35)^2}(3)^n u[-n-1]$$

(b)

$$X(z) = e^{z^{-1}} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \ldots$$

Therefore, $x[n] = \frac{1}{n!}u[n]$.

(c)

$$X(z) = \frac{z^3 - 2z}{z-2} = z^2 + 2z + \frac{2}{1-2z^{-1}} \qquad |z| < 2$$

Therefore,

$$x[n] = \delta[n+2] + 2\delta[n+1] - 2(2)^n u[-n-1]$$

3.34. (a)

$$nx[n] \Leftrightarrow -z\frac{d}{dx}X(z)$$

$$x[n-n_0] \Leftrightarrow z^{-n_0}X(z)$$

$$X(z) = \frac{3z^{-3}}{(1-\frac{1}{4}z^{-1})^2} = 12z^{-2}\left[-z\frac{d}{dz}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\right]$$

$x[n]$ is left-sided. Therefore, $X(z)$ corresponds to:

$$x[n] = -12(n-2)\left(\frac{1}{4}\right)^{n-2}u[-n+1]$$

(b)

$$X(z) = \sin(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}z^{2k+1} \qquad \text{ROC includes } |z| = 1$$

Therefore,

$$x[n] = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\delta[n+2k+1]$$

Which is stable.

(c)

$$X(z) = \frac{z^7-2}{1-z^{-7}} = z^7 - \frac{1}{1-z^{-7}} \qquad |z| > 1$$

$$X(z) = z^7 - \sum_{n=0}^{\infty} z^{-7n}$$

Therefore,

$$x[n] = \delta[n+7] - \sum_{n=0}^{\infty}\delta[n-7k]$$

3.35.

$$X(z) = e^z + e^{1/z} \quad z \neq 0$$

$$X(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n + \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^{-n} + \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \Longrightarrow x[n] = \frac{1}{|n|!} + \delta[n]$$

3.36.

$$X(z) = \log_2(\frac{1}{2} - z) \qquad |z| < \frac{1}{2}$$

(a)

$$X(z) = \log(1-2z) = -\sum_{i=1}^{\infty}\frac{(2z)^i}{i} = -\sum_{\ell=-\infty}^{-1}\frac{1}{-\ell}(2z)^{-\ell} = \sum_{\ell=-\infty}^{1}\frac{1}{\ell}\left(\frac{1}{2}\right)^{\ell} z^{-\ell}$$

Therefore,

$$x[n] = \frac{1}{n}\left(\frac{1}{2}\right)^n u[-n-1]$$

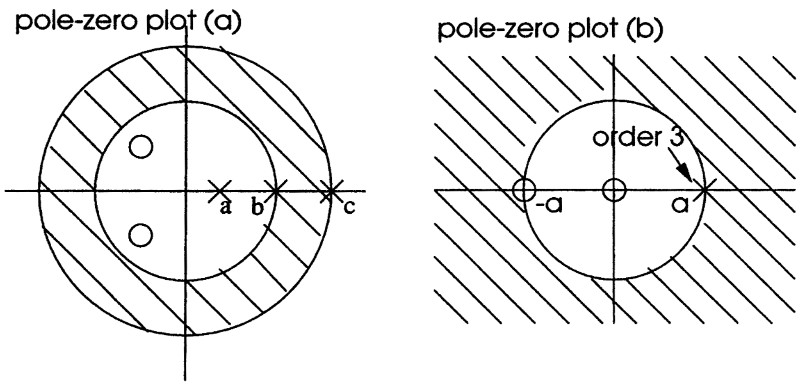
(b)

$$\begin{aligned}
nx[n] &\Leftrightarrow -z\frac{d}{dz}\log(1-2z) = -z\left(\frac{1}{1-2z}\right)(-2) = z^{-1}\left(\frac{-1}{1-\frac{1}{2}z^{-1}}\right), \qquad |z| < \frac{1}{2} \\
nx[n] &= \left(\frac{1}{2}\right)^n u[-n-1] \\
x[n] &= \frac{1}{n}\left(\frac{1}{2}\right)^n u[-n-1]
\end{aligned}$$

3.37. (a)

$$\begin{aligned} x[n] &= a^n u[n] + b^n u[n] + c^n u[-n-1] \qquad |a| < |b| < |c| \\ X(z) &= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} - \frac{1}{1-cz^{-1}} \quad |b| < |z| < |c| \\ X(z) &= \frac{1 - 2cz^{-1} + (bc + ac - ab)z^{-2}}{(1-az^{-1})(1-bz^{-1})(1-cz^{-1})} \quad |b| < |z| < |c| \end{aligned}$$

Poles: a, b, c,
Zeros: z_1, z_2, ∞ where z_1 and z_2 are roots of numerator quadratic.



(b)

$$\begin{aligned} x[n] &= n^2 a^n u[n] \\ x_1[n] &= a^n u[n] \Leftrightarrow X_1(z) = \frac{1}{1-az^{-1}} \quad |z| > a \\ x_2[n] = nx_1[n] &= na^n u[n] \Leftrightarrow X_2(z) = -z\frac{d}{dz}X_1(z) = \frac{az^{-1}}{(1-az^{-1})^2} \quad |z| > a \\ x[n] = nx_2[n] &= n^2 a^n u[n] \Leftrightarrow -z\frac{d}{dz}X_2(z) = -z\frac{d}{dz}\left(\frac{az^{-1}}{(1-az^{-1})^2}\right) \qquad |z| > a \\ X(z) &= \frac{-az^{-1}(1+az^{-1})}{(1-az^{-1})^3} \qquad |z| > a \end{aligned}$$

(c)

$$\begin{aligned} x[n] &= e^{n^4}\left(\cos\frac{\pi}{12}n\right)u[n] - e^{n^4}\left(\cos\frac{\pi}{12}n\right)u[n-1] \\ &= e^{n^4}\left(\cos\frac{\pi}{12}n\right)(u[n] - u[n-1]) = \delta[n] \end{aligned}$$

Therefore, $X(z) = 1$ for all $|z|$.

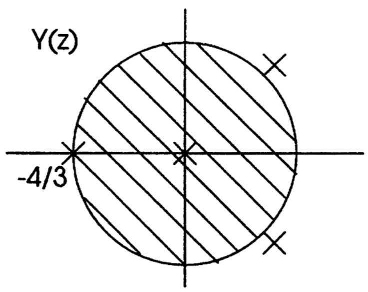
3.38. From the pole-zero diagram

$$X(z) = \frac{z}{(z^2 - z + \frac{1}{2})(z + \frac{3}{4})} \qquad |z| > \frac{3}{4}$$

$$\begin{aligned}
y[n] &= x[-n+3] = x[-(n-3)] \\
&\Rightarrow \; Y(z) = z^{-3}X(z^{-1}) = \frac{z^{-3}z^{-1}}{(z^{-2} - z^{-1} + \frac{1}{2})(z^{-1} + \frac{3}{4})} \\
&= \frac{8/3}{z(2 - 2z + z^2)(\frac{4}{3} + z)}
\end{aligned}$$

Poles at $0, -\frac{4}{3}, 1 \pm j$, zeros at ∞

$x[n]$ causal $\Rightarrow x[-n+3]$ is left-sided $\Rightarrow$ ROC is $0 < |z| < 4/3$.



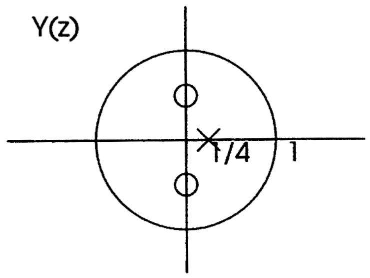
3.39. From pole-zero diagram

$$X(z) = \frac{z^2+1}{z-\frac{1}{2}}$$

(a)

$$y[n] = \left(\frac{1}{2}\right)^n x[n] \Rightarrow Y(z) = X(2z) = \frac{4z^2+1}{2z-\frac{1}{2}}$$

zeros $\pm\frac{1}{2}j$
poles $\frac{1}{4}, \infty$



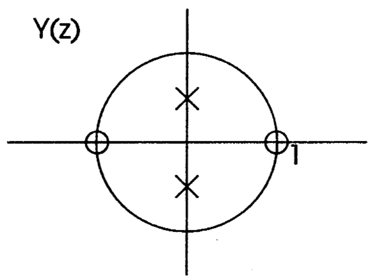
(b)

$$w[n] = \cos\left(\frac{\pi n}{2}\right) x[n] = \frac{1}{2}(e^{j\pi n/2} + e^{-j\pi n/2})x[n]$$

$$W(z) = \frac{1}{2}X(e^{-j\pi/2}z) + \frac{1}{2}X(e^{j\pi/2}z) = \frac{1}{2}X(-jz) + \frac{1}{2}X(jz)$$

$$W(z) = \frac{1}{2}\left(\frac{-z^2+1}{-jz-\frac{1}{2}}\right) + \frac{1}{2}\left(\frac{-z^2+1}{jz-\frac{1}{2}}\right) = \frac{z^2-1}{2(z^2+\frac{1}{4})}$$

poles at $\pm\frac{1}{2}j$
zeros at ± 1



3.40.

$$H(z) = \frac{3 - 7z^{-1} + 5z^{-2}}{1 - \frac{5}{2}z^{-1} + z^{-2}} = 5 + \frac{1}{1 - 2z^{-1}} - \frac{3}{1 - \frac{1}{2}z^{-1}}$$

$$h[n] \text{ stable } \Rightarrow h[n] = 5\delta[n] - 2^n u[-n-1] - 3\left(\frac{1}{2}\right)^n u[n]$$

(a)

$$\begin{aligned}
y[n] &= h[n] * x[n] = \sum_{k=-\infty}^{n} h[k] \\
&= \begin{cases} -\sum_{k=-\infty}^{n} 2^k = -2^{n+1} & n < 0 \\ -\sum_{k-\infty}^{-1} 2^k + 5 - \sum_{k=0}^{n} 3\left(\frac{1}{2}\right)^k = 4 - 3\frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} = -2 + 3\left(\frac{1}{2}\right)^n & n \geq 0 \end{cases} \\
&= -2u[n] + 3\left(\frac{1}{2}\right)^n u[n] - 2^{n+1}u[-n-1]
\end{aligned}$$

(b)

$$\begin{aligned}
Y(z) &= \frac{1}{1 - z^{-1}} H(z) = -2\frac{1}{1 - z^{-1}} + 2\frac{1}{1 - 2z^{-1}} + 3\frac{1}{1 - \frac{1}{2}z^{-1}}, \qquad \frac{1}{2} < |z| < 2 \\
y[n] &= -2u[n] - 2(2)^n u[-n-1] + 3\left(\frac{1}{2}\right)^n u[n]
\end{aligned}$$

3.41.

$$
\begin{aligned}
H(z) &= \frac{1-z^3}{1-z^4} = z^{-1}\left(\frac{1-z^{-3}}{1-z^{-4}}\right) \qquad |z|>1 \\
u[n] &\Leftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z|>1 \\
U(z)H(z'40 &= \frac{z^{-1}-z^{-4}}{(1-z^{-4})(1-z^{-1})} \\
&= \frac{z^{-1}}{1-z^{-1}} - \frac{z^{-4}}{1-z^{-4}} \qquad |z|>1 \\
u[n]*h[n] &= u[n-1] - \sum_{k=0}^{\infty}\delta[n-4-4k]
\end{aligned}
$$

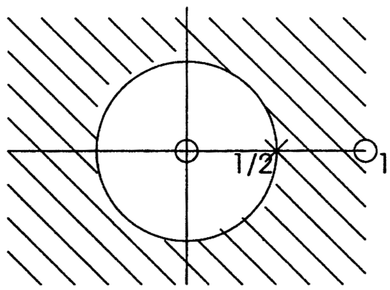
3.42.

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1] = 4\left(\frac{1}{2}\right)^{n+1} u[n+1] \Leftrightarrow Y(z) = \frac{4z}{1 - \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

(a)

$$H(z) = \frac{Y(z)}{X(Z)} = \frac{4z(1 - z^{-1})}{1 - \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$



(b)

$$\begin{aligned} H(z) &= \frac{4z}{1 - \frac{1}{2}z^{-1}} - \frac{4}{1 - \frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2} \\ h[n] &= 4\left(\frac{1}{2}\right)^{n+1} u[n+1] - 4\left(\frac{1}{2}\right)^{n} u[n] \\ &= 4\delta[n+1] - 2\left(\frac{1}{2}\right)^{n} u[n] \end{aligned}$$

(c) The ROC of $H(z)$ includes $|z| = 1 \Rightarrow$ stable.

(d) From part (b) we see that $h[n]$ starts at $n = -1 \Rightarrow$ not causal

3.43.

$$X(z) = \frac{\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{4}}{1 - 2z^{-1}}$$

has poles at $z = \frac{1}{2}$ and $z = 2$.

Since the unit circle is in the region of convergence $X(z)$ and $x[n]$ have both a causal and an anticausal part. The causal part is "outside" the pole at $\frac{1}{2}$. The anticausal part is "inside" the pole at 2, therefore, $x[0]$ is the sum of the two parts

$$x[0] = \lim_{z\to\infty} \frac{\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \lim_{z\to 0} \frac{\frac{1}{4}z}{z - 2} = \frac{1}{3} + 0 = \frac{1}{3}$$

3.44. (a) After writing the following equalities:

$$\begin{aligned} V(z) &= X(z) - W(z) \\ W(z) &= V(z)H(z) + E(z) \end{aligned}$$

we solve for W(z):

$$W(z) = \frac{H(z)}{1+H(z)}X(z) + \frac{1}{1+H(z)}E(z)$$

(b)

$$\begin{aligned} H_1(z) &= \frac{H(z)}{1+H(z)} = \frac{\frac{z^{-1}}{1-z^{-1}}}{1+\frac{z^{-1}}{1-z^{-1}}} = z^{-1} \\ H_2(z) &= \frac{1}{1+\frac{z^{-1}}{1-z^{-1}}} = 1 - z^{-1} \end{aligned}$$

(c) $H(z)$ is not stable due to its pole at $z = 1$, but $H_1(z)$ and $H_2(z)$ are.

3.45. (a) Yes, $h[n]$ is BIBO stable if its ROC includes the unit circle. Hence, the system is stable if $r_{min} < 1$ and $r_{max} > 1$.

(b) Let's consider the system step by step.

(i) First, $v[n] = \alpha^{-n}x[n]$. By taking the z-transform of both sides, $V(z) = X(\alpha z)$.

(ii) Second, $v[n]$ is filtered to get $w[n]$. So $W(z) = H(z)V(z) = H(z)X(\alpha z)$.

(iii) Finally, $y[n] = \alpha^n w[n]$. In the z-transform domain, $Y(z) = W(z/\alpha) = H(z/\alpha)X(z)$.

In conclusion, the system is LTI, with system function $G(z) = H(z/\alpha)$ and $g[n] = \alpha^n h[n]$.

(c) The ROC of $G(z)$ is $\alpha r_{min} < |z| < \alpha r_{max}$. We want $r_{min} < 1/\alpha$ and $r_{max} > 1/\alpha$ for the system to be stable.

3.46. (a) $h[n]$ is the response of the system when $x[n] = \delta[n]$. Hence,

$$h[n] + \sum_{k=1}^{10} \alpha_k h[n-k] = \delta[n] + \beta\delta[n-1],$$

Further, since the system is causal, $h[n] = 0$ for $n < 0$. Therefore,

$$h[0] + \sum_{k=1}^{10} \alpha_k h[-k] = h[0] = \delta[0] = 1.$$

(b) At $n = 1$,

$$h[1] + \alpha_1 h[0] = \delta[1] + \beta\delta[0] \qquad \Longrightarrow \alpha_1 = \frac{\beta - h[1]}{h[0]} = \beta - h[1]$$

(c) How can we extend h[n] for $n > 10$ and still have it compatible with the difference equation for S? Note that the difference equation can describe systems up to order 10. If we choose

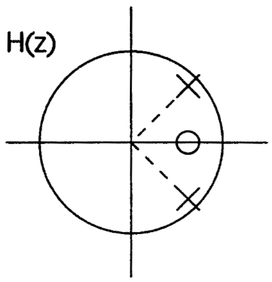
$$h[n] = (0.9)^n \cos(\frac{\pi}{4}n)u[n],$$

we only need a second order difference equation:

$$\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = 0.$$

The z-transform of $h[n]$ can be found from the z-transform table:

$$H(z) = \frac{1 - \frac{0.9}{\sqrt{2}}}{(1 - 0.9e^{j\pi/4}z^{-1})(1 - 0.9e^{-j\pi/4}z^{-1})}$$

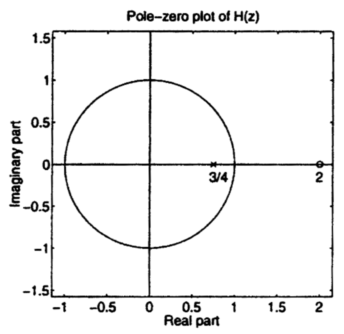


3.47. (a)

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-2z^{-1}}, \qquad \frac{1}{2} < |z| < 2$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{\frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})}}{\frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}} \\ &= \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4} \end{aligned}$$



(b)

$$h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]$$

(c)

$$y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$$

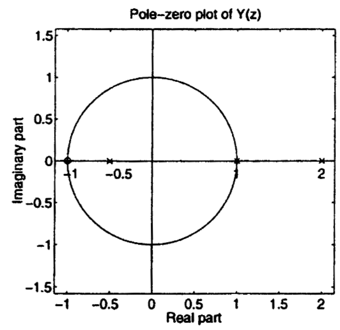
(d) The system is stable because the ROC includes the unit circle. It is also causal since $h[n] = 0$ for $n < 0$.

3.48. (a)

$$X(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}}$$

The ROC is $\frac{1}{2} < |z| < 2$.

(b) The following figure shows the pole-zero plot of $Y(z)$. Since $X(z)$ has poles at 0.5 and 2, the poles at 1 and -0.5 are due to $H(z)$. Since $H(z)$ is causal, its ROC is $|z| > 1$. The ROC of $Y(z)$ must contain the intersection of the ROC of $X(z)$ and the ROC of $H(z)$. Hence the ROC of $Y(z)$ is $1 < |z| < 2$.



(c)

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{\frac{1+z^{-1}}{(1-z^{-1})(1+\frac{1}{2}z^{-1})(1-2z^{-1})}}{\frac{1}{(1-12z^{-1})(1-2z^{-1})}} \\ &= \frac{(1+z^{-1})(1-\frac{1}{2}z^{-1})}{(1-z^{-1})(1-\frac{1}{2}z^{-1})} \\ &= 1 + \frac{\frac{2}{3}}{1-z^{-1}} + \frac{-\frac{2}{3}}{1+\frac{1}{2}z^{-1}} \end{aligned}$$

Taking the inverse z-transform, we find

$$h[n] = \delta[n] + \frac{2}{3}u[n] - \frac{2}{3}(-\frac{1}{2})^n u[n]$$

(d) Since $H(z)$ has a pole on the unit circle, the system is not stable.

3.49. (a)

$$\begin{aligned} ny[n] &= x[n] \\ -z\frac{dY(z)}{dz} &= X(z) \\ Y(z) &= -\int z^{-1}X(z)dz \end{aligned}$$

(b) To apply the results of part (a), we let $x[n] = u[n-1]$, and $w[n] = y[n]$.

$$\begin{aligned} W(z) &= -\int z^{-1}\frac{z^{-1}}{1-z^{-1}}dz \\ &= -\int \frac{1}{z(z-1)}dz \\ &= -\int \frac{-1}{z} + \frac{1}{z-1}dz \\ &= \ln(z) - \ln(z-1) \end{aligned}$$

3.50. (a) Since $y[n]$ is stable, its ROC contains the unit-circle. Hence, $Y(z)$ converges for $\frac{1}{2} < |z| < 2$.

(b) Since the ROC is a ring on the z-plane, $y[n]$ is a two-sided sequence.

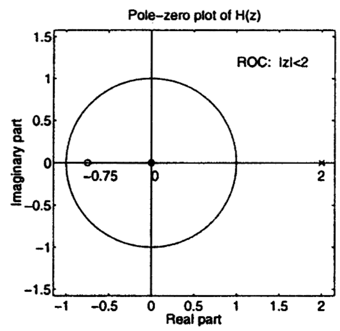
(c) $x[n]$ is stable, so its ROC contains the unit-circle. Also, it has a zero at ∞ so the ROC includes ∞. ROC: $|z| > \frac{3}{4}$.

(d) Since the ROC of $x[n]$ includes ∞, $X(z)$ contains no positive powers of z, and so $x[n] = 0$ for $n < 0$. Therefore $x[n]$ is causal.

(e)

$$\begin{aligned} x[0] &= X(z)|_{z=\infty} \\ &= \frac{A(1-\frac{1}{4}z^{-1})}{(1+\frac{3}{4}z^{-1})(1-\frac{1}{2}z^{-1})}|_{z=\infty} \\ &= 0 \end{aligned}$$

(f) $H(z)$ has zeros at -.75 and 0, and poles at 2 and ∞. Its ROC is $|z| < 2$.



(g) Since the ROC of $h[n]$ includes 0, $H(z)$ contains no negative powers of z, which implies that $h[n] = 0$ for $n > 0$. Therefore $h[n]$ is anti-causal.

3.51

$$y[n] = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)y[n-k] + \sum_{k=0}^{M}\left(\frac{b_k}{b_0}\right)x[n-k]$$

A. Using Equation (3.76), the unilateral z-transform of $y[n-k]$ is

$$\sum_{m=1}^{k} y[m-k-1]z^{-m+1} + z^{-k}\mathcal{Y}(z).$$

Applying the unilateral z-transform to the difference equation gives

$$\mathcal{Y}(z) = -\sum_{k=1}^{N}\left(\frac{a_k}{a_0}\right)\left\{\sum_{m=1}^{k} y[m-k-1]z^{-m+1} + z^{-k}\mathcal{Y}(z)\right\}$$
$$+\sum_{k=0}^{M}\left(\frac{b_k}{a_0}\right)z^{-k}\mathcal{X}(z),$$

assuming that $x[n]$ is suddenly applied at $n=0$.

Solving for $\mathcal{Y}(z)$ gives

$$\mathcal{Y}(z) = -\frac{\sum_{k=1}^{N} a_k\left(\sum_{m=1}^{k} y[m-k-1]z^{-m+1}\right)}{\sum_{k=0}^{N} a_k z^{-k}} + \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}\mathcal{X}(z),$$

QED.

B. If $x[n]=0$ for all n, then $\mathcal{X}(z)=0$. This gives

$$\mathcal{Y}(z) = \mathcal{Y}_{ZIR}(z) = -\frac{\sum_{k=1}^{N} a_k\left(\sum_{m=1}^{k} y[m-k-1]z^{-m+1}\right)}{\sum_{k=0}^{N} a_k z^{-k}},$$

which depends only on the initial conditions $y[-1], y[-2], \dots, y[-N]$. The inverse z-transform is the "zero-input response" $y_{ZIR}[n]$.

If $y[-1] = y[-2] = \dots = y[-N] = 0$, then

$$\mathcal{Y}(z) = \mathcal{Y}_{ZICR}(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \mathcal{X}(z),$$

which depends only on the input $x[n]$. The inverse z-transform is the "zero-initial-condition response" $y_{ZICR}[n]$.

In general we have $\mathcal{Y}(z) = \mathcal{Y}_{ZIR}(z) + \mathcal{Y}_{ZICR}(z)$. Since the z-transform is linear, this implies $y[n] = y_{ZIR}[n] + y_{ZICR}[n]$, as was to have been shown.

C. When the initial conditions are all zero, $\mathcal{Y}(z) = \mathcal{Y}_{ZICR}(z)$ as shown in part B. Applying the bilateral z-transform to the difference equation gives

$$Y(z) = -\sum_{k=1}^{N} \left(\frac{a_k}{a_0}\right) z^{-k} Y(z) + \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) z^{-k} X(z).$$

Solving for $Y(z)$ gives

$$Y(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} X(z),$$

Now if $x[n] = 0$ for $n < 0$, then $X(z)$ is identical to $\mathcal{X}(z)$. In this case we have $Y(z) = \mathcal{Y}_{ZICR}(z)$.

4.1.

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(2\pi(100)n\frac{1}{400}\right) \\ &= \sin\left(\frac{\pi}{2}n\right) \end{aligned}$$

4.2. The discrete-time sequence

$$x[n] = \cos(\frac{\pi n}{4})$$

results by sampling the continuous-time signal

$$x_c(t) = \cos(\Omega_o t).$$

Since $\omega = \Omega T$ and $T = 1/1000$ seconds, the signal frequency could be:

$$\Omega_o = \frac{\pi}{4} \cdot 1000 = 250\pi$$

or possibly:

$$\Omega_o = (2\pi + \frac{\pi}{4}) \cdot 1000 = 2250\pi.$$

4.3. (a) Since $x[n] = x_c(nT)$,

$$\begin{aligned} \frac{\pi n}{3} &= 4000\pi n T \\ T &= \frac{1}{12000} \end{aligned}$$

(b) No. For example, since

$$\cos(\frac{\pi}{3}n) = \cos(\frac{7\pi}{3}n),$$

T can be 7/12000.

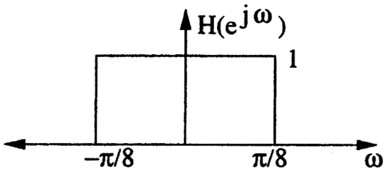
4.4. (a) Letting $T = 1/100$ gives

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(20\pi n \frac{1}{100}\right) + \cos\left(40\pi n \frac{1}{100}\right) \\ &= \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right) \end{aligned}$$

(b) No, another choice is $T = 11/100$:

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \sin\left(20\pi n \frac{11}{100}\right) + \cos\left(40\pi n \frac{11}{100}\right) \\ &= \sin\left(\frac{11\pi n}{5}\right) + \cos\left(\frac{22\pi n}{5}\right) \\ &= \sin\left(\frac{\pi n}{5}\right) + \cos\left(\frac{2\pi n}{5}\right) \end{aligned}$$

4.5. A plot of $H(e^{j\omega})$ appears below.



(a)

$$x_c(t) = 0, \quad , |\Omega| \geq 2\pi \cdot 5000$$

The Nyquist rate is 2 times the highest frequency. $\Rightarrow T = \frac{1}{10{,}000}$ sec. This avoids all aliasing in the C/D converter.

(b)

$$\begin{aligned} \frac{1}{T} &= 10kHz \\ \omega &= T\Omega \\ \frac{\pi}{8} &= \frac{1}{10,000}\Omega_c \\ \Omega_c &= 2\pi \cdot 625\text{rad/sec} \\ f_c &= 625Hz \end{aligned}$$

(c)

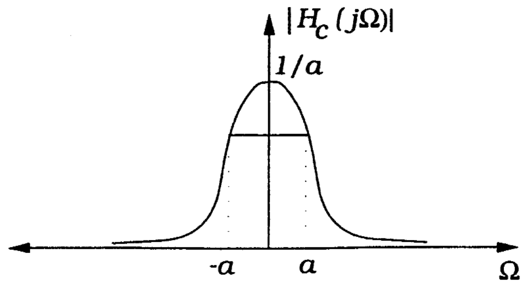
$$\begin{aligned} \frac{1}{T} &= 20kHz \\ \omega &= T\Omega \\ \frac{\pi}{8} &= \frac{1}{20,000}\Omega_c \\ \Omega_c &= 2\pi \cdot 1250\text{rad/sec} \\ f_c &= 1250Hz \end{aligned}$$

4.6. (a) The Fourier transform of the filter impulse response

$$\begin{aligned} H_c(j\Omega) &= \int_{-\infty}^{\infty} h_c(t)e^{-j\Omega t}\,dt \\ &= \int_0^{\infty} a^{-at}e^{-j\Omega t}\,dt \\ &= \frac{1}{a+j\Omega} \end{aligned}$$

So, we take the magnitude

$$|H_c(j\Omega)| = \left(\frac{1}{a^2+\Omega^2}\right)^{\frac{1}{2}}$$



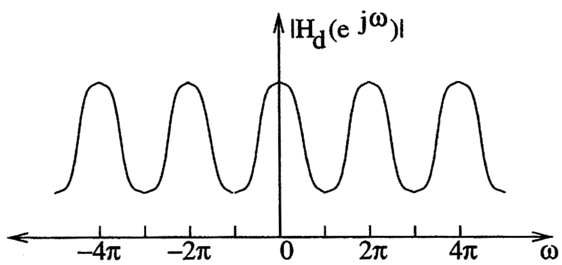
(b) Sampling the filter impulse response in (a), the discrete-time filter is described by

$$\begin{aligned} h_d[n] &= Te^{-anT}u[n] \\ H_d(e^{j\omega}) &= \sum_{n=0}^{\infty} Te^{-anT}e^{-j\omega n} \\ &= \frac{T}{1-e^{-aT}e^{-j\omega}} \end{aligned}$$

Taking the magnitude of this response

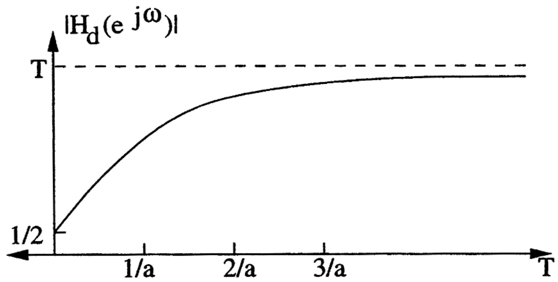
$$|H_d(e^{j\omega})| = \frac{T}{(1-2e^{-aT}\cos(\omega)+e^{-2aT})^{\frac{1}{2}}}.$$

Note that the frequency response of the discrete-time filter is periodic, with period 2π.



(c) The minimum occurs at $\omega = \pi$. The corresponding value of the frequency response magnitude is

$$\begin{aligned} |H_d(e^{j\pi})| &= \frac{T}{(1+2e^{-aT}+e^{-2aT})^{\frac{1}{2}}} \\ &= \frac{T}{1+e^{-aT}}. \end{aligned}$$



4.7. The continuous-time signal contains an attenuated replica of the original signal with a delay of τ_d.

$$x_c(t) = s_c(t) + \alpha s_c(t - \tau_d)$$

(a) Taking the Fourier transform of the analog signal:

$$X_c(j\Omega) = S_c(j\Omega) \cdot (1 + \alpha e^{-j\tau_d \Omega})$$

Note that $X_c(j\Omega)$ is zero for $|\Omega| > \pi/T$. Sampling the continuous-time signal yields the discrete-time sequence, $x[n]$. The Fourier transform of the sequence is

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{T}\sum_{r=-\infty}^{\infty} S_c\left(\frac{j\omega}{T} + j\frac{2\pi r}{T}\right) \\ &+ \frac{\alpha}{T}\sum_{r=-\infty}^{\infty} S_c\left(\frac{j\omega}{T} + j\frac{2\pi r}{T}\right) e^{-j\tau_d\left(\frac{\omega}{T} + \frac{2\pi r}{T}\right)} \end{aligned}$$

(b) The desired response:

$$H(j\Omega) = \begin{cases} 1 + \alpha e^{-j\tau_d\Omega}, & \text{for } |\Omega| \le \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

Using $\omega = \Omega T$, we obtain a discrete-time system which simulates the above response:

$$H(e^{j\omega}) = 1 + \alpha e^{-j\frac{\tau_d \omega}{T}}$$

(c) We need to take the inverse Fourier transform of the discrete-time impulse response of part (b).

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (1 + \alpha e^{-j\frac{\tau_d\omega}{T}}) e^{j\omega n}\, d\omega \end{aligned}$$

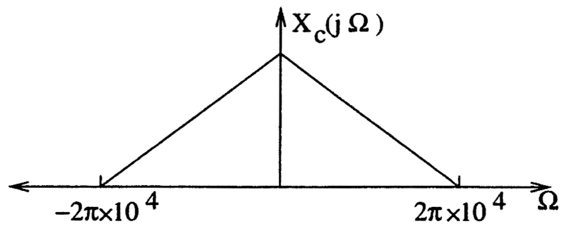
(i) Consider the case when $\tau_d = T$:

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (e^{j\omega n} + \alpha e^{j\omega(n-1)})\, d\omega \\ &= \frac{\sin(\pi n)}{\pi n} + \frac{\alpha \sin[\pi(n-1)]}{\pi(n-1)} \\ &= \delta[n] + \alpha\delta[n-1] \end{aligned}$$

(ii) For $\tau_d = T/2$:

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (e^{j\omega n} + \alpha e^{j\omega(n-\frac{1}{2})})\, d\omega \\ &= \frac{\sin(\pi n)}{\pi n} + \frac{\alpha \sin[\pi(n-\frac{1}{2})]}{\pi(n-\frac{1}{2})} \\ &= \delta[n] + \frac{\alpha \sin[\pi(n-\frac{1}{2})]}{\pi(n-\frac{1}{2})} \end{aligned}$$

4.8. A plot of $X_c(j\Omega)$ appears below.



(a) For $x_c(t)$ to be recoverable from $x[n]$, the transform of the discrete signal must have no aliasing. When sampling, the radian frequency is related to the analog frequency by

$$\omega = \Omega T.$$

No aliasing will occur if the sampling interval satisfies the Nyquist Criterion. Thus, for the band-limited signal, $x_c(t)$, we should select T as:

$$T \leq \frac{1}{2 \times 10^4}.$$

(b) Assuming that the system is linear and time-invariant, the convolution sum describes the input-output relationship.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

We are given

$$\begin{aligned} y[n] &= T \sum_{k=-\infty}^{n} x[k] \\ &= T \sum_{k=-\infty}^{\infty} x[k]u[n-k] \end{aligned}$$

Hence, we may infer that the impulse response of the system

$$h[n] = T \cdot u[n].$$

(c) We use the expression for $y[n]$ as given and examine the limit

$$\begin{aligned} \lim_{n\to\infty} y[n] &= \lim_{n\to\infty} T \cdot \sum_{k=-\infty}^{n} x[k] \\ &= T \cdot \sum_{k=-\infty}^{\infty} x[k] \end{aligned}$$

Recall the analysis equation for the Fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Hence,

$$\lim_{n\to\infty} y[n] = T \cdot X(e^{j\omega})|_{\omega=0}$$

(d) We use the result from part (c). Noting that

$$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c(\frac{j\omega}{T} + \frac{j2\pi r}{T}).$$

Thus, we have

$$T \cdot X(e^{j\omega})|_{\omega=0} = \sum_{r=-\infty}^{\infty} X_c(\frac{j2\pi r}{T})$$

From the given information, we seek a value of T such that:

$$\begin{aligned}\sum_{r=-\infty}^{\infty} X_c(\frac{j2\pi r}{T}) &= \int_{-\infty}^{\infty} x_c(t)\,dt \\ &= X_c(j\Omega)|_{\Omega=0}\end{aligned}$$

For the final equality to be true, there must be no contribution from the terms for which $r \neq 0$. That is, we require no aliasing at $\Omega = 0$. Since we are only interested in preserving the spectral component at $\Omega = 0$, we may sample at a rate which is lower than the Nyquist rate. The maximum value of T to satisfy these conditions is

$$T \leq \frac{1}{1 \times 10^4}.$$

4.9. (a) Since $X(e^{j\omega}) = X(e^{j(\omega-\pi)})$, $X(e^{j\omega})$ is periodic with period π.

(b) Using the inverse DTFT,

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j(\omega-\pi)})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j\omega})e^{j(\omega+\pi)n}d\omega \\
&= \frac{1}{2\pi}e^{j\pi n}\int_{\langle 2\pi\rangle} X(e^{j\omega})e^{j\omega n}d\omega \\
&= (-1)^n x[n].
\end{aligned}$$

All odd samples of $x[n] = 0$, because $x[n] = -x[n]$. Hence $x[3] = 0$.

(c) Yes, $y[n]$ contains all even samples of $x[n]$, and all odd samples of $x[n]$ are 0.

$$x[n] = \begin{cases} y[n/2], & n \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

4.10. Use $x[n] = x_c(nT)$, and simplify:

(a) $x[n] = \cos(2\pi n/3)$.

(b) $x[n] = \sin(4\pi n/3) = -\sin(2\pi n/3)$

(c) $x[n] = \frac{\sin(2\pi n/5)}{\pi n/5000}$

4.11. (a) Pick T such that

$$x[n] = x_c(nT) = \sin(10\pi nT) = \sin(\pi n/4) \qquad \Longrightarrow T = 1/40$$

There are other choices. For example, by realizing that $\sin(\pi n/4) = \sin(9\pi n/4)$, we find $T = 9/40$.

(b) Choose $T = 1/20$ to make $x[n] = x_c(nT)$. This is unique.

4.12. (a) Notice first that $H(e^{j\omega}) = 10j\omega, -\pi \le \omega < \pi$.

(i) After sampling,

$$
\begin{aligned}
x[n] &= \cos(\frac{3\pi}{5}n), \\
y[n] &= |H(e^{j\frac{3\pi}{5}})|\cos(\frac{3\pi}{5}n + \angle H(e^{j\frac{3\pi}{5}})) \\
&= 6\pi\cos(\frac{3\pi}{5}n + \frac{\pi}{2}) \\
&= -6\pi\sin(\frac{3\pi}{5}n) \\
y_c(t) &= -6\pi\sin(6\pi t).
\end{aligned}
$$

(ii) After sampling, $x[n] = \cos(\frac{7\pi}{5}n) = \cos(\frac{3\pi}{5}n)$, so again, $y_c(t) = -6\pi\sin(6\pi t)$.

(b) $y_c(t)$ is what you would expect from a differentiator in the first case but not in the second case. This is because aliasing has occurred in the second case.

4.13. (a)

$$\begin{aligned} x_c(t) &= \sin(\frac{\pi}{20}t) \\ y_c(t) &= \sin(\frac{\pi}{20}(t-5)) \\ &= \sin(\frac{\pi}{20}t - \frac{\pi}{4}) \\ y[n] &= \sin(\frac{\pi n}{2} - \frac{\pi}{4}) \end{aligned}$$

(b) We get the same result as before:

$$\begin{aligned} x_c(t) &= \sin(\frac{\pi}{10}t) \\ y_c(t) &= \sin(\frac{\pi}{10}(t-2.5)) \\ &= \sin(\frac{\pi}{10}t - \frac{\pi}{4}) \\ y[n] &= \sin(\frac{\pi n}{2} - \frac{\pi}{4}) \end{aligned}$$

(c) The sampling period T is not limited by the continuous time system $h_c(t)$.

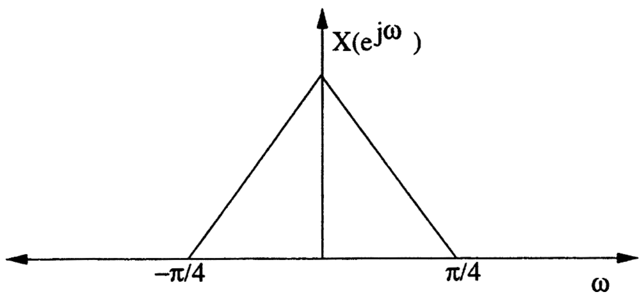
4.14. There is no loss of information if $X(e^{j\omega/2})$ and $X(e^{j(\omega/2-\pi)})$ do not overlap. This is true for (b), (d), (e).

4.15. The output $x_r[n] = x[n]$ if no aliasing occurs as result of downsampling. That is, $X(e^{j\omega}) = 0$ for $\pi/3 \leq |\omega| \leq \pi$.

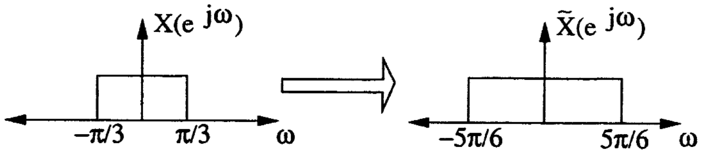
(a) $x[n] = \cos(\pi n/4)$. $X(e^{j\omega})$ has impulses at $\omega = \pm\pi/4$, so there is no aliasing. $x_r[n] = x[n]$.

(b) $x[n] = \cos(\pi n/2)$. $X(e^{j\omega})$ has impulses at $\omega = \pm\pi/2$, so there is aliasing. $x_r[n] \neq x[n]$.

(c) A sketch of $X(e^{j\omega})$ is shown below. Clearly there will be no aliasing and $x_r[n] = x[n]$.



4.16. (a) In the frequency domain, we have



$$\frac{M}{L} = \frac{5\pi/6}{\pi/3} = \frac{5}{2}$$

This is unique.

(b) One choice is

$$\frac{M}{L} = \frac{\pi/2}{3\pi/4} = \frac{2}{3}$$

However, this is not unique. We can also write $\tilde{x}_d[n] = \cos(\frac{5\pi}{2}n)$, so another choice is

$$\frac{M}{L} = \frac{5\pi/2}{3\pi/4} = \frac{10}{3}$$

4.17. (a) In the frequency domain,

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < 2\pi/3 \\ 0, & 2\pi/3 < |\omega| < \pi \end{cases}$$

After the sampling rate change,

$$\tilde{X}_d(e^{j\omega}) = \begin{cases} 4/3, & |\omega| < \pi/2 \\ 0, & \pi/2 < |\omega| < \pi \end{cases},$$

which leads to

$$x[n] = \frac{4}{3}\frac{\sin(\pi n/2)}{\pi n}$$

(b) Upsampling by 3 and low-pass filtering $x[n] = \sin(3\pi n/4)$ results in $\sin(\pi n/4)$. Downsampling by 5 gives us $\tilde{x}_d[n] = \sin(5\pi n/4) = -\sin(3\pi n/4)$.

4.18. For the condition to be satisfied, we have to ensure that $\omega_0/L \le \min(\pi/L, \pi/M)$, so that the lowpass filtering does not cut out part of the spectrum.

(a) $\omega_0/2 \le \pi/3 \Longrightarrow \omega_{0,max} = 2\pi/3$.

(b) $\omega_0/3 \le \pi/5 \Longrightarrow \omega_{0,max} = 3\pi/5$.

(c) Since $L > M$, there is no chance of aliasing. Hence $\omega_{0,max} = \pi$.

4.19. The nyquist sampling property must be satisfied: $T \leq \pi/\Omega_0$.

4.20. (a) The Nyquist sampling property must be satisfied: $T \leq \pi/\Omega_0 \Longrightarrow F_s \geq 2000$.

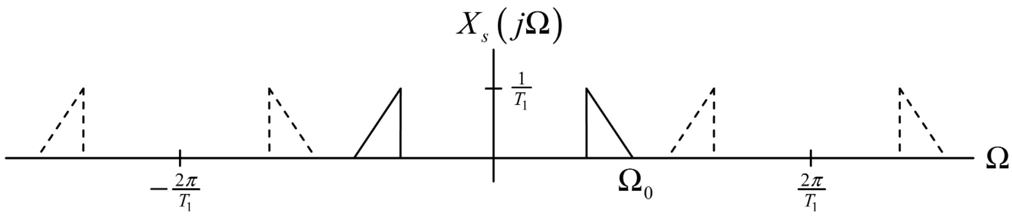
(b) We'd have to sample so that $X(e^{j\omega})$ lies between $|\omega| < \pi/2$. So $F_s \geq 4000$.

4.21

A. The impulse-train signal $x_s(t)$ has spectrum $X_s(j\Omega)$ given by

$$X_s(j\Omega) = \tfrac{1}{T}\sum_{k=-\infty}^{\infty} X\left[j\left(\Omega - k\tfrac{2\pi}{T_1}\right)\right].$$

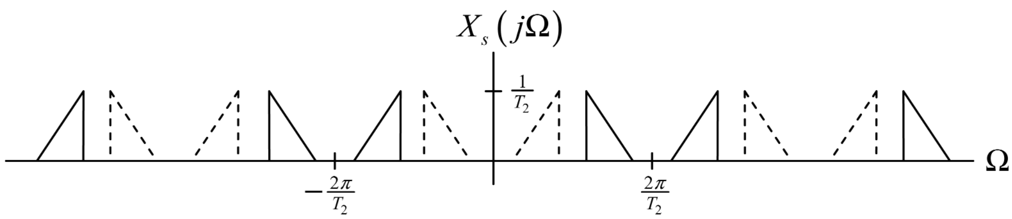
An example is shown below.



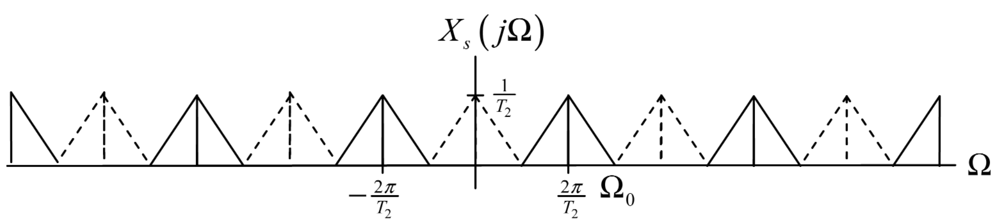
We will have $x_r(t) = x_c(t)$ provided $T_1 \le \frac{\pi}{\Omega_0}$.

B. We will have $x_o(t) = x_c(t)$ under any of the following circumstances:

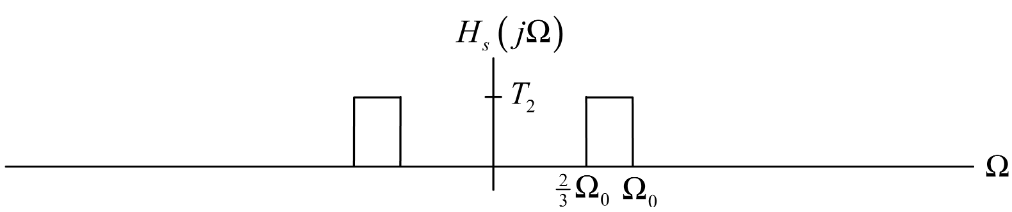
1. As illustrated above, $T_2 \le \frac{\pi}{\Omega_0}$.
2. As illustrated below, $\frac{1.5\pi}{\Omega_0} \le T_2 \le \frac{2\pi}{\Omega_0}$.



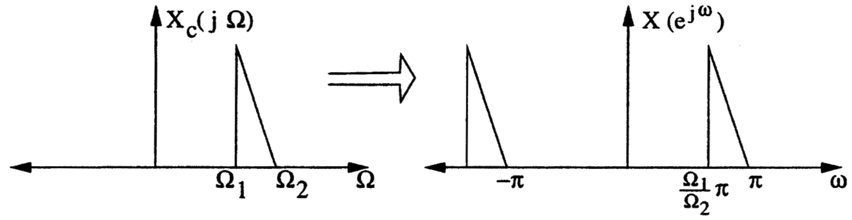
3. As illustrated below, $T_2 = \frac{3\pi}{\Omega_0}$.



The frequency response of the filter that is needed to recover $x_c(t)$ is shown below.

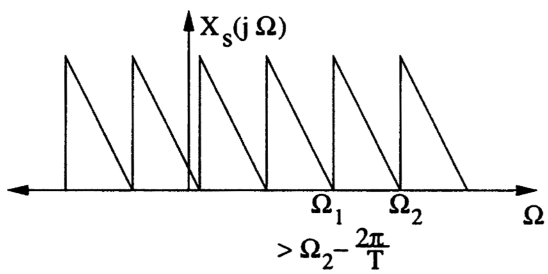


4.22. (a) Keeping in mind that after sampling, $\omega = \Omega T$, the Fourier transform of $x[n]$ is

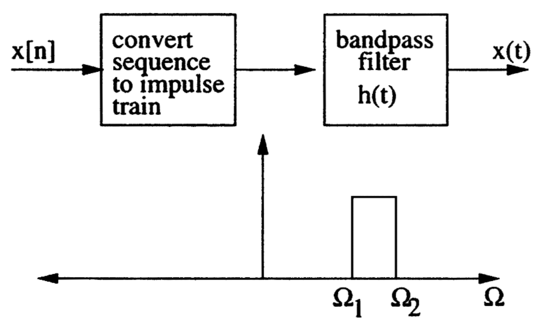


(b) A straight-forward application of the Nyquist criterion would lead to an incorrect conclusion that the sampling rate is at least twice the maximum frequency of $x_c(t)$, or $2\Omega_2$. However, since the spectrum is bandpass, we only need to ensure that the replications in frequency which occur as a result of sampling do not overlap with the original. (See the following figure of $X_s(j\Omega)$.) Therefore, we only need to ensure

$$\Omega_2 - \frac{2\pi}{T} < \Omega_1 \Longrightarrow T < \frac{2\pi}{\Delta\Omega}$$

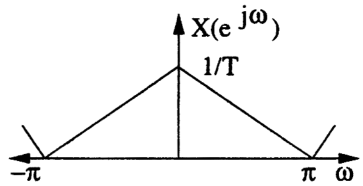


(c) The block diagram along with the frequency response of $h(t)$ is shown here:

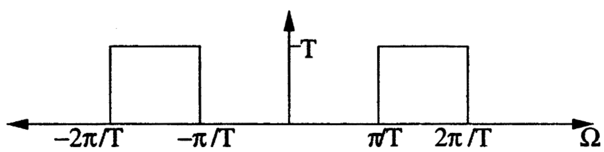


4.23. (a)

$$\omega = \Omega T, \qquad T = \frac{2\pi}{\Omega_0}$$



(b) To recover simply filter out the undesired parts of $X(e^{j\omega})$.



(c)

$$T \leq \frac{2\pi}{\Omega_0}$$

4.24

(a) Given

$$x[n] = \cos(\omega_0 n), \quad \omega_0 = \Omega_0 T < \pi,$$

we have from Table 2.3,

$$X(e^{j\omega}) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0), \quad |\omega| < \pi.$$

(b) Eq. (4.46) gives $H(e^{j\omega}) = \dfrac{j\omega}{T}, \quad |\omega| < \pi$. Then

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) \\ &= \frac{j\omega}{T}\left[\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right] \\ &= \frac{j\omega_0}{T}\pi\delta(\omega-\omega_0) - \frac{j\omega_0}{T}\pi\delta(\omega+\omega_0), \quad |\omega| < \pi. \end{aligned}$$

(c) From Eq. (4.32),

$$\begin{aligned} Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T}) &= \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \dfrac{\pi}{T} \\ 0, & \text{otherwise} \end{cases} \\ &= j\omega_0\pi\delta(\Omega T - \omega_0) - j\omega_0\pi\delta(\Omega T + \omega_0) \\ &= j\omega_0\pi\delta\left[(\Omega - \omega_0/T)T\right] - j\omega_0\pi\delta\left[(\Omega + \omega_0/T)T\right]. \end{aligned}$$

(d) The inverse Fourier transform of $\delta(\Omega T)$ is the constant $1/(2\pi T)$. We then have

$$\begin{aligned} y_r(t) &= \frac{j\omega_0}{2T}e^{j\omega_0 t/T} - \frac{j\omega_0}{2T}e^{-j\omega_0 t/T} \\ &= -\Omega_0\left(\frac{e^{j\omega_0 t/T} - e^{-j\omega_0 t/T}}{j2}\right) \\ &= -\Omega_0 \sin(\Omega_0 t), \end{aligned}$$

as was to have been shown.

4.25. Appears in: Fall04 PS1, Fall02 PS1.

Problem

Note that in OSB and 6.341 Ω denotes continuous-time frequency and ω denotes discrete-time frequency.

Figure 1 shows a continuous-time filter that is implemented using an LTI discrete-time filter with frequency response $H(e^{j\omega})$.

(a) If the CTFT of $x_c(t)$, namely $X_c(j\Omega)$, is as shown in Figure 2 and $\omega_c = \frac{\pi}{5}$, sketch and label $X(e^{j\omega})$, $Y(e^{j\omega})$ and $Y_c(j\Omega)$ for each of the following cases:

(i) $1/T_1 = 1/T_2 = 2 \times 10^4$

(ii) $1/T_1 = 4 \times 10^4, \;\; 1/T_2 = 10^4$

(iii) $1/T_1 = 10^4, \;\; 1/T_2 = 3 \times 10^4$

(b) For $1/T_1 = 1/T_2 = 6 \times 10^3$, and for input signals $x_c(t)$ whose spectra are bandlimited to $|\Omega| < 2\pi \times 5 \times 10^3$ (but otherwise unconstrained), what is the maximum choice of the cutoff frequency ω_c of the filter $H(e^{j\omega})$ for which the overall system is LTI? For this maximum choice of ω_c, specify $H_c(j\Omega)$.

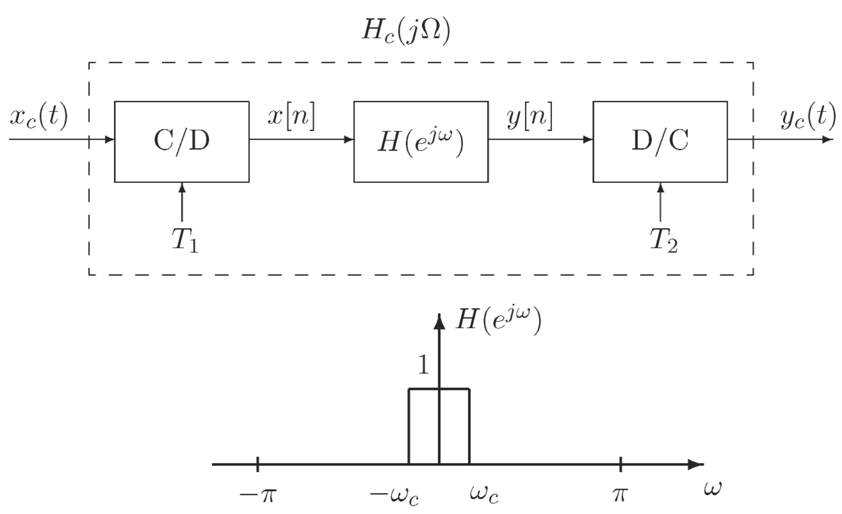


Figure 1: Problem .

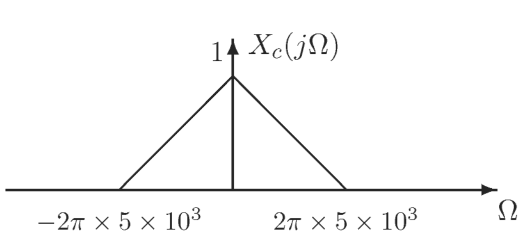
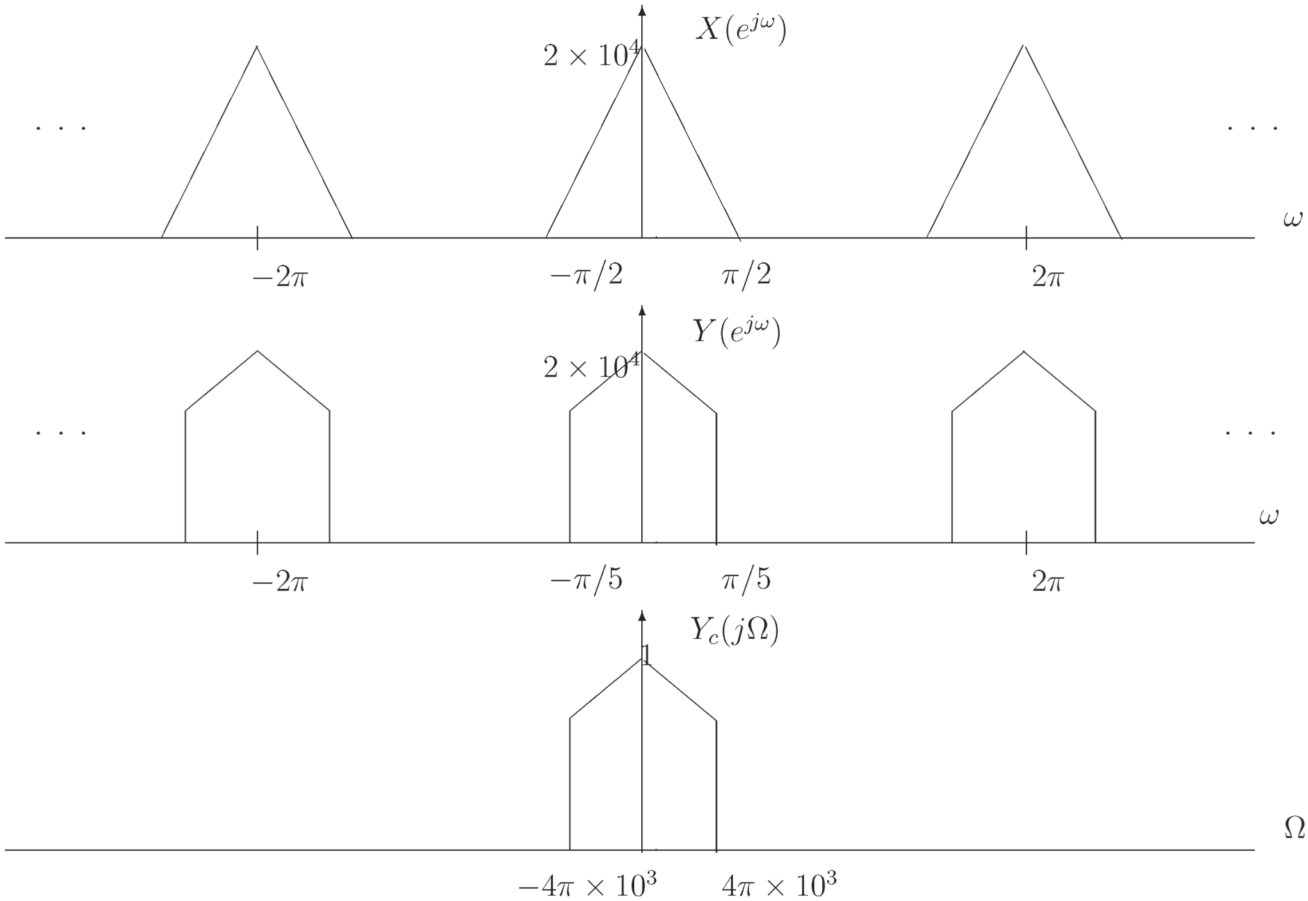


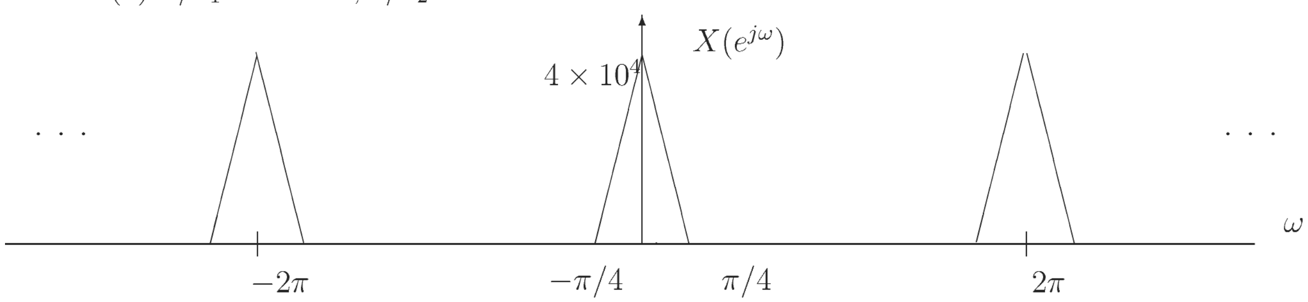
Figure 2: Problem , part (a).

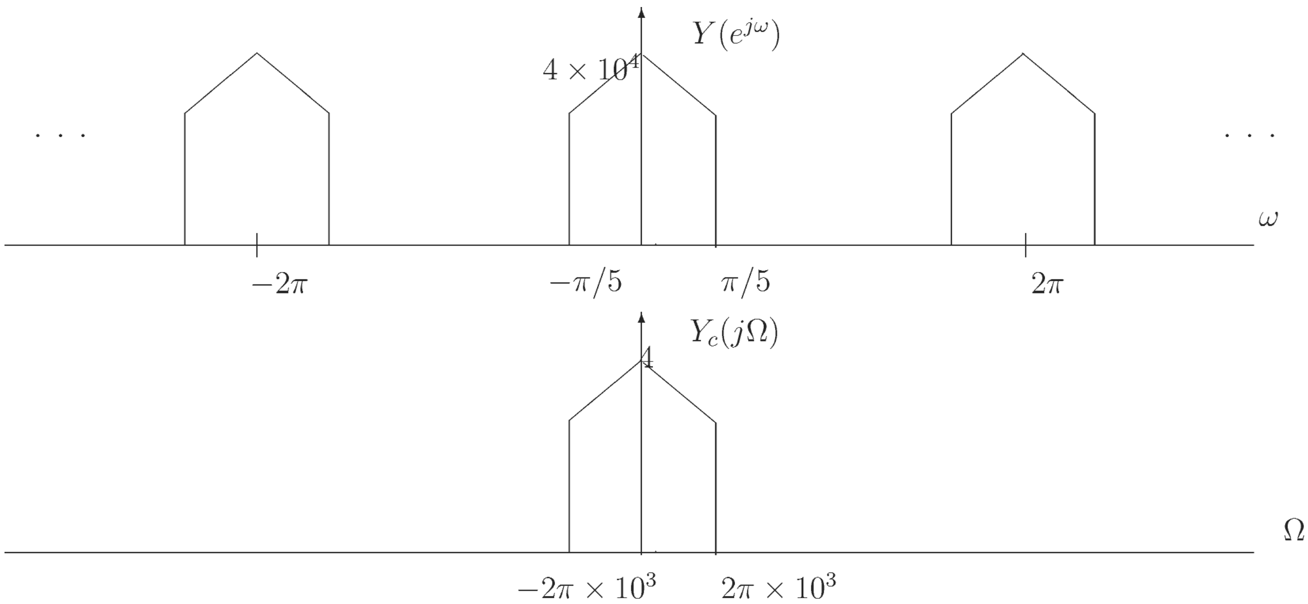
Solution from Fall04 PS1

(a) (i) $1/T_1 = 1/T_2 = 2 \times 10^4$

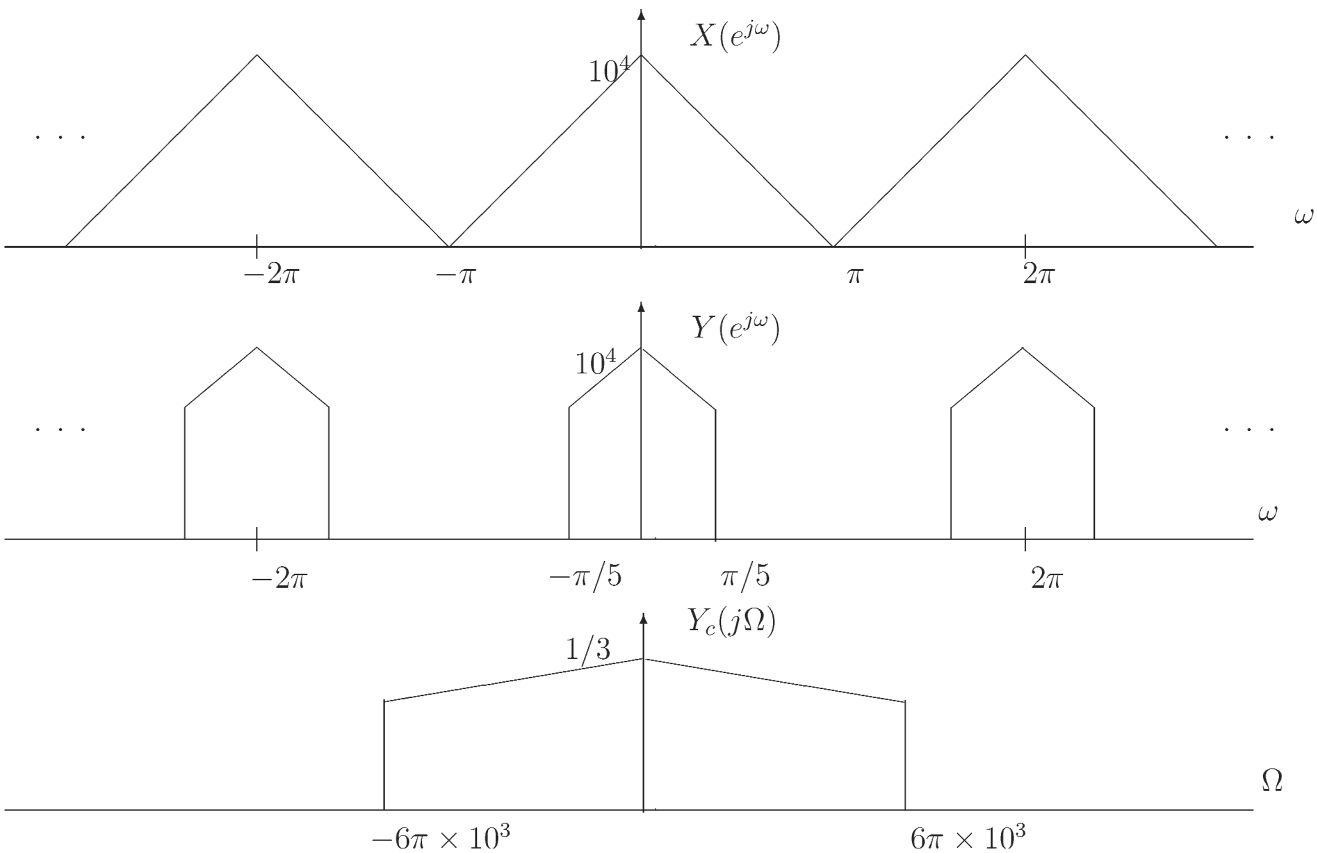


(ii) $1/T_1 = 4 \times 10^4$, $1/T_2 = 10^4$



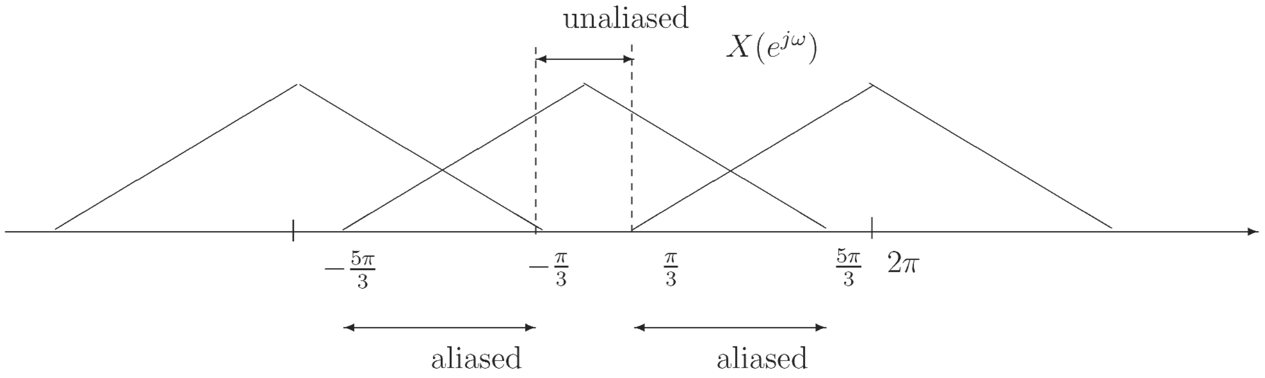


(iii) $1/T_1 = 10^4$, $1/T_2 = 3 \times 10^4$



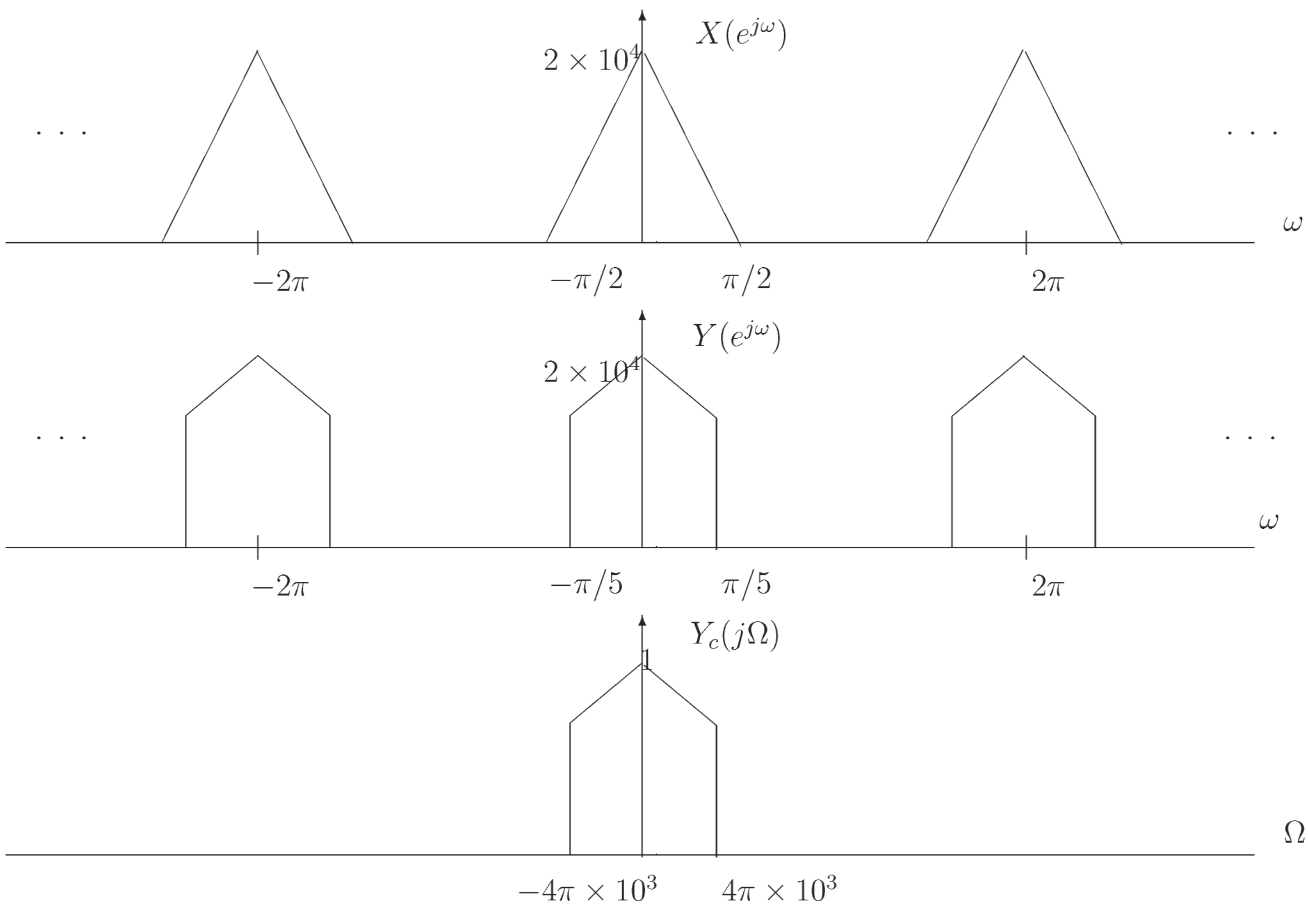
(b) From the figure below, it can be seen that the only portion of the spectrum which remains unaffected by the aliasing is $|\omega| < \pi/3$. So if we choose $\omega_c < \pi/3$, the overall system is LTI with a frequency response of

$$H_c(j\Omega) = \begin{cases} 1 & \text{for } |\Omega| < \omega_c \times 6 \times 10^3 \\ 0 & \text{otherwise.} \end{cases}$$

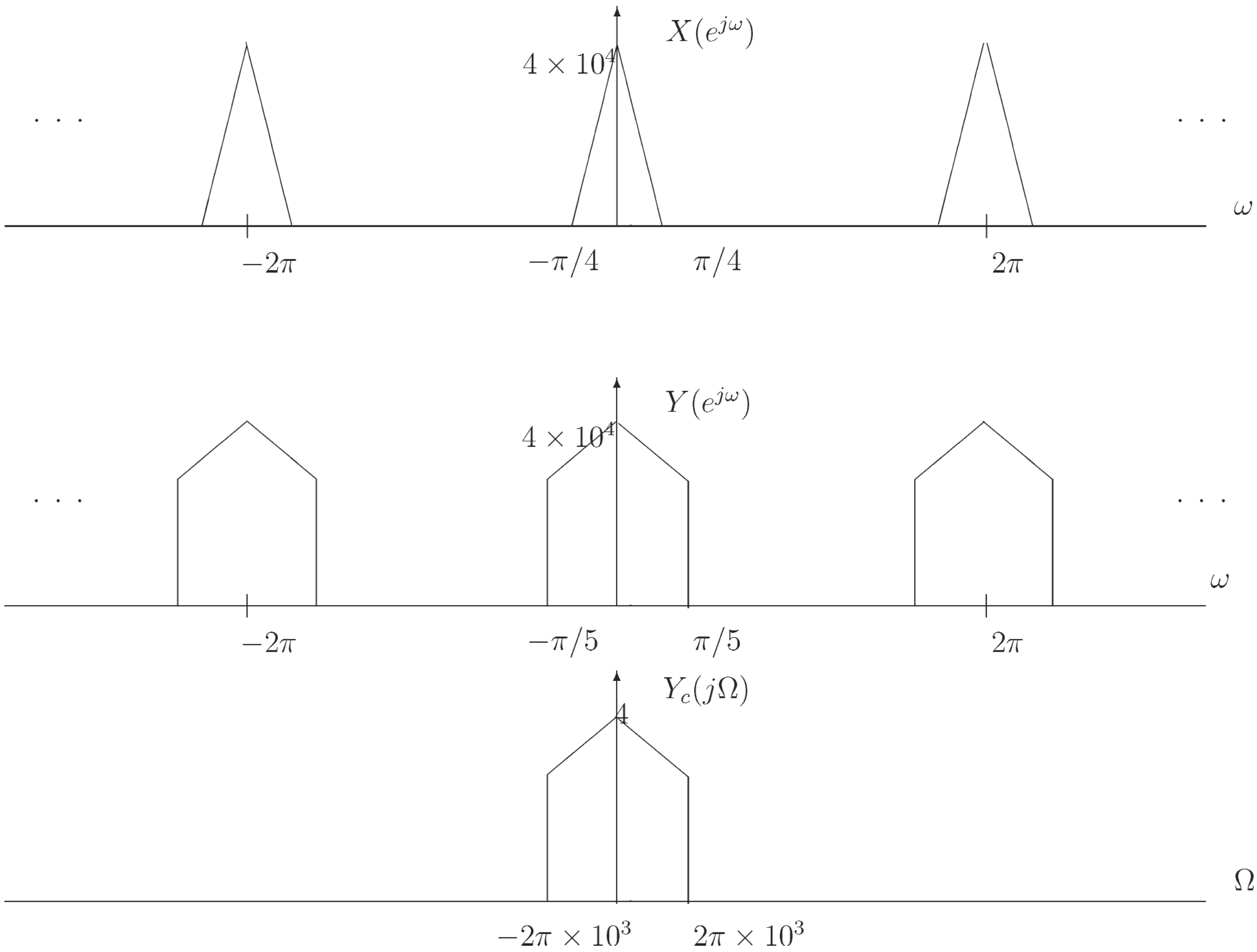


Solution from Fall02 PS1

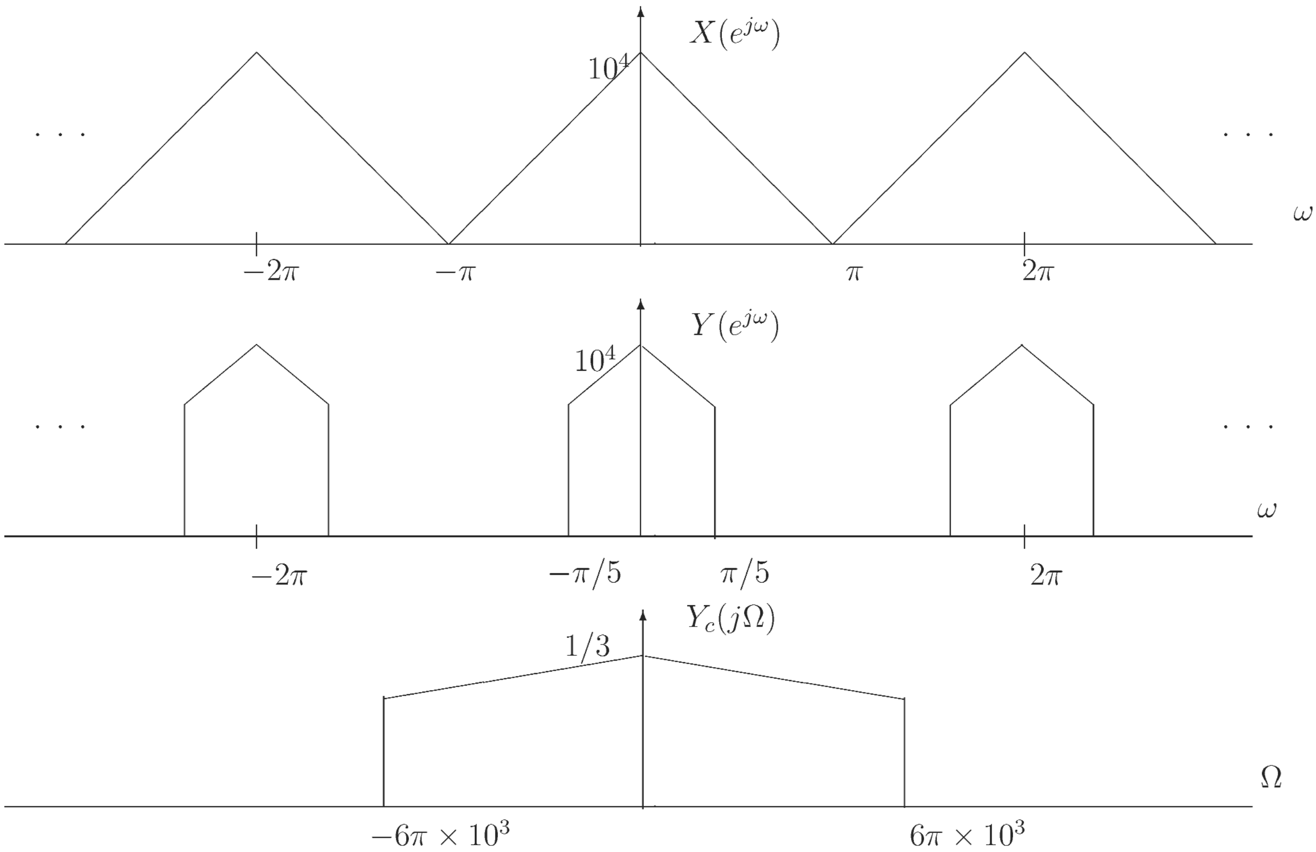
(a) (i) $1/T_1 = 1/T_2 = 2 \times 10^4$



(ii) $1/T_1 = 4 \times 10^4$, $1/T_2 = 10^4$

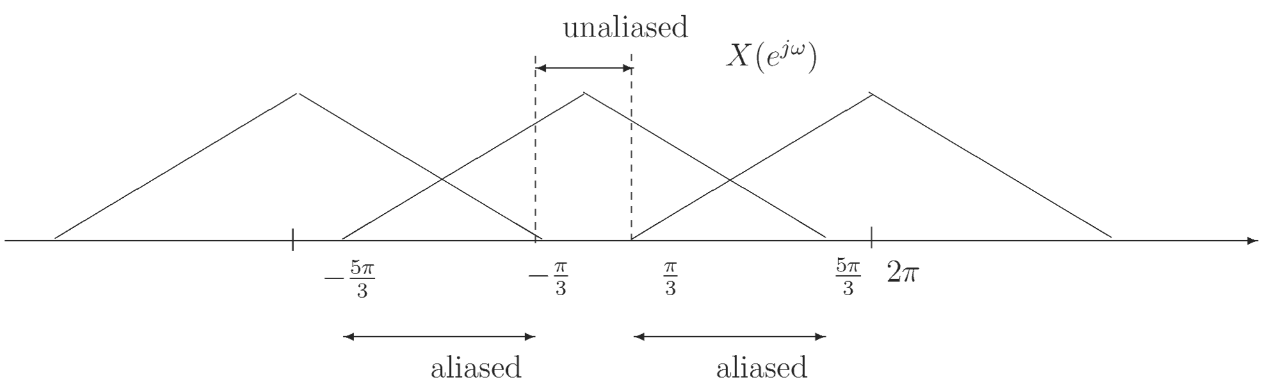


(iii) $1/T_1 = 10^4$, $1/T_2 = 3 \times 10^4$



(b) From the figure below, it can be seen that the only portion of the spectrum which remains unaffected by the aliasing is $|\omega| < \pi/3$. So if we choose $\omega_c < \pi/3$, the overall system is LTI with a frequency response of

$$H_c(j\Omega) = \begin{cases} 1 & \text{for } |\Omega| < \omega_c \times 6 \times 10^3 \\ 0 & \text{otherwise.} \end{cases}$$



4.26

A. The input signal is sampled at a rate high enough to avoid aliasing. Then

$$X_d\left(e^{j\omega}\right)=\frac{1}{T}X_c\left(j\frac{\omega}{T}\right),\quad |\omega|<\pi .$$

Now

$$\begin{aligned}Y_d\left(e^{j\omega}\right)&=H_d\left(e^{j\omega}\right)X_d\left(e^{j\omega}\right)\\&=\frac{1}{T}H_d\left(e^{j\omega}\right)X_c\left(j\frac{\omega}{T}\right),\quad |\omega|<\pi\end{aligned}$$

The D/C converter includes an ideal lowpass filter of bandwidth $\frac{\pi}{T}$ and gain T. Therefore

$$\begin{aligned}Y_c(j\Omega)&=\begin{cases}TY_d\left(e^{j\Omega T}\right), & |\Omega|<\dfrac{\pi}{T}\\ 0, & \text{otherwise}\end{cases}\\&=H_d\left(e^{j\Omega T}\right)X_c(j\Omega).\end{aligned}$$

The continuous-time frequency response of the end-to-end system is given by

$$\begin{aligned}H_c(j\Omega)&=\begin{cases}\dfrac{Y_c(j\Omega)}{X_c(j\Omega)}, & |\Omega|<\dfrac{\pi}{T}\\ 0, & \text{otherwise}\end{cases}\\&=\begin{cases}H_d\left(e^{j\Omega T}\right), & |\Omega|<\dfrac{\pi}{T}\\ 0, & \text{otherwise}\end{cases}\\&=\begin{cases}\dfrac{e^{j\Omega T/2}-e^{-j\Omega T/2}}{T}, & |\Omega|<\dfrac{\pi}{T}\\ 0, & \text{otherwise}\end{cases}\\&=\begin{cases}\dfrac{j2}{T}\sin(\Omega T/2), & |\Omega|<\dfrac{\pi}{T}\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

B. We are given $x_c(t)=\dfrac{\sin(\Omega_M t)}{\Omega_M t}$, with $\Omega_M=\dfrac{\pi}{T}$. Then

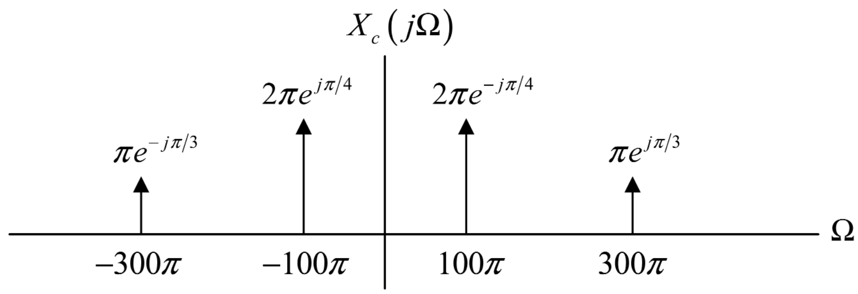
$$\begin{aligned}x_d[n]&=\frac{\sin(\Omega_M nT)}{\Omega_M nT}\\&=\frac{\sin(\pi n)}{\pi n}\\&=0.\end{aligned}$$

But then $y_d[n]=0$ and $y_c(t)=0$ as well.

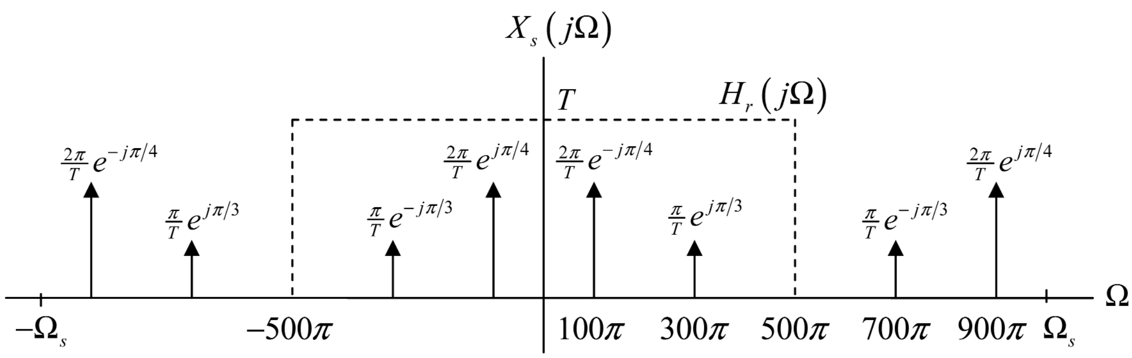
4.27

A.

$$X_c(j\Omega) = 2\pi e^{-j\pi/4}\delta(\Omega - 100\pi) + 2\pi e^{j\pi/4}\delta(\Omega + 100\pi)$$
$$+ \pi e^{j\pi/3}\delta(\Omega - 300\pi) + \pi e^{-j\pi/3}\delta(\Omega + 300\pi).$$



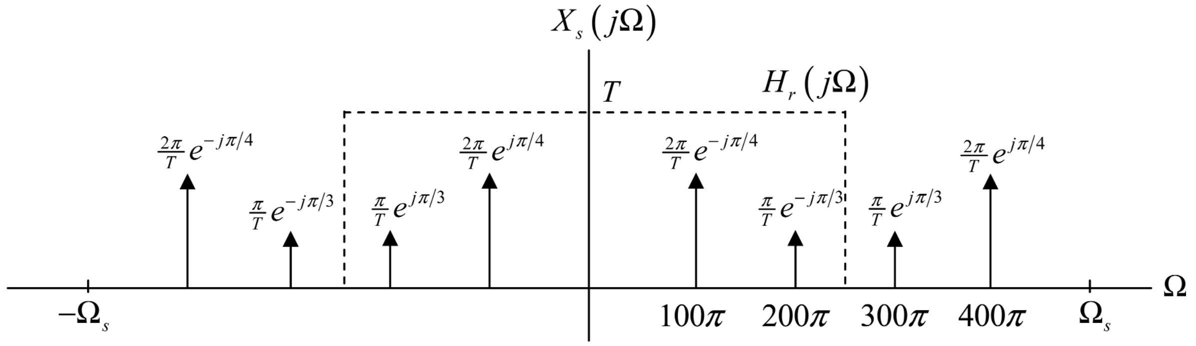
B. If $f_s = 1/T = 500$ samples/s then $\Omega_s = 2\pi/T = 1000\pi$ rad/s.



There is no aliasing, so $x_r(t) = x_c(t)$; that is,

$$x_r(t) = 2\cos(100\pi t - \pi/4) + \cos(300\pi t + \pi/3).$$

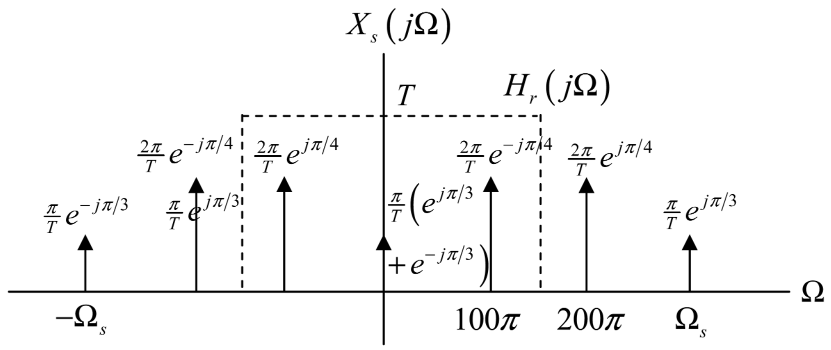
C. If $f_s = 1/T = 250$ samples/s then $\Omega_s = 2\pi/T = 500\pi$ rad/s.



Now there is aliasing and

$$x_r(t) = 2\cos(100\pi t - \pi/4) + \cos(200\pi t - \pi/3).$$

D. We want to sample the component at 300π rad/s exactly once per cycle, so that all the samples have the same value. At $\Omega_s = 300\pi$ rad/s we have

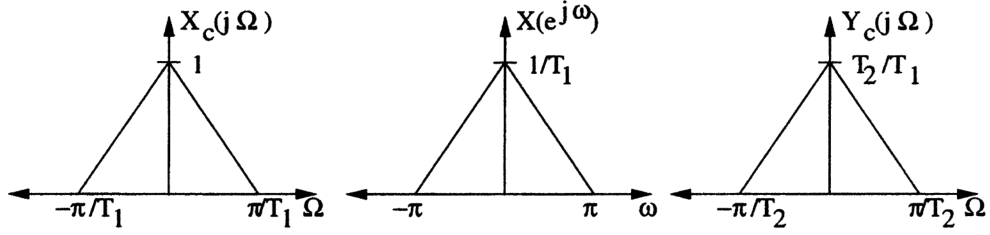


Now

$$x_r(t) = \cos(\pi/3) + 2\cos(100\pi - \pi/4)$$
$$= 1/2 + 2\cos(100\pi - \pi/4).$$

We have $A = 1/2$.

4.28. In the frequency domain, we have



$$x_c(t) = 0, \qquad |\Omega| \geq \frac{\pi}{T_1}$$

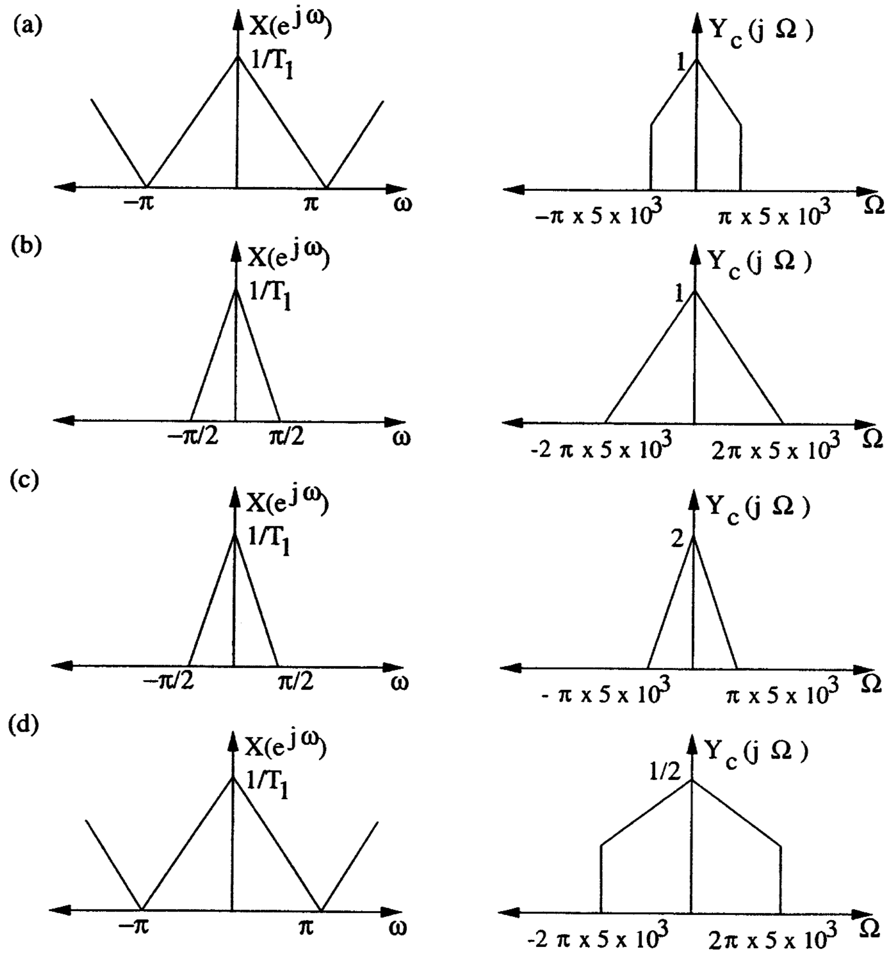
Therefore, since we are sampling this $x_c(t)$ at the Nyquist frequency $x[n]$ will be full band and unaliased.

$$x[n] = x_c(nT_1)$$

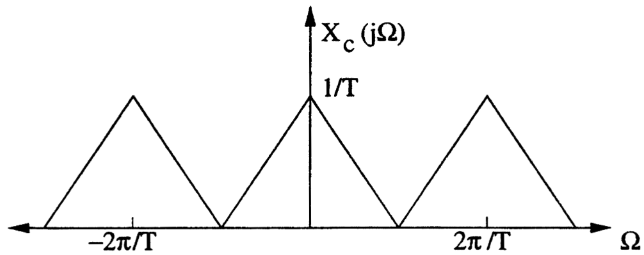
$y_c(t)$ is a band-limited interpolation of $x[n]$ at a different period. Since no aliasing occurs at $x[n]$, the spectrum of $y_c(t)$ will be a frequency axis scaling of the spectrum of $x_c(t)$ for $T_1 > T_2$ or $T_1 < T_2$. As we show in the figure,

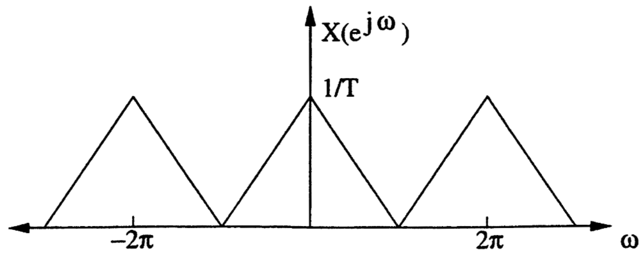
$$y_c(t) = \frac{T_2}{T_1} x_c\left(\frac{T_2}{T_1} t\right)$$

4.29. The Fourier transform of $y_c(t)$ is sketched below for each case.



4.30. (a) $x_s(t) = x_c(t)s(t) \Rightarrow X_s(j\Omega) * s(j\Omega)$





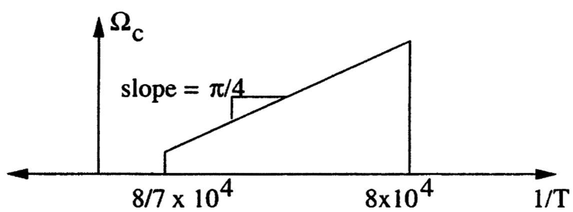
(b) Since $H_d(e^{j\omega})$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$, we don't care about any signal aliasing that occurs in the region $\frac{\pi}{4} \leq \omega \leq \pi$. We require:

$$\begin{aligned} \frac{2\pi}{T} - 2\pi \cdot 10000 &\geq \frac{\pi}{4T} \\ \frac{1}{T} &\geq \frac{8}{7} \cdot 10000 \\ T &\leq \frac{7}{8} \times 10^{-4}\text{sec} \end{aligned}$$

Also, once all of the signal lies in the range $|\omega| \leq \frac{\pi}{4}$, the filter will be ineffective, i.e., $\frac{\pi}{4} \leq T(2\pi \times 10^4)$. So, $T \geq 12.5\mu$sec.

(c)

$$\Omega = \frac{\omega}{T} \Rightarrow \Omega_c = \frac{\pi}{4T}$$



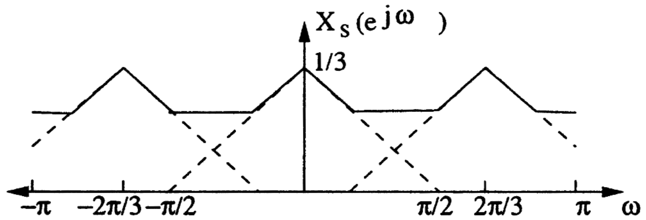
4.31. First we show that $X_s(e^{j\omega})$ is just a sum of shifted versions of $X(e^{j\omega})$:

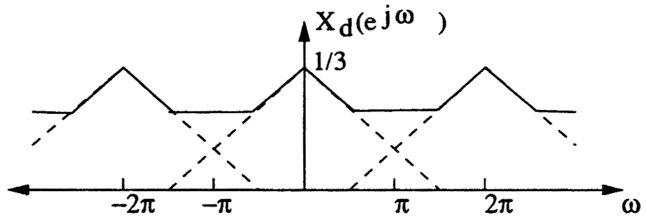
$$\begin{aligned}
x_s[n] &= \begin{cases} x[n], & n = Mk, \quad k = 0, \pm 1, \pm 2 \\ 0, & \text{otherwise} \end{cases} \\
&= \left(\frac{1}{M} \sum_{k=0}^{M-1} e^{j(2\pi kn/M)} \right) x[n] \\
X_s(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_s[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{M} \sum_{k=0}^{M-1} x[n] e^{j(2\pi kn/M)} e^{-j\omega n} \\
&= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{\infty} x[n] e^{-j[\omega - (2\pi k/M)]n} \\
&= \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j[\omega - (2\pi k/M)]} \right)
\end{aligned}$$

Additionally, $X_d(e^{j\omega})$ is simply $X_s(e^{j\omega})$ with the frequency axis expanded by a factor of M:

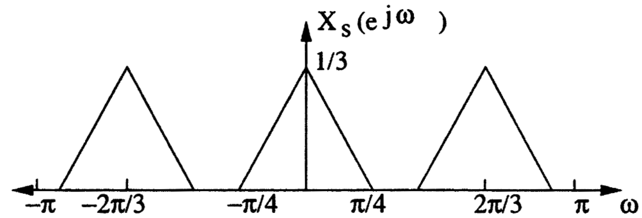
$$\begin{aligned}
X_d(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} X_s[Mn] e^{-j\omega n} \\
&= \sum_{l=-\infty}^{\infty} x_s[l] e^{-j(\omega/M)l} \\
&= X_s\left(e^{j(\omega/M)} \right)
\end{aligned}$$

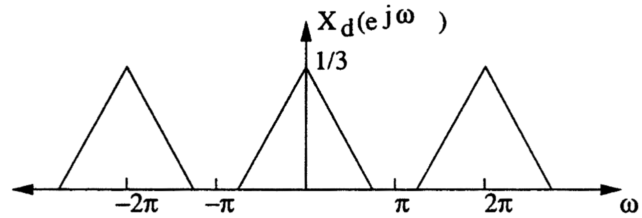
(a) (i) $X_s(e^{j\omega})$ and $X_d(e^{j\omega})$ are sketched below for $M = 3$, $\omega_H = \pi/2$.





(ii) $X_s(e^{j\omega})$ and $X_d(e^{j\omega})$ are sketched below for $M = 3$, $\omega_H = \pi/4$.

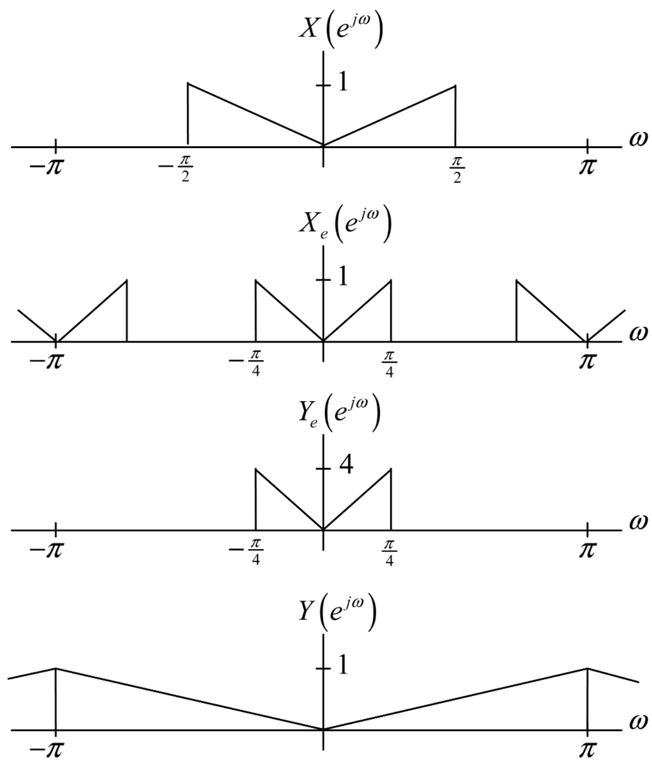




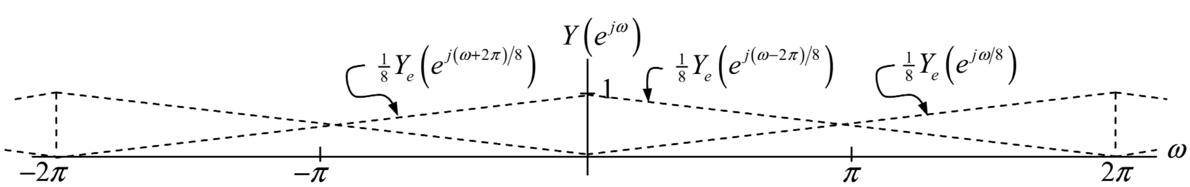
(b) From the definition of $X_s(e^{j\omega})$, we see that there will be no aliasing if the signal is bandlimited to π/M. In this problem, $M = 3$. Thus the maximum value of ω_H is $\pi/3$.

4.32

A. With $M = 2$ and $N = 4$,



B. With $M = 2$ and $N = 8$, $X_e(e^{j\omega})$ and $Y_e(e^{j\omega})$ remain as in part A, except that $Y_e(e^{j\omega})$ now has a peak value of 8. After expanding we have



We see that $Y(e^{j\omega}) = 1$ for all ω. Inverse transforming gives $y[n] = \delta[n]$ in this case.

4.33

$$x[n] \longrightarrow \boxed{\uparrow 5} \longrightarrow \boxed{\downarrow 3} \longrightarrow \boxed{\downarrow 5} \longrightarrow \boxed{\uparrow 3} \longrightarrow y[n]$$

Since 3 and 5 are relatively prime, the order of the two operations in the center can be interchanged. This gives

$$x[n] \longrightarrow \boxed{\uparrow 5} \longrightarrow \boxed{\downarrow 5} \longrightarrow \boxed{\downarrow 3} \longrightarrow \boxed{\uparrow 3} \longrightarrow y[n]$$

Expanding by 5 and immediately compressing by 5 produces no net effect. We have

$$x[n] \longrightarrow \boxed{\downarrow 3} \xrightarrow{x_1[n]} \boxed{\uparrow 3} \longrightarrow y[n]$$

Compressing by 3 produces

$$x_1[n] = x[3n].$$

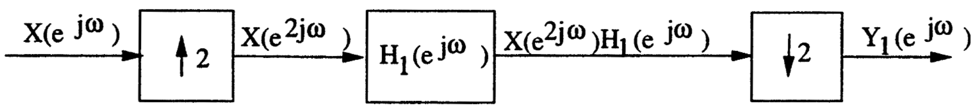
Expanding by 3 now gives

$$y[n] = \begin{cases} x_1[n/3], & n = 3k, \quad k \text{ any integer} \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$y[n] = \begin{cases} x[n], & n = 3k, \quad k \text{ any integer} \\ 0, & \text{otherwise.} \end{cases}$$

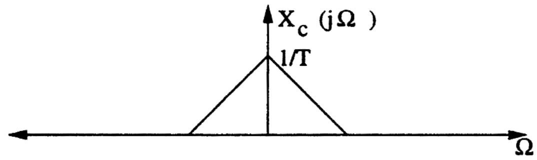
4.34. We can analyze the system in the frequency domain:



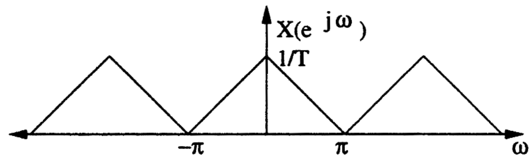
$Y_1(e^{j\omega})$ is $X(e^{2j\omega})H_1(e^{j\omega})$ downsampled by 2:

$$
\begin{aligned}
Y_1(e^{j\omega}) &= \frac{1}{2}\left\{X(e^{2j\omega/2})H_1(e^{j\omega/2}) + X(e^{(2j(\omega-2\pi)/2})H_1(e^{j(\omega-2\pi)/2})\right\} \\
&= \frac{1}{2}\left\{X(e^{j\omega})H_1(e^{j\omega/2}) + X(e^{j(\omega-2\pi)})H_1(e^{j(\frac{\omega}{2}-\pi)})\right\} \\
&= \frac{1}{2}\left\{H_1(e^{j\omega/2}) + H_1(e^{j(\frac{\omega}{2}-\pi)})\right\} X(e^{j\omega}) \\
&= H_2(e^{j\omega})X(e^{j\omega}) \\
H_2(e^{j\omega}) &= \frac{1}{2}\left\{H_1(e^{j\omega/2}) + H_1(e^{j(\frac{\omega}{2}-\pi)})\right\}
\end{aligned}
$$

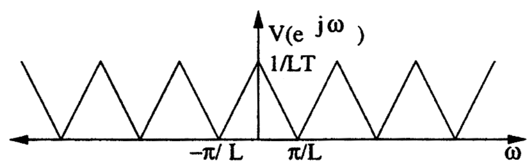
4.35. $X_c(j\Omega)$ is drawn below.



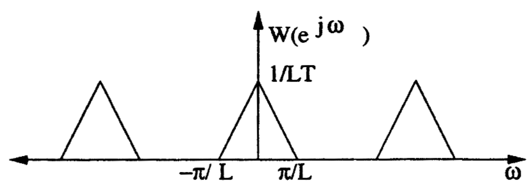
$x_c(t)$ is sampled at sampling period T, so there is no aliasing in $x[n]$.



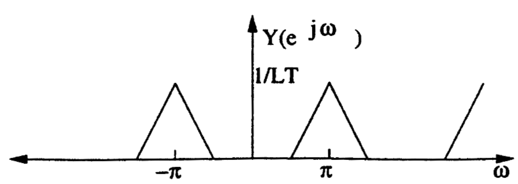
Inserting $L-1$ zeros between samples compresses the frequency axis.



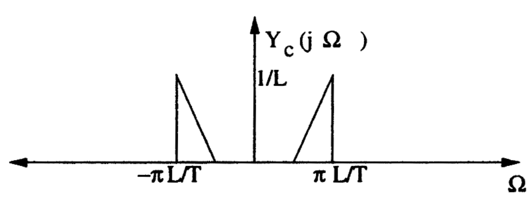
The filter $H(e^{j\omega})$ removes frequency components between π/L and π.



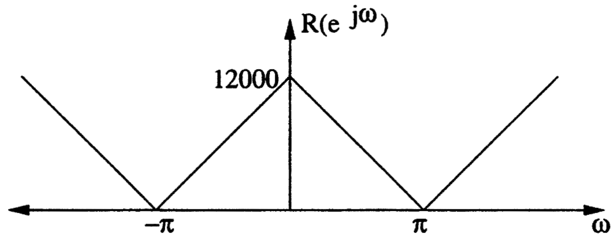
The multiplication by $(-1)^n$ shifts the center of the frequency band from 0 to π.

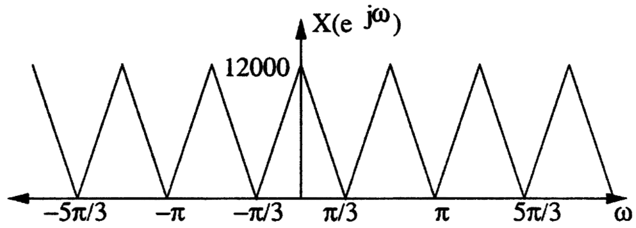


The D/C conversion maps the range $-\pi$ to π to the range $-\pi/T$ to π/T.



4.36. (a) See the following figure:

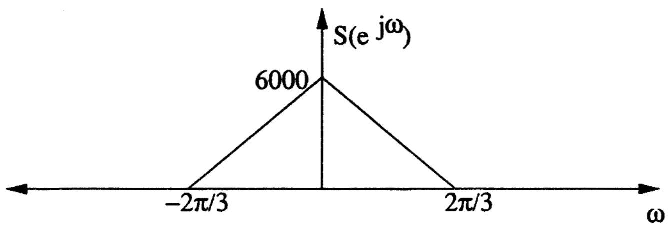




(b) For this to be true, $H(e^{j\omega})$ needs to filter out $X(e^{j\omega})$ for $\pi/3 \le |\omega| \le \pi$. Hence let $\omega_0 = \pi/3$. Furthermore, we want

$$\frac{\pi/2}{T_2} = 2\pi(1000) \Longrightarrow T_2 = 1/6000$$

(c) Matching the following figure of $S(e^{j\omega})$ with the figure for $R_c(j\Omega)$, and remembering that $\Omega = \omega/T$, we get $T_3 = (2\pi/3)/(2000\pi) = 1/3000$.



4.37. Problem 3 in Spring2004 Midterm exam.

Problem

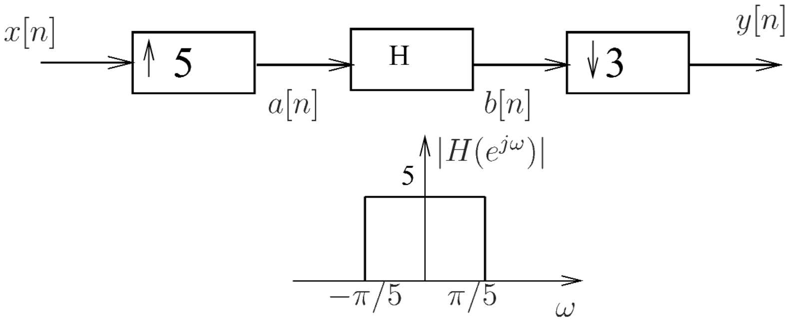
We have a discrete-time signal, $x[n]$, arriving from a source at a rate of $\frac{1}{T_1}$ samples per second. We want to digitally resample it to create a signal $y[n]$ that has $\frac{1}{T_2}$ samples per second, where $T_2 = \frac{3}{5}T_1$.

(a) Draw a block diagram of a discrete-time system to perform the resampling. Specify the input/output relationship for all the boxes in the Fourier domain.

(b) For an input signal $x[n] = \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases}$, determine y[n].

Solution from Spring04 midterm

Part(a):



The input-output relationships for the boxes in the above figure are as follows:

$$A(e^{j\omega}) = X(e^{j5\omega})$$

$$B(e^{j\omega}) = \begin{cases} 5A(e^{j\omega}), & |w| < \pi/5 \\ 0, & \text{otherwise} \end{cases}$$

$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} B(e^{\frac{j\omega - 2\pi k}{3}})$$

Part(b):

$$A(e^{j\omega}) = 1$$

$$B(e^{j\omega}) = \begin{cases} 5, & |w| < \pi/5 \\ 0, & \text{otherwise} \end{cases}$$

$$Y(e^{j\omega}) = \begin{cases} 5/3, & |w| < 3\pi/5 \\ 0, & \text{otherwise} \end{cases}$$

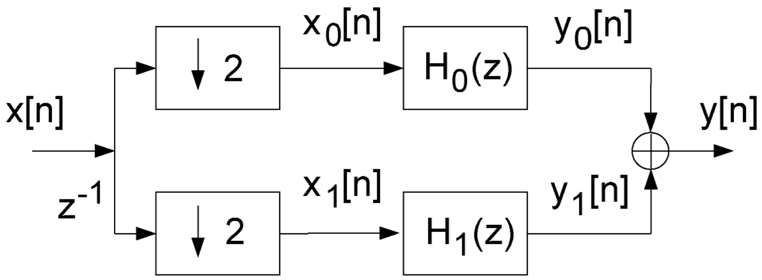
Taking the inverse Fourier Transform:

$$y[n] = \frac{5}{3}\frac{\sin(3\pi/5n)}{\pi n} \tag{1}$$

4.38. Appears in: Fall05 PS6, Fall04 PS5, Fall02 PS5.

Problem

Consider the decimation filter structure shown below:

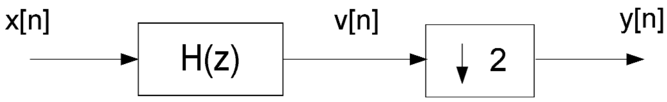


where $y_0[n]$ and $y_1[n]$ are generated according to the following forward recursions:

$$\begin{aligned} y_0[n] &= \frac{1}{4}y_0[n-1] - \frac{1}{3}x_0[n] + \frac{1}{8}x_0[n-1] \\ y_1[n] &= \frac{1}{4}y_1[n-1] + \frac{1}{12}x_1[n] \end{aligned}$$

(a) How many multiplies per output sample does the implementation of the filter structure require? Consider a divide to be equivalent to a multiply.

The decimation filter can also be implemented as shown below,



where $v[n] = av[n-1] + bx[n] + cx[n-1]$.

(b) Determine a, b, and c.

(c) How many multiplies per output sample does this second implementation require?

Solution from Fall05 PS6

(a) There is one output sample generated for every pair of input samples. Even input samples require 3 multiplies and odd input samples require 2 multiplies. Thus each pair requires 5 multiplies.

(b) Applying the compressor identity to the previous structure results in:

$$H(z) = H_0(z^2) + z^{-1}H_1(z^2).$$

From the difference equations in the previous part we have:

$$H_0(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-1}}{1 - \frac{1}{4}z^{-1}},$$

and

$$H_1(z) = \frac{\frac{1}{12}}{1 - \frac{1}{4}z^{-1}}.$$

Thus,

$$H(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-2} + \frac{1}{12}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{-\frac{1}{3}(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{-\frac{1}{3} + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}.$$

Therefore, $a = 1/2, b = -1/3$ and $c = 1/4$.

(c) In this implementation 3 multiplies are required for every input sample. For every output sample we need to calculate 2 values of $v[n]$. Altogether we need 6 multiplies per output sample.

Solution from Fall04 PS5

(a) There is one output sample generated for every pair of input samples. Even input samples require 3 multiplies and odd input samples require 2 multiplies. Thus each pair requires 5 multiplies.

(b) Applying the compressor identity to the previous structure results in:

$$H(z) = H_0(z^2) + z^{-1}H_1(z^2).$$

From the difference equations in the previous part we have:

$$H_0(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-1}}{1 - \frac{1}{4}z^{-1}},$$

and

$$H_1(z) = \frac{\frac{1}{12}}{1 - \frac{1}{4}z^{-1}}.$$

Thus,

$$H(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-2} + \frac{1}{12}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{-\frac{1}{3}(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{-\frac{1}{3} + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}.$$

Therefore, $a = 1/2, b = -1/3$ and $c = 1/4$.

(c) In this implementation 3 multiplies are required for every input sample. For every output sample we need to calculate 2 values of $v[n]$. Altogether we need 6 multiplies per output sample.

Solution from Fall02 PS5

1. There is one output sample generated for every pair of input samples. Even input samples require 3 multiplies and odd input samples require 2 multiplies. Thus each pair requires 5 multiplies.

2. Applying the compressor identity to the previous structure results in:

$$H(z) = H_0(z^2) + z^{-1}H_1(z^2).$$

From the difference equations in the previous part we have:

$$H_0(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-1}}{1 - \frac{1}{4}z^{-1}},$$

and

$$H_1(z) = \frac{\frac{1}{12}}{1 - \frac{1}{4}z^{-1}}.$$

Thus,

$$H(z) = \frac{-\frac{1}{3} + \frac{1}{8}z^{-2} + \frac{1}{12}z^{-1}}{1 - \frac{1}{4}z^{-2}} = \frac{-\frac{1}{3}(1 - \frac{3}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{-\frac{1}{3} + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}.$$

Therefore, $a = 1/2, b = -1/3$ and $c = 1/4$.

3. In this implementation 3 multiplies are required for every input sample. For every output sample we need to calculate 2 values of $v[n]$. Altogether we need 6 multiplies per output sample.

4.39. Appears in: Spring05 PS3.

Problem

Consider the two systems of Figure **??**.

(a) For $M = 2$, $L = 3$, and any arbitrary $x[n]$, will $y_A[n] = y_B[n]$? If your answer is yes, justify your answer. If your answer is no, clearly explain or give a counterexample.

(b) (Optional) How must M and L be related to guarantee $y_A[n] = y_B[n]$ for arbitrary $x[n]$?

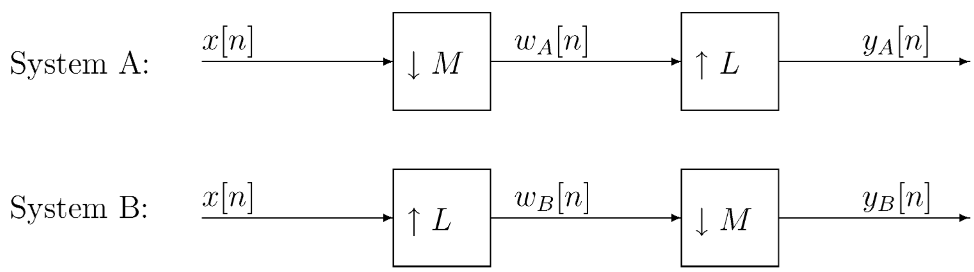


Figure 1: Systems compared in Problem 3.8.

Solution from Spring05 PS3

(a) The following equations describe the stages of System A:

$$w_A[n] = x[2n]$$

$$y_A[n] = \begin{cases} w_A[\frac{n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

The following equations describe the stages of System B:

$$w_B[n] = \begin{cases} x[\frac{n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

$$y_B[n] = w_B[2n]$$

Therefore,

$$y_A[n] = \begin{cases} x[\frac{2n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$y_B[n] = \begin{cases} x[\frac{2n}{3}] & \text{if } \frac{2n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Because for all integer values of n for which $\frac{n}{3}$ is an integer, $\frac{2n}{3}$ is also an integer and vice-versa, the systems are equivalent.

(b) More generally, the systems can be described by the following equations:

$$y_A[n] = \begin{cases} x[\frac{Mn}{L}] & \text{if } \frac{n}{L} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

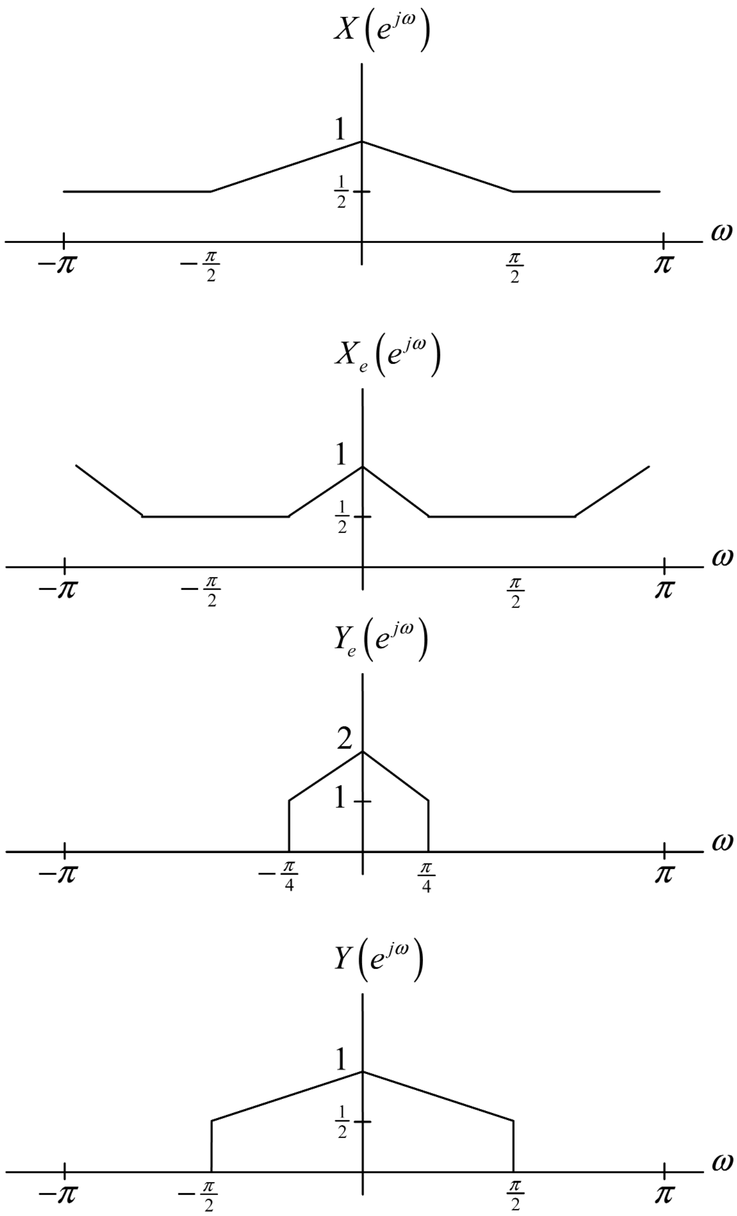
$$y_B[n] = \begin{cases} x[\frac{Mn}{L}] & \text{if } \frac{Mn}{L} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the two systems are equivalent if for all integer values of n where $\frac{Mn}{L}$ is an integer, $\frac{n}{L}$ is also an integer, and if for all integer values of n where $\frac{n}{L}$ is an integer, $\frac{Mn}{L}$ is also an integer. Since we are guaranteed that for each n which gives integer values of $\frac{n}{L}$, $\frac{Mn}{L}$ must also be an integer (since we're only considering integer M and L), we need only to show that every time $\frac{Mn}{L}$ is an integer, $\frac{n}{L}$ is an integer in order to have an equivalence between the two systems.

For arbitrary integer n, $\frac{Mn}{L}$ is an integer if and only if Mn is an integral multiple of L. This only occurs whenever Mn contains all of L's prime factors. Likewise, $\frac{n}{L}$ is an integer if and only if n contains all of L's prime factors. It is therefore true that in order for the systems to be equivalent, Mn containing all of L's prime factors must imply that n contains all of L's prime factors. This is guaranteed to be true if M and L share no prime factors in common besides 1. (This condition will ensure that any prime factors which Mn has in common with L, besides 1, must have come exclusively from n.) Therefore, the two systems are equivalent if the greatest common factor of M and L is 1 (M and L are co-prime).

4.40

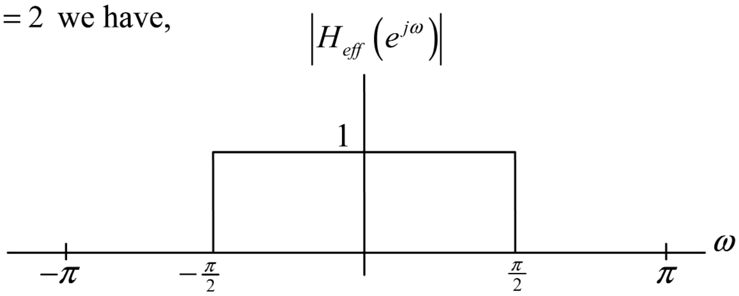
A.



B.

$$
\begin{aligned}
\varepsilon &= \sum_{n=-\infty}^{\infty} \left| x[n] - y[n] \right|^2 \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| X\left(e^{j\omega}\right) - Y\left(e^{j\omega}\right) \right|^2 d\omega \\
&= \frac{1}{2\pi} 2 \int_{\pi/2}^{\pi} (1/2)^2 \, d\omega \\
&= 1/8.
\end{aligned}
$$

C. For $M=2$ we have,



D. When $M=6$ we have $X_e\left(e^{j\omega}\right)=X\left(e^{j6\omega}\right)$. Then

$$\begin{aligned} Y_e\left(e^{j\omega}\right) &= H\left(e^{j\omega}\right)X_e\left(e^{j\omega}\right) \\ &= H\left(e^{j\omega}\right)X\left(e^{j6\omega}\right) \\ &= \begin{cases} 6X\left(e^{j6\omega}\right), & |\omega| \le \dfrac{\pi}{4} \\ 0, & \dfrac{\pi}{4} < |\omega| \le \pi. \end{cases} \end{aligned}$$

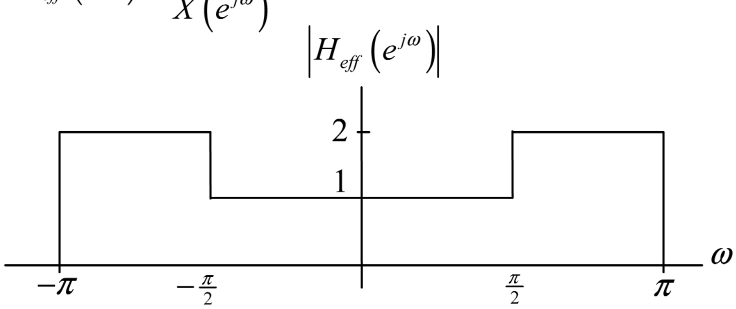
Now

$$Y\left(e^{j\omega}\right)=\begin{cases} X\left(e^{j6(\omega+2\pi)/6}\right)+X\left(e^{j6\omega/6}\right), & -\pi \le \omega \le -\dfrac{\pi}{2} \\ X\left(e^{j6\omega/6}\right), & -\dfrac{\pi}{2} < \omega < \dfrac{\pi}{2} \\ X\left(e^{j6\omega/6}\right)+X\left(e^{j6(\omega-2\pi)/6}\right), & \dfrac{\pi}{2} \le \omega \le \pi. \end{cases}$$

But $X\left(e^{j(\omega\pm 2\pi)}\right)=X\left(e^{j\omega}\right)$. Thus

$$Y\left(e^{j\omega}\right)=\begin{cases} 2X\left(e^{j\omega}\right), & -\pi \le \omega \le -\dfrac{\pi}{2} \\ X\left(e^{j\omega}\right), & -\dfrac{\pi}{2} < \omega < \dfrac{\pi}{2} \\ 2X\left(e^{j\omega}\right), & \dfrac{\pi}{2} \le \omega \le \pi. \end{cases}$$

Finally, $H_{eff}\left(e^{j\omega}\right)=\dfrac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}$. The magnitude is plotted below.



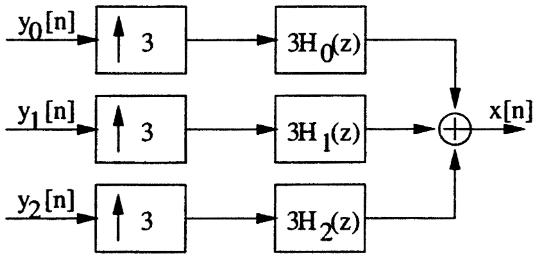
4.41. (a) Notice that

$$\begin{aligned} y_0[n] &= x[3n] \\ y_1[n] &= x[3n+1] \\ y_2[n] &= x[3n+2], \end{aligned}$$

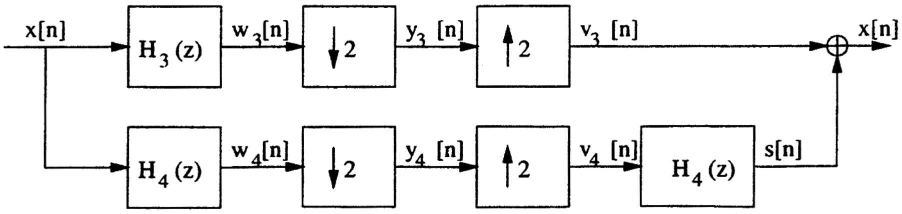
and therefore,

$$x[n] = \begin{cases} y_0[n/3], & n = 3k \\ y_1[(n-1)/3], & n = 3k+1 \\ y_2[(n-2)/3], & n = 3k+2 \end{cases}$$

(b) Yes. Since the bandwidth of the filters are $2\pi/3$, there is no aliasing introduced by downsampling. Hence to reconstruct $x[n]$, we need the system shown in the following figure:



(c) Yes, $x[n]$ can be reconstructed from $y_3[n]$ and $y_4[n]$ as demonstrated by the following figure:



In the following discussion, let $x_e[n]$ denote the even samples of $x[n]$, and $x_o[n]$ denote the odd samples of $x[n]$:

$$x_e[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$x_o[n] = \begin{cases} 0, & n \text{ even} \\ x[n], & n \text{ odd} \end{cases}$$

In the figure, $y_3[n] = x[2n]$, and hence,

$$\begin{aligned} v_3[n] &= \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ &= x_e[n] \end{aligned}$$

Furthermore, it can be verified using the IDFT that the impulse response $h_4[n]$ corresponding to $H_4(e^{j\omega})$ is

$$h_4[n] = \begin{cases} -2/(j\pi n), & n \text{ odd} \\ 0, & \text{otherwise} \end{cases}$$

Notice in particular that every other sample of the impulse response $h_4[n]$ is zero. Also, from the form of $H_4(e^{j\omega})$, it is clear that $H_4(e^{j\omega})H_4(e^{j\omega}) = 1$, and hence $h_4[n] * h_4[n] = \delta[n]$. Therefore,

$$\begin{aligned} v_4[n] &= \begin{cases} y_4[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ &= \begin{cases} w_4[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ &= \begin{cases} (x * h_4)[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ &= x_o[n] * h_4[n] \end{aligned}$$

where the last equality follows from the fact that $h_4[n]$ is non-zero only in the odd samples. Now, $s[n] = v_4[n] * h_4[n] = x_o[n] * h_4[n] * h_4[n] = x_o[n]$, and since $x[n] = x_e[n] + x_o[n]$, $s[n] + v_3[n] = x[n]$.

4.42

A. We are given

$$h[n]=\begin{cases}\left(\frac{1}{2}\right)^n, & 0\le n\le 11\\ 0, & \text{otherwise.}\end{cases}$$

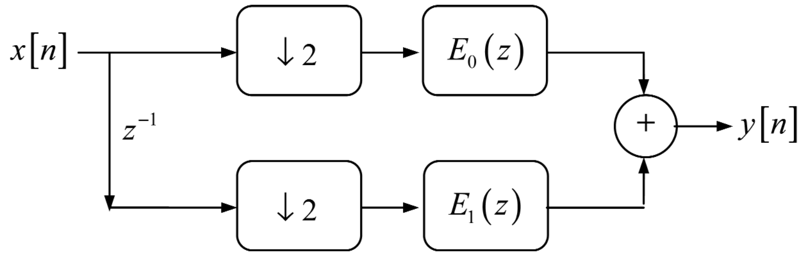
Then

$$E_0(z)=1+\frac{1}{2^2}z^{-1}+\frac{1}{2^4}z^{-2}+\frac{1}{2^6}z^{-3}+\frac{1}{2^8}z^{-4}+\frac{1}{2^{10}}z^{-5},$$

and

$$E_1(z)=\frac{1}{2}+\frac{1}{2^3}z^{-1}+\frac{1}{2^5}z^{-2}+\frac{1}{2^7}z^{-3}+\frac{1}{2^9}z^{-4}+\frac{1}{2^{11}}z^{-5}.$$

A compressor structure with two polyphase components is shown below.



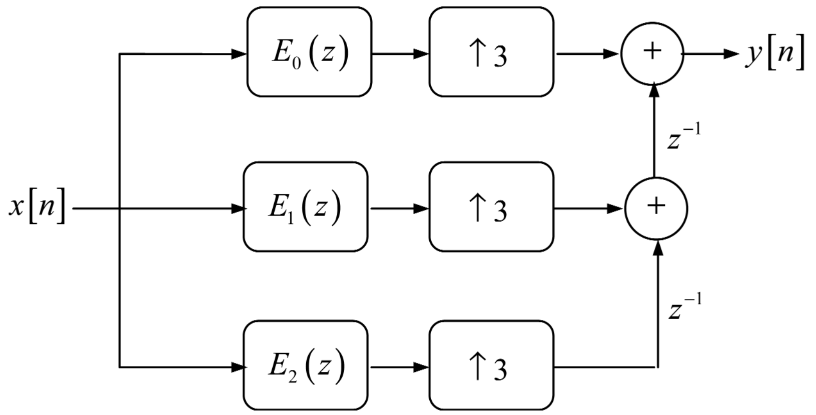
B. Using the same $h[n]$,

$$E_0(z)=1+\frac{1}{2^3}z^{-1}+\frac{1}{2^6}z^{-2}+\frac{1}{2^9}z^{-3}$$

$$E_1(z)=\frac{1}{2}+\frac{1}{2^4}z^{-1}+\frac{1}{2^7}z^{-2}+\frac{1}{2^{10}}z^{-3}$$

$$E_2(z)=\frac{1}{2^2}+\frac{1}{2^5}z^{-1}+\frac{1}{2^8}z^{-2}+\frac{1}{2^{11}}z^{-3}.$$

An expander structure with three polyphase components is shown below.



4.43

It is not possible to find a choice for $H_2(z)$ that will guarantee that $y_2[n] = y_1[n]$ whenever $x_2[n] = x_1[n]$. One way to show this is to observe that $H_1(z) = 1 + z^{-3}$, so that $H_1(e^{j\omega}) = 1 + e^{-j3\omega}$. This implies that $H_1(e^{j0}) = 2$, while $H_1(e^{j\pi}) = 0$. This in turn means that $Y_1(e^{j0})$ may not equal 0, but $Y_1(e^{j\pi}) = 0$.

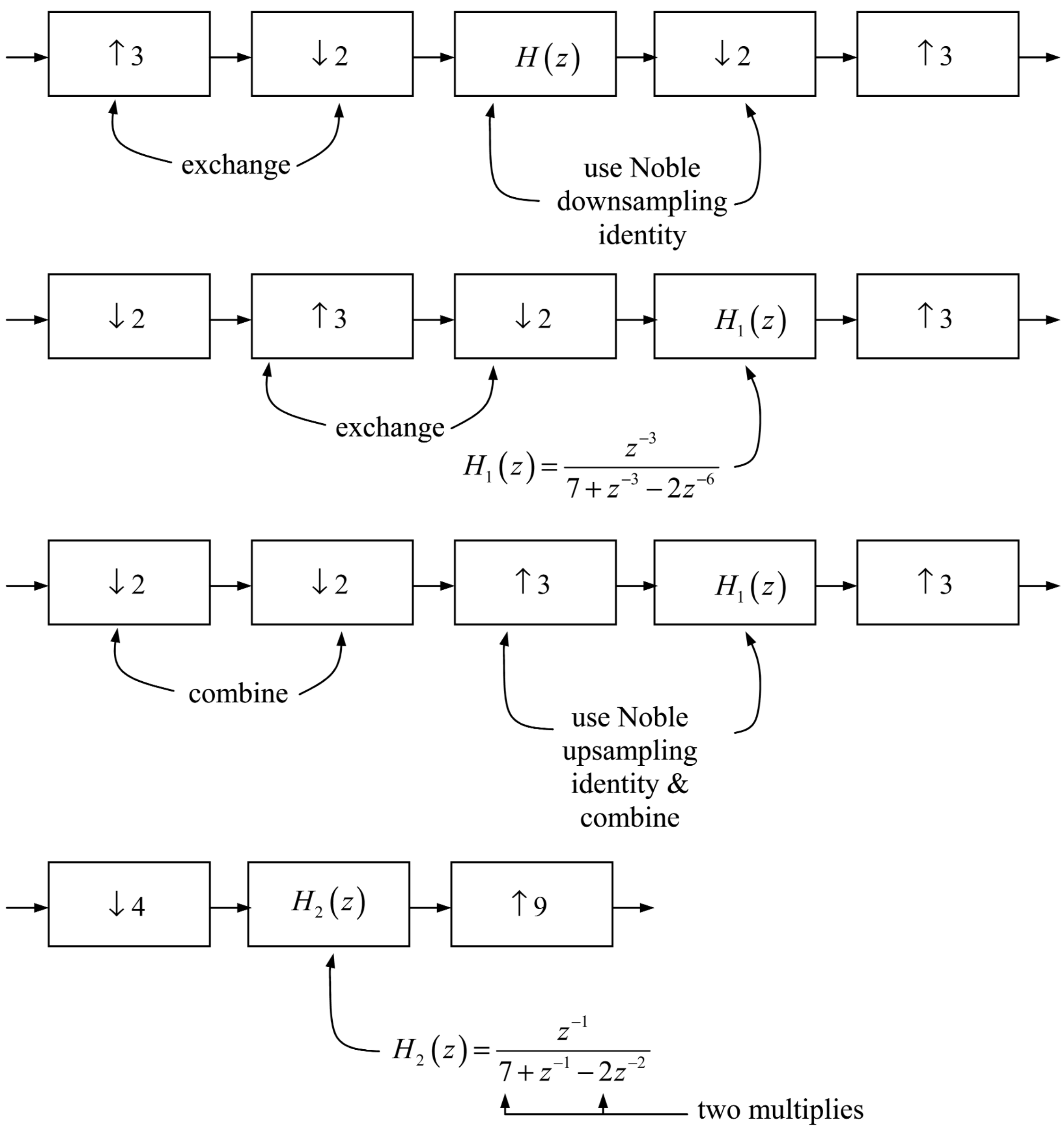
Now consider System 2. For this system

$$\begin{aligned} Y_2(e^{j\omega}) &= W_2(e^{j2\omega}) \\ &= H_2(e^{j2\omega}) X_2(e^{j2\omega}). \end{aligned}$$

To guarantee $Y_2(e^{j\omega}) = Y_1(e^{j\omega})$ for all ω we must allow $Y_2(e^{j0})$ to possibly have a nonzero value. This implies that $H_2(e^{j0}) \neq 0$. We must also insure that $Y_2(e^{j\pi}) = 0$, and this implies that $H_2(e^{j2\pi}) = 0$. The frequency resoponse $H_2(e^{j\omega})$ must be periodic in ω, however, with period 2π. Consequently the required conditions on $H_2(e^{j\omega})$ cannot be satisfied.

4.44

We need to compress as much as possible, filter, and then expand. To start, the expand by 3 and compress by 2 blocks at the input can be exchanged.



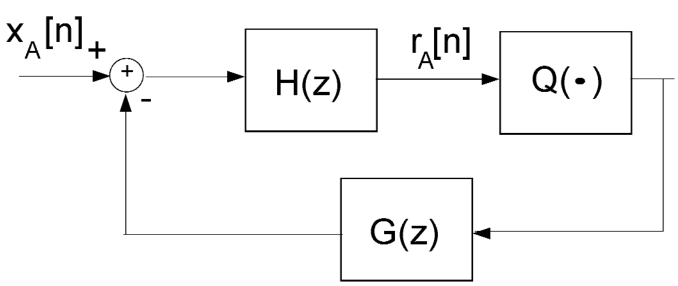
Total: $2/9$ multiplications per output sample.

4.45. Appears in: Fall02 PS3, Spring00 PS4.

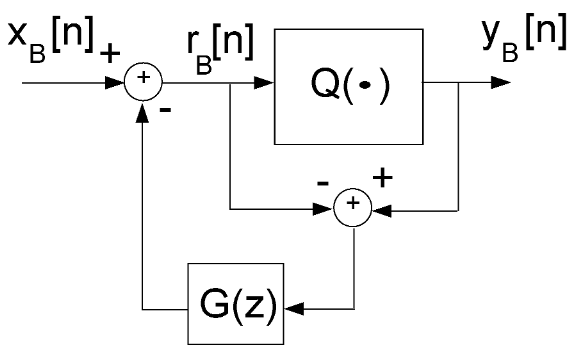
Problem

Consider the following two systems:

System A:



System B:



where $Q(\cdot)$ represents a quantizer which is the same in both system
can $H(z)$ always be specified so that the two systems are equivalent (
$x_A[n] = x_B[n]$) for any arbitrary quantizer $Q(\cdot)$? If so, specify $H(z)$
your reasoning.

Solution from Fall02 PS3

Since $Q(\cdot)$ can do anything, we need to let the transfer function from $x[n]$ to $r[n]$ and from $y[n]$ to $r[n]$ be the same in both systems. In other words $H_{x_A r_A}(z) = H_{x_B r_B}(z)$ and $H_{y_A r_A}(z) = H_{y_B r_B}(z)$, where

$$\begin{aligned} H_{x_A r_A}(z) &= H(z) \\ H_{x_B r_B}(z) &= \frac{1}{1 - G(z)} \\ H_{y_A r_A}(z) &= -G(z)H(z) \\ H_{y_B r_B}(z) &= -\frac{G(z)}{1 - G(z)}. \end{aligned}$$

Both conditions are satisfied if we let $H(z) = \frac{1}{1-G(z)}$. Thus the two systems are equivalent if we choose $H(z)$ appropriately.

Solution from Spring00 PS4

Since $Q(\cdot)$ can do anything, we need to let the transfer function from $x[n]$ to $r[n]$ and from $y[n]$ to $r[n]$ be the same in both systems. In other words $H_{x_A r_A}(z) = H_{x_B r_B}(z)$ and $H_{y_A r_A}(z) = H_{y_B r_B}(z)$, where

$$\begin{aligned} H_{x_A r_A}(z) &= H(z) \\ H_{x_B r_B}(z) &= \frac{1}{1 - G(z)} \\ H_{y_A r_A}(z) &= -G(z)H(z) \\ H_{y_B r_B}(z) &= -\frac{G(z)}{1 - G(z)}. \end{aligned}$$

Both conditions are satisfied if we let $H(z) = \frac{1}{1-G(z)}$. Thus the two systems are equivalent if we choose $H(z)$ appropriately.

4.46

A. This proposed identity is not valid.
Consider as an input $\delta[n]$.

If $\delta[n]$ is compressed by a factor of two, the result is $\delta[n]$. If $\delta[n]$ is applied to a half-sample delay, the result is nonzero for some values of n.

On the other hand, if $\delta[n]$ is delayed by a single sample, the result is $\delta[n-1]$. Then compressing by a factor of two yield zero for all values of n.

B. This proposed identity is not valid.
Consider as input $\delta[n-1]$ and suppose $h[n]=\delta[n-1]$.
If $\delta[n-1]$ is delayed one sample, the result is $\delta[n-2]$. Compressing by a factor of two yields $\delta[n-1]$. The response of the filter is $\delta[n-2]$. Expanding by a factor of two gives $\delta[n-4]$. Finally, advancing one sample produces $\delta[n-3]$.

On the other hand, if $\delta[n-1]$ is advanced one sample, the result is $\delta[n]$. Compressing by a factor of two yields $\delta[n]$ again. Since $h[n+1]=\delta[(n+1)-1]=\delta[n]$, the filter response is $\delta[n]$. Expanding by a factor of two gives $\delta[n]$, and delaying by two samples produces $\delta[n-2]$.

C. This proposed identity is valid. The validity is demonstrated by looking in the frequency domain.
Let the input to the first system be $v[n]$ with DTFT $V(e^{j\omega})$. Expanding by a factor of L produces $V(e^{j\omega L})$. The response of system A is then $\left(V(e^{j\omega L})\right)^L$.

Now consider the second system with the same input $v[n]$. The response of block A is $\left(V(e^{j\omega})\right)^L$. Expanding by a factor of L produces $\left(V(e^{j\omega L})\right)^L$.

4.47. (a) Notice first that

$$X_c(j\Omega) = \begin{cases} F_c(j\Omega)|H_{aa}(j\Omega)|e^{-j\Omega^3}, & |\Omega| \le 400\pi \\ E_c(j\Omega)|H_{aa}(j\Omega)|e^{-j\Omega^3}, & 400\pi \le |\Omega| \le 800\pi \\ 0, & \text{otherwise} \end{cases}$$

For the given $T = 1/800$, there is no aliasing from the C/D conversion. Hence, the equivalent CT transfer function $H_c(j\Omega)$ can be written as

$$H_c(j\Omega) = \begin{cases} H(e^{j\omega})|_{\omega=\Omega T}, & |\Omega| \le \pi/T \\ 0, & \text{otherwise} \end{cases}$$

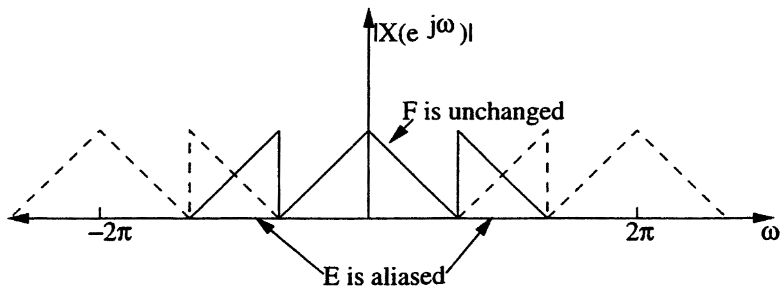
Furthermore, since $Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)$, the desired tranfer function is

$$H_c(j\Omega) = \begin{cases} e^{j\Omega^3}, & |\Omega| \le 400\pi \\ 0, & \text{otherwise} \end{cases}$$

Combining the two previous equations, we find

$$H(e^{j\omega}) = \begin{cases} e^{j(800\omega)^3}, & |\omega| \le \pi/2 \\ 0, & \pi/2 \le |\omega| \le \pi \end{cases}$$

(b) Some aliasing will occur if $2\pi/T < 1600\pi$. However, this is fine as long as the aliasing affects only $E_c(j\Omega)$ and not $F_c(j\Omega)$, as we show below:



In order for the aliasing to not affect $F_c(j\Omega)$, we require

$$\frac{2\pi}{T} - 800\pi \ge 400\pi \Longrightarrow \frac{2\pi}{T} \ge 1200\pi$$

The minimum $\frac{2\pi}{T}$ is 1200π. For this choice, we get

$$H(e^{j\omega}) = \begin{cases} e^{j(600\omega)^3}, & |\omega| \le 2\pi/3 \\ 0, & 2\pi/3 \le |\omega| \le \pi \end{cases}$$

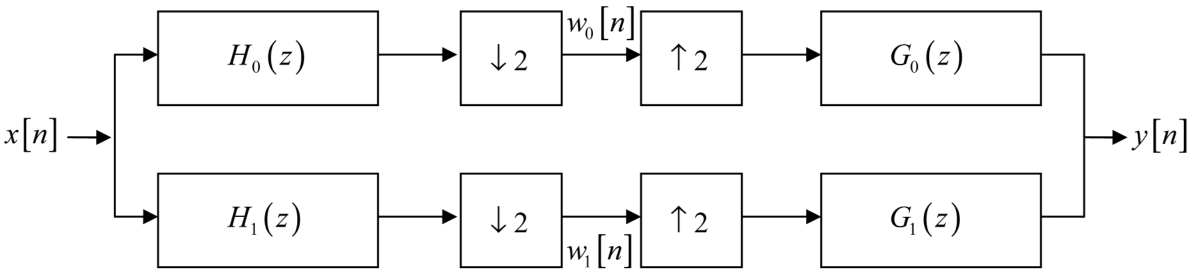
4.48

(a) $B(z)+B(-z)=2c$ means that $b[n]+(-1)^n b[n]=2c\,\delta[n]$, for all n. For $n=0$, $b[0]+b[0]=2c$. For $n\neq 0$, $b[n]+(-1)^n b[n]=0$. For n even, $(-1)^n=1$, so $b[n]+(-1)^n b[n]=b[n]+b[n]=2b[n]$. For n odd, $b[n]+(-1)^n b[n]=b[n]-b[n]=0$.

So, we have the following constraints: $b[n]=0$ for even $n\neq 0$, $b[0]=c$, and $b[n]$ can be anything for odd n.

(b) It is possible. One example is $h[n]=\delta[n]+\delta[n-1]$. Thus $b[n]=h[n]*h[-n]=\delta[n+1]+2\delta[n]+\delta[n-1]$.

(c) Let $H_0(z)=H(z)$, and $H_1(z)=z^{N-1}H(-z^{-1})$.



Then

$$W_0(e^{j\omega})=\frac{1}{2}\left[X(e^{j\omega/2})H_0(e^{j\omega/2})+X(e^{j(\omega-2\pi)/2})H_0(e^{j(\omega-2\pi)/2})\right]$$

$$W_1(e^{j\omega})=\frac{1}{2}\left[X(e^{j\omega/2})H_1(e^{j\omega/2})+X(e^{j(\omega-2\pi)/2})H_1(e^{j(\omega-2\pi)/2})\right]$$

Now, $Y(e^{j\omega})=W_0(e^{j2\omega})G_0(e^{j\omega})+W_1(e^{j2\omega})G_1(e^{j\omega})$:

$$\begin{aligned}Y(e^{j\omega})&=\frac{1}{2}\left[X(e^{j\omega})H_0(e^{j\omega})+X(e^{j(\omega-\pi)})H_0(e^{j(\omega-\pi)})\right]G_0(e^{j\omega})\\&\quad+\frac{1}{2}\left[X(e^{j\omega})H_1(e^{j\omega})+X(e^{j(\omega-\pi)})H_1(e^{j(\omega-\pi)})\right]G_1(e^{j\omega})\\&=\frac{1}{2}\left[H_0(e^{j\omega})G_0(e^{j\omega})+H_1(e^{j\omega})G_1(e^{j\omega})\right]X(e^{j\omega})\\&\quad+\frac{1}{2}\left[H_0(e^{j(\omega-\pi)})G_0(e^{j\omega})+H_1(e^{j(\omega-\pi)})G_1(e^{j\omega})\right]X(e^{j(\omega-\pi)}).\end{aligned}$$

We want to get rid of the term in the output which is multiplied by $X\left(e^{j(\omega-\pi)}\right)$, which is the result of aliasing. With this term present the system will not be LTI. Thus we must have

$$H_0\left(e^{j(\omega-\pi)}\right)G_0\left(e^{j\omega}\right)+H_1\left(e^{j(\omega-\pi)}\right)G_1\left(e^{j\omega}\right)=0.$$

This condition can be satisfied with the following choice for $G_0(z)$ and $G_1(z)$:
$G_0\left(e^{j\omega}\right)=2H_1\left(e^{j(\omega-\pi)}\right)$ and $G_1\left(e^{j\omega}\right)=-2H_0\left(e^{j(\omega-\pi)}\right)$. Looking up our definition of $H_1(z)$, we have

$$G_0(z)=2H_1(-z)=2(-z)^{N-1}H\left(z^{-1}\right)$$
$$G_1(z)=-2H_0(z)=-2H(-z).$$

With this choice of $G_0(z)$ and $G_1(z)$, the aliasing resulting from decimation in the analysis section of the QMF filter bank is perfectly cancelled by the synthesis part. The factors of 2 are optional – they compensate for $\frac{1}{2}$ introduced by the downsampler.

(d)

$$\begin{aligned}
Y(z)&=\frac{1}{2}\left[H_0(z)G_0(z)+H_1(z)G_1(z)\right]X(z)\\
&=\frac{1}{2}\left[2H(z)(-z)^{N-1}H\left(z^{-1}\right)+2z^{N-1}H\left(-z^{-1}\right)\left(-H(-z)\right)\right]X(z)\\
&=\left[(-1)^{N-1}z^{N-1}H(z)H\left(z^{-1}\right)-z^{N-1}H(-z)H\left(-z^{-1}\right)\right]X(z)\\
&=z^{N-1}\left[(-1)^{N-1}H(z)H\left(z^{-1}\right)-H(-z)H\left(-z^{-1}\right)\right]X(z).
\end{aligned}$$

Recall that we are given that $H(z)H\left(z^{-1}\right)+H(-z)H\left(-z^{-1}\right)=c$. For N even $(-1)^{N-1}=-1$, and we have

$$Y(z)=z^{N-1}\left[-H(z)H\left(z^{-1}\right)-H(-z)H\left(-z^{-1}\right)\right]X(z)=-cz^{N-1}X(z).$$

Thus for even N, the output of the QMF is simply a scaled shifted version of the input. Therefore, the overall system does indeed reconstruct the input perfectly, but only for even N.

4.49

Suppose $x[n]$ has spectrum $X(e^{j\omega})$. At the output of the first $H_0(e^{j\omega})$ stage, the spectrum is

$$H_0(e^{j\omega})X(e^{j\omega}) = \begin{cases} X(e^{j\omega}), & |\omega| < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\omega| < \pi. \end{cases}$$

After downsampling, we have $\frac{1}{2}X(e^{j\omega/2}),\quad |\omega| < \pi$. At the output of the $Q_0(e^{j\omega})$ stage we have $\frac{1}{2}X(e^{j\omega/2})Q_0(e^{j\omega})$. The next step is upsampling. This produces $\frac{1}{2}X(e^{j\omega})Q_0(e^{j2\omega}),\quad |\omega| < \frac{\pi}{2}$. The upsampled signal is passed through a second $H_0(e^{j\omega})$ stage. At the output of the upper branch we have the spectrum

$$\begin{cases} \frac{1}{2}X(e^{j\omega})Q_0(e^{j2\omega}), & |\omega| < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\omega| < \pi. \end{cases}$$

At the output of the first $H_1(e^{j\omega})$ stage in the lower branch, the spectrum is

$$H_1(e^{j\omega})X(e^{j\omega}) = \begin{cases} 0, & |\omega| < \frac{\pi}{2} \\ X(e^{j\omega}), & \frac{\pi}{2} < |\omega| < \pi. \end{cases}$$

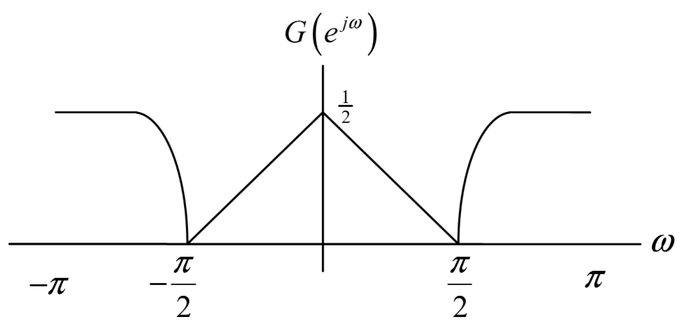
After downsampling, we have $\frac{1}{2}X(e^{j\omega/2}),\quad \pi < |\omega| < 2\pi$. At the output of the $Q_1(e^{j\omega})$ stage we have $\frac{1}{2}Q_1(e^{j\omega})X(e^{j\omega/2}),\quad \pi < |\omega| < 2\pi$. The next stage is upsampling; upsampling produces $\frac{1}{2}Q_1(e^{j2\omega})X(e^{j\omega}),\quad \frac{\pi}{2} < |\omega| < \pi$. The upsampled signal is passed through a second $H_1(e^{j\omega})$ stage. At the output of the lower branch we have the spectrum

$$\begin{cases} 0, & 0<|\omega|<\dfrac{\pi}{2} \\ \tfrac{1}{2}Q_1\left(e^{j2\omega}\right)X\left(e^{j\omega}\right), & \dfrac{\pi}{2}<|\omega|<\pi. \end{cases}$$

Finally, we combine the two branches and divide by $X\left(e^{j\omega}\right)$ to obtain the frequency response

$$G\left(e^{j\omega}\right)=\begin{cases} \tfrac{1}{2}Q_0\left(e^{j2\omega}\right), & 0<|\omega|<-\dfrac{\pi}{2} \\ \tfrac{1}{2}Q_1\left(e^{j2\omega}\right), & \dfrac{\pi}{2}<|\omega|<\pi. \end{cases}$$

The result is sketched below.



4.50

(a)

(a1) If $H_0(z)$ is linear phase, then $H_0(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta}$, where $A(e^{j\omega})$ is real-valued and α and β are constants. In this case we have $H_0^2(e^{j\omega}) = A^2(e^{j\omega})e^{-j2\alpha\omega + j2\beta}$ and $H_0(e^{j(\omega-\pi)}) = A(e^{j(\omega-\pi)})e^{-j\alpha(\omega-\pi)+j\beta}$, so that $H_0^2(e^{j(\omega-\pi)}) = A^2(e^{j(\omega-\pi)})e^{-j2\alpha(\omega-\pi)+j2\beta} = A^2(e^{j(\omega-\pi)})e^{-j2\alpha\omega + j2(\beta-\alpha\pi)}$. Now $T(e^{j\omega})$ is given by

$$
\begin{aligned}
T(e^{j\omega}) &= \tfrac{1}{2}\left(H_0^2(e^{j\omega}) - H_0^2(e^{j(\omega-\pi)})\right) \\
&= \tfrac{1}{2}A^2(e^{j\omega})e^{-j2\alpha\omega+j2\beta} - \tfrac{1}{2}A^2(e^{j(\omega-\pi)})e^{-j2\alpha\omega+j2(\beta-\alpha\pi)} \\
&= \left[\tfrac{1}{2}A^2(e^{j\omega}) - \tfrac{1}{2}A^2(e^{j(\omega-\pi)})e^{-j2\pi\alpha}\right]e^{-j2\alpha\omega+j2\beta}.
\end{aligned}
$$

We see that $T(z)$ will be linear phase if $\left[\tfrac{1}{2}A^2(e^{j\omega}) - \tfrac{1}{2}A^2(e^{j(\omega-\pi)})e^{-j2\pi\alpha}\right]$ is real-valued. A sufficient condition is that α is an integer multiple of $\tfrac{1}{2}$.

(a2) If $E_0(z)$ and $E_1(z)$ are linear phase then $E_0(e^{j\omega}) = A_0(e^{j\omega})e^{-j\alpha_0\omega + j\beta_0}$ and $E_1(e^{j\omega}) = A_1(e^{j\omega})e^{-j\alpha_1\omega + j\beta_1}$. In this case $T(e^{j\omega})$ is given by

$$
\begin{aligned}
T(e^{j\omega}) &= 2e^{-j\omega}E_0(e^{j2\omega})E_1(e^{j2\omega}) \\
&= 2e^{-j\omega}A_0(e^{j2\omega})e^{-j2\alpha_0\omega+j2\beta_0}A_1(e^{j2\omega})e^{-j2\alpha_1\omega+j2\beta_1} \\
&= 2A_0(e^{j2\omega})A_1(e^{j2\omega})e^{-j[2(\alpha_0+\alpha_1)+1]\omega+j2(\beta_0+\beta_1)}.
\end{aligned}
$$

We see in this case that $T(z)$ is linear phase.

(b) Given $h_0[n]=\delta[n]+\delta[n-1]+\frac{1}{4}\delta[n-2]$,

(b1) We have $H_0(z)=1+z^{-1}+\frac{1}{4}z^{-2}$. Then $H_1(z)=H_0(-z)=1-z^{-1}+\frac{1}{4}z^{-2}$. Also, $G_0(z)=H_0(z)$ and $G_1(z)=-H_1(z)$. These give $h_1[n]=\delta[n]-\delta[n-1]+\frac{1}{4}\delta[n-2]$, $g_0[n]=\delta[n]+\delta[n-1]+\frac{1}{4}\delta[n-2]$, and $g_1[n]=-\delta[n]+\delta[n-1]-\frac{1}{4}\delta[n-2]$.

(b2) We have $e_0[n]=h_0[2n]=\delta[n]+\frac{1}{4}\delta[n-1]$ and $e_1[n]=h_0[2n+1]=\delta[n]$.

(b3) Now $E_0(z)=1+\frac{1}{4}z^{-1}$ and $E_1(z)=1$. Therefore

$$\begin{aligned} T(z) &= 2z^{-1}E_0\left(z^2\right)E_1\left(z^2\right) \\ &= 2z^{-1}\left(1+\tfrac{1}{4}z^{-2}\right)\cdot 1 \\ &= 2z^{-1}+\tfrac{1}{2}z^{-3}. \end{aligned}$$

Also, $t[n]=2\delta[n-1]+\frac{1}{2}\delta[n-3]$.

4.51.

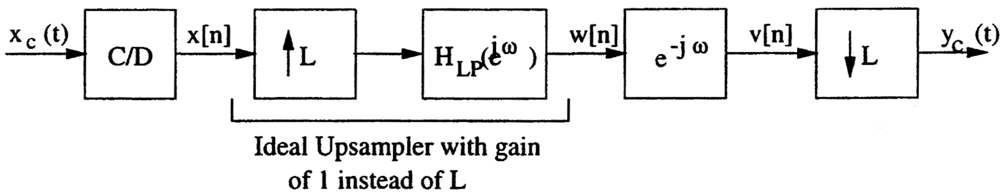
$$\begin{aligned} y[n] &= x^2[n] \\ Y(e^{j\omega}) &= X(e^{j\omega}) * X(e^{j\omega}) \end{aligned}$$

therefore, $Y(e^{j\omega})$ will occupy twice the frequency band that $X(e^{j\omega})$ does if no aliasing occurs.
If $Y(e^{j\omega}) \neq 0, \quad -\pi < \omega < \pi$, then $X(e^{j\omega}) \neq 0, \quad -\frac{\pi}{2} < \omega < \frac{\pi}{2}$ and so $X(j\Omega) = 0, \quad |\Omega| \geq 2\pi(1000)$.
Since $\omega = \Omega T$,

$$\begin{aligned} \frac{\pi}{2} &\geq T \cdot 2\pi(1000) \\ T &\leq \frac{1}{4000} \end{aligned}$$

4.52. Split $H(e^{j\omega})$ into a lowpass and a delay.

$$\begin{aligned} H(e^{j\omega}) &= H_{LP}(e^{j\omega})e^{-j\omega} \\ H_{LP}(e^{j\omega}) &= \begin{cases} 1, & |\omega| < \frac{\pi}{L} \\ 0, & \frac{\pi}{L} < |\omega| \le \pi \end{cases} \end{aligned}$$



Then we analyze the system as follows:

$$\begin{aligned} x[n] &= x_c(nT) \quad \text{no aliasing assumed} \\ w[n] &= \frac{1}{L}x_c(n\frac{T}{L}) \quad \text{rate change} \\ v[n] &= w[n-1] = \frac{1}{L}x_c\left(n\frac{T}{L} - \frac{T}{L}\right), \quad \text{delay at higher rate} \\ y[n] &= v[nL] = \frac{1}{L}x_c\left(nT - \frac{T}{L}\right) \end{aligned}$$

ch04 171-202.qxd 4/15/10 3:20 PM Page 202

5.1.

$$y[n] = \begin{cases} 1, & 0 \leq n \leq 10, \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

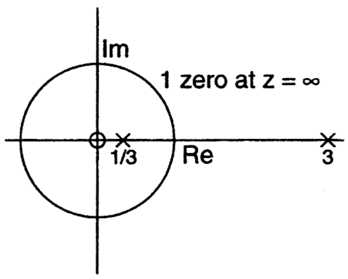
$$Y(e^{j\omega}) = e^{-j5\omega} \frac{\sin \frac{11}{2}\omega}{\sin \frac{\omega}{2}}$$

This $Y(e^{j\omega})$ is full band. Therefore, since $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$, the only possible $x[n]$ and ω_c that could produce $y[n]$ is $x[n] = y[n]$ and $\omega_c = \pi$.

5.2. We have $y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n]$ or $z^{-1}Y(z) - \frac{10}{3}Y(z) + zY(z) = X(z)$. So,

$$\begin{aligned} H(z) &= \frac{1}{z^{-1} - \frac{10}{3} + z} \\ &= \frac{z}{(z - \frac{1}{3})(z - 3)} \\ &= \frac{-\frac{1}{8}}{z - \frac{1}{3}} + \frac{\frac{9}{8}}{z - 3} \end{aligned}$$

(a)



(b)

$$H(z) = \frac{-\frac{1}{8}z^{-1}}{1 - \frac{1}{3}z^{-1}} + \frac{\frac{9}{8}z^{-1}}{1 - 3z^{-1}}$$

Stable $\Rightarrow$ ROC is $\frac{1}{3} \leq |z| \leq 3$. Therefore,

$$h[n] = -\frac{1}{8}\left(\frac{1}{3}\right)^{n-1} u[n-1] - \frac{9}{8}(3)^{n-1}u[-n]$$

5.3.

$$y[n-1] + \frac{1}{3}y[n-2] = x[n]$$

$$z^{-1}Y(z) + \frac{1}{3}z^{-2}Y(z) = X(z)$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} + \frac{1}{3}z^{-2}} \\ H(z) &= \frac{z}{1 + \frac{1}{3}z^{-1}} \end{aligned}$$

i) $\frac{1}{3} < |z|$, $h[n] = (-\frac{1}{3})^{n+1}u[n+1] \Rightarrow$ **answer (a)**
ii) $\frac{1}{3} > |z|$,

$$\begin{aligned} h[n] &= -\left(-\frac{1}{3}\right)^{n+1} u[-n-2] \\ &= -\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^{n} u[-n-2] \\ &= \frac{1}{3}\left(-\frac{1}{3}\right)^{n} u[-n-2] \Rightarrow \textbf{answer (d)} \end{aligned}$$

5.4. (a)

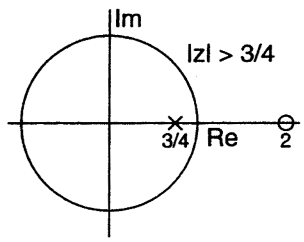
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + (2)^n u[-n-1]$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{z}{z-2}, \qquad \frac{1}{2} < |z| < 2$$

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n]$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}, \qquad \frac{3}{4} < |z|$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} \cdot \frac{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}{-\frac{3}{2}z^{-1}} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \quad |z| > \frac{3}{4}$$



(b)

$$H(z) = \frac{1}{1-\frac{3}{4}z^{-1}} - \frac{2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]$$

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$$

$$Y(z) - \frac{3}{4}z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$$

$$y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$$

(d) **The ROC is outside $|z| = \frac{3}{4}$, which includes the unit circle. Therefore the system is stable. The $h[n]$ we found in part (b) tells us the system is also causal.**

5.5.

$$y[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n] + u[n]$$

$$Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

$$x[n] = u[n]$$

$$X(z) = \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 - \frac{19}{6}z^{-1} + \frac{2}{3}z^{-2}}{1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}}, \qquad |z| > \frac{1}{3}$$

(a) Cross multiplying and equating z^{-1} with a delay in time:

$$y[n] - \frac{7}{12}y[n-1] + \frac{1}{12}y[n-2] = 3x[n] - \frac{19}{6}x[n-1] + \frac{2}{3}x[n-2]$$

(b) Using partial fractions on $H(z)$ we get:

$$H(z) = \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{z^{-1}}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}} - \frac{z^{-1}}{1-\frac{1}{4}z^{-1}} + 1, \qquad |z| > \frac{1}{3}$$

So,

$$h[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{3}\right)^{n-1} u[n-1] + \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{4}\right)^{n-1} u[n-1] + \delta[n]$$

(c) Since the ROC of $H(z)$ includes $|z| = 1$ the system is stable.

5.6. (a)

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$$

$$X(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \qquad \frac{1}{2} < |z| < 2$$

(b)

$$Y(z) = \frac{1-z^{-2}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

This has the same poles as the input, therefore the ROC is still $\frac{1}{2} < |z| < 2$.

(c)

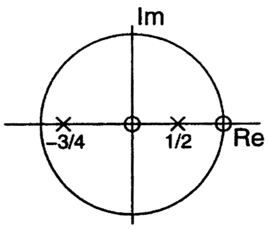
$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2} \Leftrightarrow h[n] = \delta[n] - \delta[n-2]$$

5.7. (a)

$$x[n] = 5u[n] \Leftrightarrow X(z) = \frac{5}{1 - z^{-1}}, \qquad |z| > 1$$

$$y[n] = \left(2\left(\frac{1}{2}\right)^n + 3\left(-\frac{3}{4}\right)^n\right)u[n] \Leftrightarrow Y(z) = \frac{2}{1 - \frac{1}{2}z^{-1}} + \frac{3}{1 + \frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{3}{4}z^{-1})}, \qquad |z| > \frac{3}{4}$$



(b)

$$H(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{3}{4}z^{-1})} = \frac{-\frac{2}{5}}{(1 - \frac{1}{2}z^{-1})} + \frac{\frac{7}{5}}{(1 + \frac{3}{4}z^{-1})}, \qquad |z| > \frac{3}{4}$$

$$h[n] = -\frac{2}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{7}{5}\left(-\frac{3}{4}\right)^n u[n]$$

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}$$

$$Y(z) + \frac{1}{4}z^{-1}Y(z) - \frac{3}{8}z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

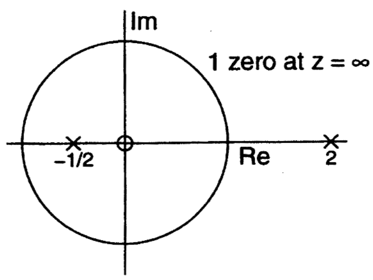
$$y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = x[n] - x[n-1]$$

5.8. (a)

$$\begin{aligned} y[n] &= \frac{3}{2}y[n-1] + y[n-2] + x[n-1] \\ Y(z) &= \frac{3}{2}z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z) \end{aligned}$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1-\frac{3}{2}z^{-1}-z^{-2}} = \frac{z^{-1}}{(1-2z^{-1})(1+\frac{1}{2}z^{-1})}, \quad |z| > 2$$



(b)

$$\begin{aligned} H(z) &= \frac{z^{-1}}{(1-2z^{-1})(1+\frac{1}{2}z^{-1})} = \frac{A}{(1-2z^{-1})} + \frac{B}{(1+\frac{1}{2}z^{-1})}, \quad |z| > 2 \\ A &= \left.\frac{z^{-1}}{(1+\frac{1}{2}z^{-1})}\right|_{z^{-1}=\frac{1}{2}} = \frac{2}{5} \\ B &= \left.\frac{z^{-1}}{(1-2z^{-1})}\right|_{z^{-1}=-2} = -\frac{2}{5} \end{aligned}$$

$$h[n] = \frac{2}{5}\left[(2)^n - \left(-\frac{1}{2}\right)^n\right]u[n]$$

(c) Use ROC of $\frac{1}{2} < |z| < 2$ since the ROC must include $|z| = 1$ for a stable system.

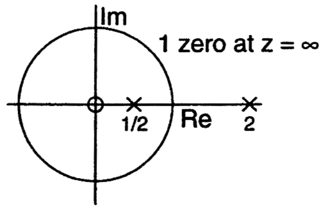
$$h[n] = -\frac{2}{5}(2)^n u[-n-1] - \frac{2}{5}\left(-\frac{1}{2}\right)^n u[n]$$

5.9.

$$y[n-1] - \frac{5}{2}y[n] + y[n+1] = x[n]$$

$$z^{-1}Y(z) - \frac{5}{2}Y(z) + zY(z) = X(z)$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} \\ &= \frac{z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} \\ &= \frac{\frac{2}{3}}{1-2z^{-1}} - \frac{\frac{2}{3}}{1-\frac{1}{2}z^{-1}} \end{aligned}$$



Three regions of convergence:

(a) $|z| < \frac{1}{2}$:

$$h[n] = -\frac{2}{3}(2)^n u[-n-1] + \frac{2}{3}\left(\frac{1}{2}\right)^n u[-n-1]$$

(b) $\frac{1}{2} < |z| < 2$:

$$h[n] = -\frac{2}{3}(2)^n u[-n-1] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

Includes $|z| = 1$, so this is stable.

(c) $|z| > 2$:

$$h[n] = \frac{2}{3}(2)^n u[n] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

ROC outside of largest pole, so this is causal.

5.10. Figure P5.16 shows two zeros and three poles inside the unit circle. Since the number of poles must equal the number of zeros, there must be an additional zero at $z = \infty$.

$H(z)$ is causal, so the ROC lies outside the largest pole and includes the unit circle. Therefore, the system is also stable.

The inverse system switches poles and zeros. The inverse system could have a ROC that includes $|z| = 1$, making it stable. However, the zero at $z = \infty$ of $H(z)$ is a pole for $H_i(z)$, so the system $H_i(z)$ cannot be causal.

5.11. (a) *It cannot be determined.* The ROC might or might not include the unit circle.

(b) *It cannot be determined.* The ROC might or might not include $z = \infty$.

(c) *False.* Given that the system is causal, we know that the ROC must be outside the outermost pole. Since the outermost pole is outside the unit circle, the ROC will not include the unit circle, and thus the system is not stable.

(d) *True.* If the system is stable, the ROC must include the unit circle. Because there are poles both inside and outside the unit circle, any ROC including the unit circle must be a ring. A ring-shaped ROC means that we have a two-sided system.

5.12. (a) *Yes.* The poles $z = \pm j(0.9)$ are inside the unit circle so the system is stable.

(b) First, factor $H(z)$ into two parts. The first should be minimum phase and therefore have all its poles and zeros inside the unit circle. The second part should contain the remaining poles and zeros.

$$H(z) = \underbrace{\frac{1+0.2z^{-1}}{1+0.81z^{-2}}}_{\text{minimum phase}} \qquad \underbrace{\frac{1-9z^{-2}}{1}}_{\substack{\text{poles \& zeros} \\ \text{outside unit circle}}}$$

Allpass systems have poles and zeros that occur in conjugate reciprocal pairs. If we include the factor $(1 - \frac{1}{9}z^{-2})$ in both parts of the equation above the first part will remain minimum phase and the second will become allpass.

$$\begin{aligned} H(z) &= \frac{(1+0.2z^{-1})(1-\frac{1}{9}z^{-2})}{1+0.81z^{-2}} \cdot \frac{1-9z^{-2}}{1-\frac{1}{9}z^{-2}} \\ &= H_1(z)H_{ap}(z) \end{aligned}$$

5.13. *An aside*: Technically, this problem is not well defined, since a pole/zero plot does not uniquely determine a system. That is, many system functions can have the same pole/zero plot. For example, consider the systems

$$\begin{aligned} H_1(z) &= z^{-1} \\ H_2(z) &= 2z^{-1} \end{aligned}$$

Both of these systems have the same pole/zero plot, namely a pole at zero and a zero at infinity. Clearly, the system $H_1(z)$ is allpass, as it passes all frequencies with unity gain (it is simply a unit delay). However, one could ask whether $H_2(z)$ is allpass. Looking at the standard definition of an allpass system provided in this chapter, the answer would be no, since the system does not pass all frequencies with *unity* gain.

A broader definition of an allpass system would be a system for which the system magnitude response $|H(e^{j\omega})| = a$, where a is a real constant. Such a system would pass all frequencies, and scale the output by a constant factor a. In a practical setting, this definition of an allpass system is satisfactory. Under this definition, both systems $H_1(z)$ and $H_2(z)$ would be considered allpass.

For this problem, it is assumed that none of the poles or zeros shown in the pole/zero plots are scaled, so this issue of using the proper definition of an allpass system does not apply. The standard definition of an allpass system is used.

Solution:

(a) Yes, the system is allpass, since it is of the appropriate form.

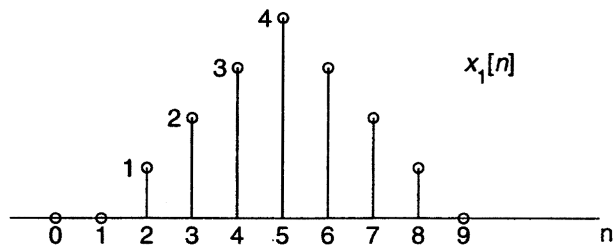
(b) No, the system is not allpass, since the zero does not occur at the conjugate reciprocal location of the pole.

(c) Yes, the system is allpass, since it is of the appropriate form.

(d) Yes, the system is allpass. This system consists of an allpass system in cascade with a pole at zero. The pole at zero is simply a delay, and does not change the magnitude spectrum.

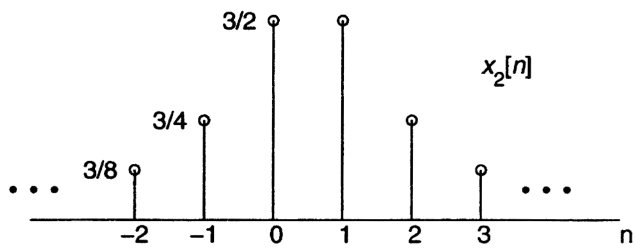
5.14. (a) By the symmetry of $x_1[n]$ we know it has linear phase. The symmetry is around $n = 5$ so the continuous phase of $X_1(e^{jw})$ is $\arg[X_1(e^{j\omega})] = -5\omega$. Thus,

$$\text{grd}[X_1(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[X_1(e^{j\omega})]\} = -\frac{d}{d\omega}\{-5\omega\} = 5$$



(b) By the symmetry of $x_2[n]$ we know it has linear phase. The symmetry is around $n = 1/2$ so we know the phase of $X_2(e^{j\omega})$ is $\arg[X_2(e^{j\omega}] = -\omega/2$. Thus,

$$\text{grd}[X_2(e^{j\omega})] = -\frac{d}{d\omega}\{\arg[X_2(e^{j\omega})]\} = -\frac{d}{d\omega}\left\{-\frac{\omega}{2}\right\} = \frac{1}{2}$$



5.15. (a) $h[n]$ is symmetric about $n = 1$.

$$\begin{aligned} H(e^{j\omega}) &= 2 + e^{-j\omega} + 2e^{-2j\omega} \\ &= e^{-j\omega}(2e^{j\omega} + 1 + 2e^{-j\omega}) \\ &= (1 + 4\cos\omega)e^{-j\omega} \end{aligned}$$

$$A(\omega) = 1 + 4\cos\omega,\ \alpha = 1,\ \beta = 0$$

Generalized Linear phase but not Linear Phase since $A(\omega)$ is not always positive.

(b) This sequence has no even or odd symmetry, so it does not possess generalized linear phase.

(c) $h[n]$ is symmetric about $n = 1$.

$$\begin{aligned} H(e^{j\omega}) &= 1 + 3e^{-j\omega} + e^{-2j\omega} \\ &= e^{-j\omega}(e^{j\omega} + 3 + e^{-j\omega}) \\ &= (3 + 2\cos\omega)e^{-j\omega} \end{aligned}$$

$$A(\omega) = 3 + 2\cos\omega,\ \alpha = 1,\ \beta = 0$$

Generalized Linear phase & Linear Phase.

(d) $h[n]$ has even symmetry.

$$\begin{aligned} H(e^{j\omega}) &= 1 + e^{-j\omega} \\ &= e^{-j(1/2)\omega}(e^{j(1/2)\omega} + e^{-j(1/2)\omega}) \\ &= 2\cos(\omega/2)e^{-j(1/2)\omega} \end{aligned}$$

$$A(\omega) = 2\cos(\omega/2),\ \alpha = \frac{1}{2},\ \beta = 0$$

Generalized Linear Phase but not Linear Phase since $A(\omega)$ is not always positive.

(e) $h[n]$ has odd symmetry.

$$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-2j\omega} \\ &= e^{-j\omega}(e^{j\omega} - e^{-j\omega}) \\ &= e^{-j\omega}2j\sin\omega \\ &= (2\sin\omega)e^{-j\omega + j\frac{\pi}{2}} \end{aligned}$$

$$A(\omega) = 2\sin\omega,\ \alpha = 1,\ \beta = \frac{\pi}{2}$$

Generalized Linear Phase but not Linear Phase since $A(\omega)$ is not always positive.

5.16. The causality of the system cannot be determined from the figure. A causal system $h_1[n]$ that has a linear phase response $\angle H(e^{jw}) = -\alpha w$, is:

$$\begin{aligned}
h_1[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \\
H_1(e^{j\omega}) &= 1 + 2e^{-j\omega} + e^{-j2\omega} \\
&= e^{-j\omega}(e^{j\omega} + 2 + e^{-j\omega}) \\
&= e^{-j\omega}(2 + 2\cos(\omega)) \\
|H_1(e^{j\omega})| &= 2 + 2\cos(\omega) \\
\angle H_1(e^{j\omega}) &= -\omega
\end{aligned}$$

An example of a non-causal system with the same phase response is:

$$\begin{aligned}
h_2[n] &= \delta[n+1] + \delta[n] + 4\delta[n-1] + \delta[n-2] + \delta[n-3] \\
H_2(e^{j\omega}) &= e^{j\omega} + 1 + 4e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} \\
&= e^{-jw}(e^{j2\omega} + e^{j\omega} + 4 + e^{-j\omega} + e^{-j2\omega}) \\
&= e^{-j\omega}(4 + 2\cos(\omega) + 2\cos(2\omega)) \\
|H_2(e^{j\omega})| &= 4 + 2\cos(\omega) + 2\cos(2\omega) \\
\angle H_2(e^{j\omega}) &= -\omega
\end{aligned}$$

Thus, both the causal sequence $h_1[n]$ and the non-causal sequence $h_2[n]$ have a linear phase response $\angle H(e^{j\omega}) = -\alpha\omega$, where $\alpha = 1$.

5.17. A minimum phase system is one which has all its poles and zeros inside the unit circle. It is causal, stable, and has a causal and stable inverse.

(a) $H_1(z)$ has a zero outside the unit circle at $z = 2$ so it is not minimum phase.

(b) $H_2(z)$ is minimum phase since its poles and zeros are inside the unit circle.

(c) $H_3(z)$ is minimum phase since its poles and zeros are inside the unit circle.

(d) $H_4(z)$ has a zero outside the unit circle at $z = \infty$ so it is not minimum phase. Moreover, the inverse system would not be causal due to the pole at infinity.

5.18. A minimum phase system with an equivalent magnitude spectrum can be found by analyzing the system function, and reflecting all poles are zeros that are outside the unit circle to their conjugate reciprocal locations. This will move them inside the unit circle. Then, all poles and zeros for $H_{min}(z)$ will be inside the unit circle. Note that a scale factor may be introduced when the pole or zero is reflected inside the unit circle.

(a) Simply reflect the zero at $z = 2$ to its conjugate reciprocal location at $z = \frac{1}{2}$. Then, determine the scale factor.

$$H_{min}(z) = 2\left(\frac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{3}z^{-1}}\right)$$

(b) First, simply reflect the zero at $z = -3$ to its conjugate reciprocal location at $z = -\frac{1}{3}$. Then, determine the scale factor. This results in

$$H_{min}(z) = 3\frac{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}{z^{-1}\left(1+\frac{1}{3}z^{-1}\right)}$$

The $\left(1+\frac{1}{3}z^{-1}\right)$ terms cancel, leaving

$$H_{min}(z) = 3\frac{\left(1-\frac{1}{2}z^{-1}\right)}{z^{-1}}$$

Note that the term $\frac{1}{z^{-1}}$ does not affect the frequency response magnitude of the system. Consequently, it can be removed. Thus, the remaining term has a zero inside the unit circle, and is therefore minimum phase. As a result, we are left with the system

$$H_{min}(z) = 3\left(1-\frac{1}{2}z^{-1}\right)$$

(c) Simply reflect the zero at 3 to its conjugate reciprocal location at $\frac{1}{3}$ and reflect the pole at $\frac{4}{3}$ to its conjugate reciprocal location at $\frac{3}{4}$. Then, determine the scale factor.

$$H_{min}(z) = \frac{9}{4}\frac{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)^2}$$

5.19. Due to the symmetry of the impulse responses, all the systems have generalized linear phase of $\arg[H(e^{j\omega})] = \beta - n_o\omega$ where n_o is the point of symmetry in the impulse response graphs. The group delay is

$$\text{grd}\left[H_i(e^{j\omega})\right] = -\frac{d}{d\omega}\left\{\arg\left[H_i(e^{j\omega})\right]\right\} = -\frac{d}{d\omega}\{\beta - n_o\omega\} = n_o$$

To find each system's group delay we need only find the point of symmetry n_o in each system's impulse response.

$$\begin{aligned} \text{grd}\left[H_1(e^{j\omega})\right] &= 2 & \qquad \text{grd}\left[H_4(e^{j\omega})\right] &= 3 \\ \text{grd}\left[H_2(e^{j\omega})\right] &= 1.5 & \qquad \text{grd}\left[H_5(e^{j\omega})\right] &= 3 \\ \text{grd}\left[H_3(e^{j\omega})\right] &= 2 & \qquad \text{grd}\left[H_6(e^{j\omega})\right] &= 3.5 \end{aligned}$$

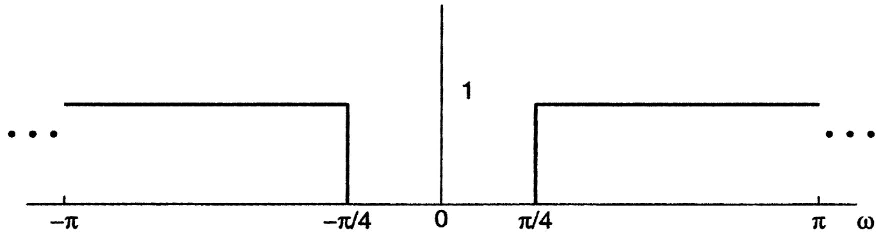
5.20. (a) *Yes.* The system function could be a generalized linear phase system implemented by a linear constant-coefficient differential equation (LCCDE) with real coefficients. The zeros come in a set of four: a zero, its conjugate, and the two conjugate reciprocals. The pole-zero plot could correspond to a Type I FIR linear phase system.

(b) *No.* This system function could not be a generalized linear phase system implemented by a LCCDE with real coefficients. Since the LCCDE has real coefficients, its poles and zeros must come in conjugate pairs. However, the zeros in this pole-zero plot do not have corresponding conjugate zeros.

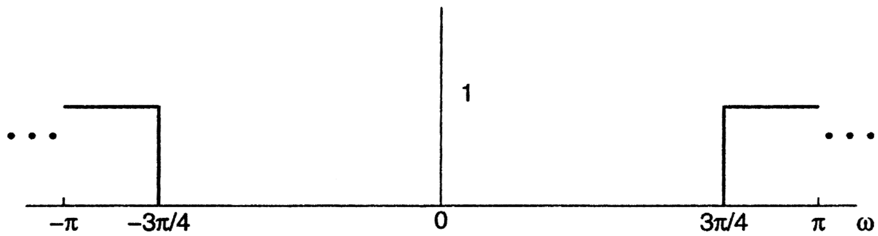
(c) *Yes.* The system function could be a generalized linear phase system implemented by a LCCDE with real coefficients. The pole-zero plot could correspond to a Type II FIR linear phase system.

5.21. $h_{lp}[n]$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$

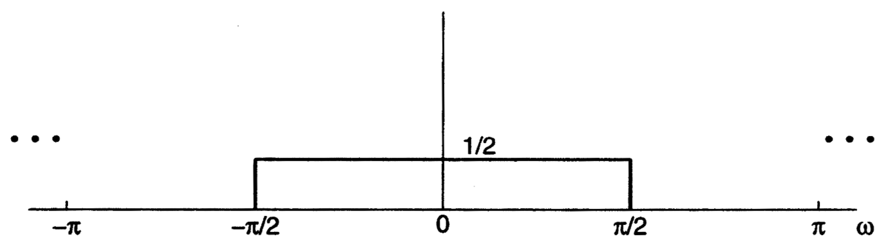
(a) $y[n] = x[n] - x[n] * h_{lp}[n] \Rightarrow H(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$
This is a highpass filter.



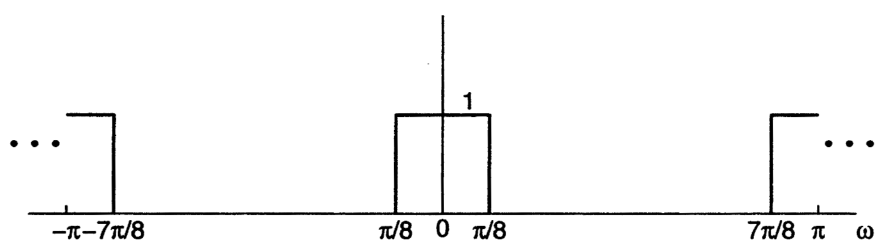
(b) $x[n]$ is first modulated by π, lowpass filtered, and demodulated by π. Therefore, $H_{lp}(e^{j\omega})$ filters the high frequency components of $X(e^{j\omega})$.
This is a highpass filter.



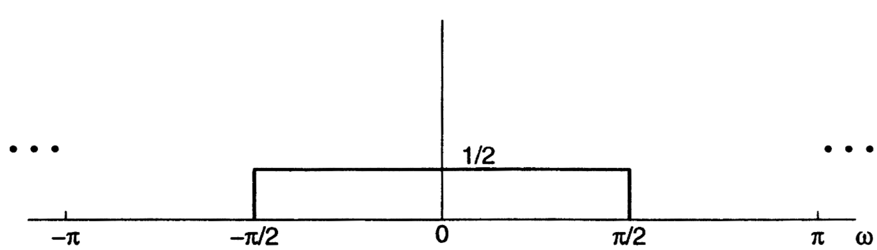
(c) $h_{lp}[2n]$ is a downsampled version of the filter. Therefore, the frequency response will be "spread out" by a factor of two, with a gain of $\frac{1}{2}$.
This is a lowpass filter.



(d) This system upsamples $h_{lp}[n]$ by a factor of two. Therefore, the frequency axis will be compressed by a factor of two.
This is a bandstop filter.



(e) This system upsamples the input before passing it through $h_{lp}[n]$. This effectively doubles the frequency bandwidth of $H_{lp}(e^{j\omega})$.
This is a lowpass filter.



5.22. Problem 2 from sp 2005 final exam
Appears in: Fall05 PS2.

Problem

Many properties of a discrete-time sequence $h[n]$ or an LTI system with impulse response $h[n]$ can be discerned from a pole-zero plot of $H(z)$. **In this problem we are concerned only with causal** $h[n]$**.** Clearly describe the z-plane characteristic that corresponds to each of the following properties:

(i) Real-valued impulse response:

(ii) Finite impulse response:

(iii) $h[n] = h[2\alpha - n]$ where 2α is an integer:

(iv) Minimum phase:

(v) All-pass:

Solution from Fall05 PS2

(i) Real-valued impulse response:

Poles that aren't real must be in complex conjugate pairs. Zeros that aren't real must be in complex conjugate pairs.

(ii) Finite impulse response:

All poles are at the origin. The ROC is the entire z-plane, except possibly $z = 0$.

(iii) $h[n] = h[2\alpha - n]$ where 2α is an integer:

Causality combined with the given symmetry property implies a finite-length $h[n]$ that can only be nonzero between time zero and time 2α. Thus we must have all poles at the origin and at most 2α zeros. The z transform of $h[2\alpha - n]$ is $z^{-2\alpha}H(1/z)$, so any zero of $H(z)$ at $c \neq 0$ must be paired with a zero at $1/c$.

(iv) Minimum phase:

All poles and zeros are inside the unit circle (so that the inverse can be stable and causal).

(v) All-pass:

Each pole is paired with a zero at the conjugate reciprocal location.

Solution from Spring05 Final

(i) Real-valued impulse response:

Poles that aren't real must be in complex conjugate pairs. Zeros that aren't real must be in complex conjugate pairs.

(ii) Finite impulse response:

All poles are at the origin.

(iii) $h[n] = h[2\alpha - n]$ where 2α is an integer:

Causality combined with the given symmetry property implies a finite-length $h[n]$ that can only be nonzero between time zero and time 2α. Thus we must have all poles at the origin and at most 2α zeros. The z transform of $h[2\alpha - n]$ is $z^{-2\alpha}H(1/z)$, so any zero of $H(z)$ at $c \neq 0$ must be paired with a zero at $1/c$.

(iv) Minimum phase:

All poles and zeros are inside the unit circle (so that the inverse can be stable and causal).

(v) All-pass:

All poles and zeros are inside the unit circle (so that the inverse can be stable and causal).

5.23.

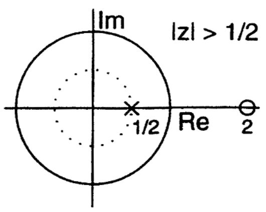
$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}} = \frac{Y(z)}{X(z)}, \qquad \text{causal, so ROC is } |z| > a$$

(a) Cross multiplying and taking the inverse transform

$$y[n] - ay[n-1] = x[n] - \frac{1}{a}x[n-1]$$

(b) Since $H(z)$ is causal, we know that the ROC is $|z| > a$. For stability, the ROC must include the unit circle. So, $H(z)$ is stable for $|a| < 1$.

(c) $a = \frac{1}{2}$



(d)

$$H(z) = \frac{1}{1 - az^{-1}} - \frac{a^{-1}z^{-1}}{1 - az^{-1}}, \qquad |z| > a$$

$$h[n] = (a)^n u[n] - \frac{1}{a}(a)^{n-1}u[n-1]$$

(e)

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} = \frac{1 - a^{-1}e^{-j\omega}}{1 - ae^{-j\omega}}$$

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = \frac{1 - a^{-1}e^{-j\omega}}{1 - ae^{-j\omega}} \cdot \frac{1 - a^{-1}e^{j\omega}}{1 - ae^{j\omega}}$$

$$\begin{aligned} |H(e^{j\omega})| &= \left(\frac{1 + \frac{1}{a^2} - \frac{2}{a}\cos\omega}{1 + a^2 - 2a\cos\omega}\right)^{\frac{1}{2}} \\ &= \frac{1}{a}\left(\frac{a^2 + 1 - 2a\cos\omega}{1 + a^2 - 2a\cos\omega}\right)^{\frac{1}{2}} \\ &= \frac{1}{a} \end{aligned}$$

5.24. (a) Type I:

$$A(\omega) = \sum_{n=0}^{M/2} a[n] \cos \omega n$$

$\cos 0 = 1$, $\cos \pi = -1$, so there are no restrictions.

Type II:

$$A(\omega) = \sum_{n=1}^{(M+1)/2} b[n] \cos \omega \left(n - \frac{1}{2}\right)$$

$\cos 0 = 1$, $\cos \left(n\pi - \frac{\pi}{2}\right) = 0$. So $H(e^{j\pi}) = 0$.

Type III:

$$A(\omega) = \sum_{n=0}^{M/2} c[n] \sin \omega n$$

$\sin 0 = 0$, $\sin n\pi = 0$, so $H(e^{j0}) = H(e^{j\pi}) = 0$.

Type IV:

$$A(\omega) = \sum_{n=1}^{(M+1)/2} d[n] \sin \omega \left(n - \frac{1}{2}\right)$$

$\sin 0 = 0$, $\sin \left(n\pi - \frac{\pi}{2}\right) \neq 0$, so just $H(e^{j0}) = 0$.

(b)

	Type I	Type II	Type III	Type IV
Lowpass	Y	Y	N	N
Bandpass	Y	Y	Y	Y
Highpass	Y	N	N	Y
Bandstop	Y	N	N	N
Differentiator	Y	N	N	Y

5.25. (a) Taking the z-transform of both sides and rearranging

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{4} + z^{-2}}{1 - \frac{1}{4}z^{-2}}$$

Since the poles and zeros {2 poles at $z = \pm 1/2$, 2 zeros at $z = \pm 2$} occur in conjugate reciprocal pairs the system is allpass. This property is easy to recognize since, as in the system above, the coefficients of the numerator and denominator z-polynomials get reversed (and in general conjugated).

(b) It is a property of allpass systems that the output energy is equal to the input energy. Here is the proof.

$$\begin{aligned}
\sum_{n=0}^{N-1} |y[n]|^2 &= \sum_{n=-\infty}^{\infty} |y[n]|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|Y(e^{j\omega})\right|^2 d\omega && \text{(by Parseval's Theorem)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})X(e^{j\omega})\right|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega && (\left|H(e^{j\omega})\right|^2 = 1 \text{ since } h[n] \text{ is allpass}) \\
&= \sum_{n=-\infty}^{\infty} |x[n]|^2 && \text{(by Parseval's theorem)} \\
&= \sum_{n=0}^{N-1} |x[n]|^2 \\
&= 5
\end{aligned}$$

5.26. *The statement is false.* A non-causal system can indeed have a positive constant group delay. For example, consider the non-causal system

$$h[n] = \delta[n+1] + \delta[n] + 4\delta[n-1] + \delta[n-2] + \delta[n-3]$$

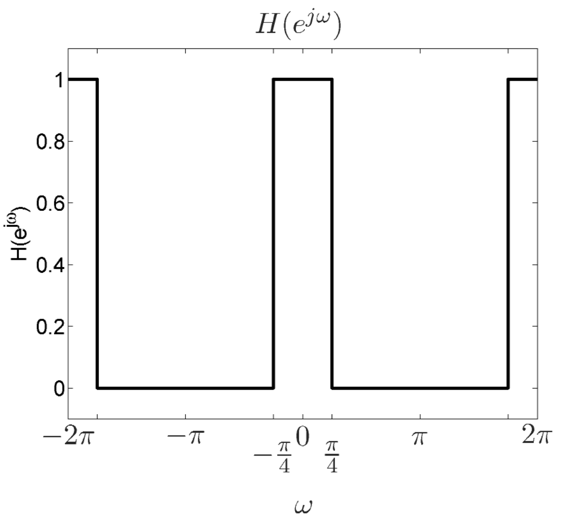
This system has the frequency response

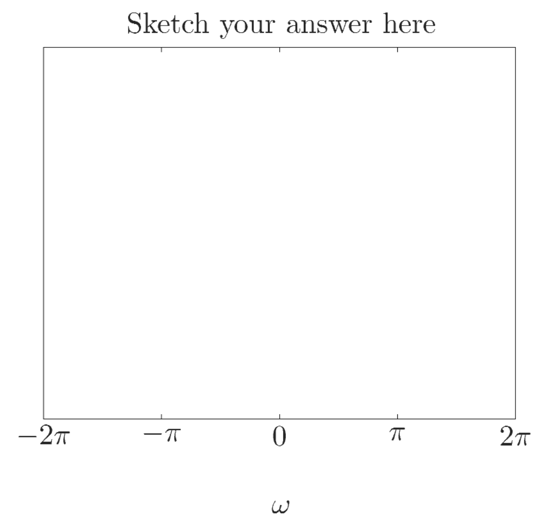
$$\begin{aligned}
H(e^{j\omega}) &= e^{j\omega} + 1 + 4e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} \\
&= e^{-j\omega}(e^{j2\omega} + e^{j\omega} + 4 + e^{-j\omega} + e^{-j2\omega}) \\
&= e^{-j\omega}(4 + 2\cos(\omega) + 2\cos(2\omega)) \\
\left|H(e^{j\omega})\right| &= 4 + 2\cos(\omega) + 2\cos(2\omega) \\
\angle H(e^{j\omega}) &= -\omega \\
\text{grd}[H(e^{j\omega})] &= 1
\end{aligned}$$

5.27. Problem 1 in Spring2005 Final exam.

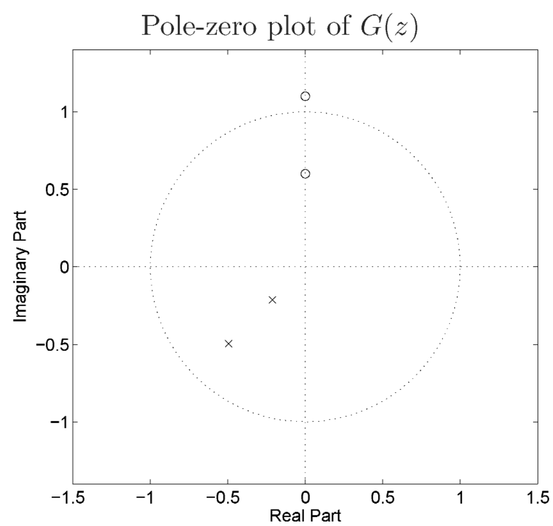
Problem

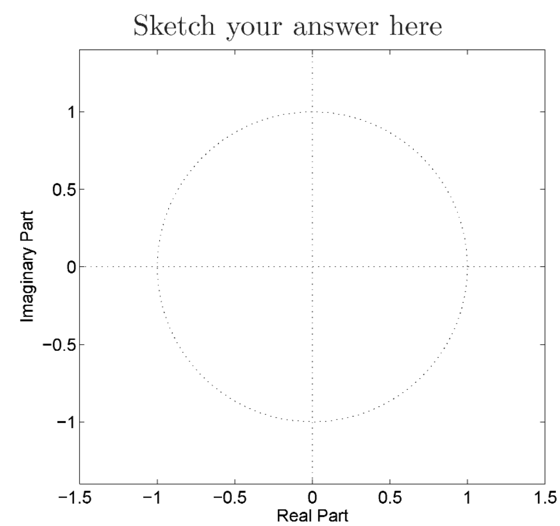
(a) An ideal lowpass filter $h[n]$ is designed with zero phase, a cutoff frequency of $\omega_c = \pi/4$, a passband gain of 1, and a stopband gain of 0. ($H(e^{j\omega})$ is shown below on the left.) Sketch the discrete-time Fourier transform of $(-1)^n h[n]$.



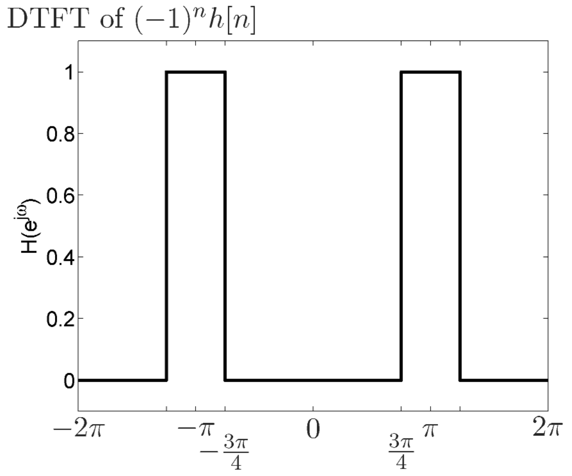


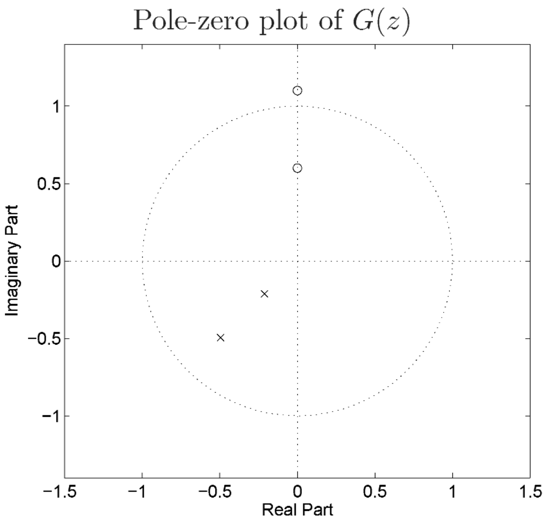
(b) A (complex) filter $g[n]$ has the pole-zero diagram shown below. Sketch the pole-zero diagram for $(-1)^n g[n]$. If there is not sufficient information provided, explain why.

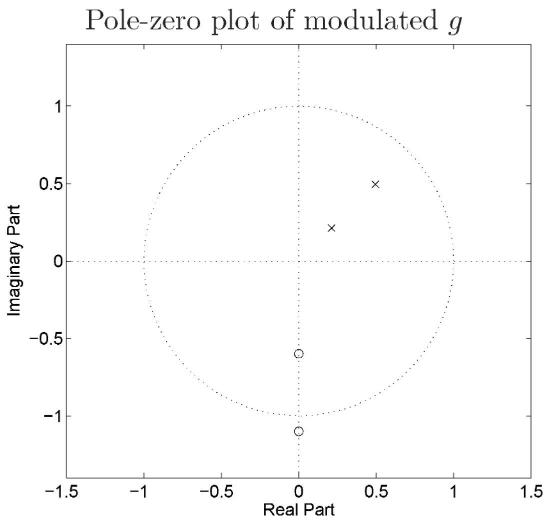




Solution from Spring2005 Final







5.28. Problem 12 from Fall 2005 Background exam

Problem

We process the signal $x[n] = \cos(\frac{\pi}{3}n)$ with a unity-gain all-pass LTI system, with frequency response $H(e^{j\omega})$ and a group delay of 4 samples at frequency $\frac{\pi}{3}$, to get the output $y[n]$. We also know that $\angle H(e^{j\frac{\pi}{3}}) = \theta$ and $\angle H(e^{-j\frac{\pi}{3}}) = -\theta$. Choose the most accurate statement:

A. $y[n] = \cos(\frac{\pi}{3}n + \theta)$

B. $y[n] = \cos(\frac{\pi}{3}(n-4) + \theta)$

C. $y[n] = \cos(\frac{\pi}{3}(n-4-\theta))$

D. $y[n] = \cos(\frac{\pi}{3}(n-4))$

E. $y[n] = \cos(\frac{\pi}{3}(n-4+\theta))$

Solution from Fall05 background exam

(Circle one) $\boxed{\text{A}}$ B C D E

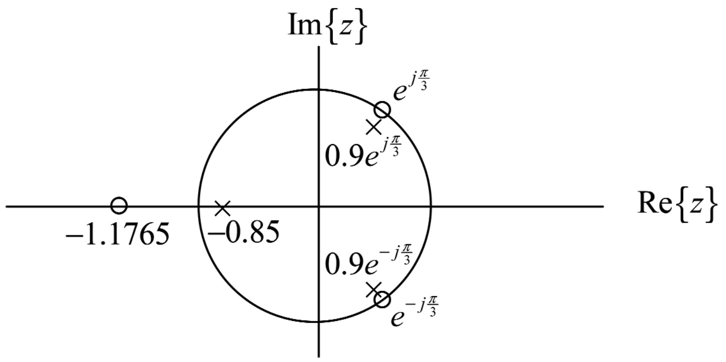
5.29

A.

$$H(z)=\frac{\left(1-e^{j\frac{\pi}{3}}z^{-1}\right)\left(1-e^{-j\frac{\pi}{3}}z^{-1}\right)\left(1+1.1765z^{-1}\right)}{\left(1-0.9e^{j\frac{\pi}{3}}z^{-1}\right)\left(1-0.9e^{-j\frac{\pi}{3}}z^{-1}\right)\left(1+0.85z^{-1}\right)}$$
$$=\frac{1+0.1765z^{-1}-0.1765z^{-2}+1.1765z^{-3}}{1-0.05z^{-1}+0.045z^{-2}+0.6885z^{-3}}$$
$$=\frac{Y(z)}{X(z)}.$$

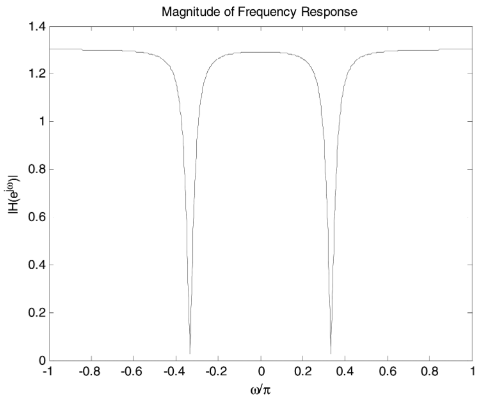
$$y[n]=0.05y[n-1]-0.45y[n-2]-0.6885y[n-3]$$
$$+x[n]+0.1765x[n-1]-0.1765x[n-2]+1.1765x[n-3].$$

B.



Since the system is causal, the ROC is the region outside the outermost pole.
$|z|>0.9$.

C.



The zeros on the unit circle null the frequency response at $\omega = \pm\pi/3$. The sharpness of the nulls depend on how close the nearby poles are to the zeros. The factor $\dfrac{1+1.1765z^{-1}}{1+0.85z^{-1}} = 1.1765\dfrac{z^{-1}+0.85}{1+0.85z^{-1}}$ is allpass and does not affect the magnitude response.

D. 1. True. The system is stable because the ROC contains the unit circle.

2. False. The impulse response must approach zero for large n because the system is stable.

3. False. The system function has a zero on the unit circle at $\omega = \pi/3$. This negates the effect of the pole, and since the pole is not on the unit circle, the pole does not cancel the zero. Instead, the sharpness of the notch depends on how close the pole is to the zero.

4. False. There is a zero outside the unit circle.

5. False. The system is not a minimum-phase system so it does not have a causal and stable inverse.

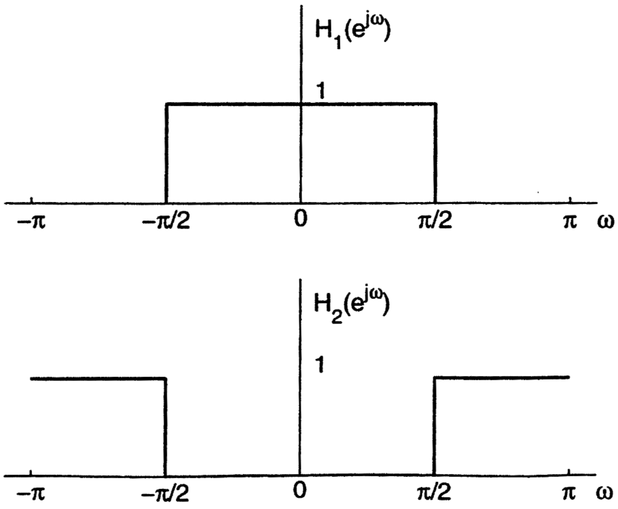
5.30. Making use of some DTFT properties can aide in the solution of this problem. First, note that

$$\begin{aligned} h_2[n] &= (-1)^n h_1[n] \\ h_2[n] &= e^{-j\pi n} h_1[n] \end{aligned}$$

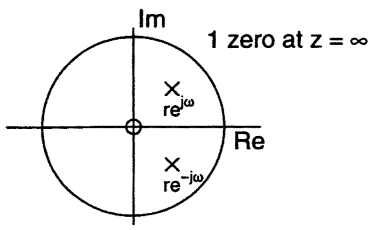
Using the DTFT property that states that modulation in the time domain corresponds to a shift in the frequency domain,

$$H_2(e^{j\omega}) = H_1(e^{j(\omega+\pi)})$$

Consequently, $H_2(e^{j\omega})$ is simply $H_1(e^{j\omega})$ shifted by π. The ideal low pass filter has now become the ideal high pass filter, as shown below.



5.31. (a) A labeled pole-zero diagram appears below.



The table of common z-transform pairs gives us

$$(r^n \sin \omega_0 n)u[n] \longleftrightarrow \frac{(r \sin \omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}, \quad |z|>r$$

which enables us to derive $h[n]$.

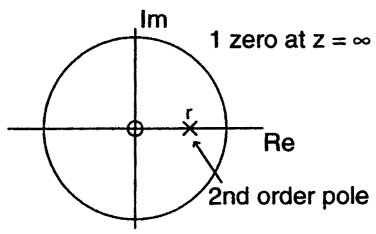
$$h[n] = \left(\frac{1}{\sin\omega_0}\right)(r^n \sin\omega_0 n)u[n]$$

(b) When $\omega_0 = 0$

$$H(z) = \frac{rz^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} = \frac{rz^{-1}}{(1-rz^{-1})^2}, \quad |z|>r$$

Again, using a table lookup gives us

$$h[n] = nr^n u[n]$$



5.32. (a)

$$H(z) = \frac{z^{-2}}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})}, \qquad \text{stable, so the ROC is } \tfrac{1}{2} < |z| < 3$$

$$x[n] = u[n] \Leftrightarrow X(z) = \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

$$Y(z) = X(z)H(z) = \frac{\frac{4}{5}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{5}}{1-3z^{-1}} - \frac{1}{1-z^{-1}}, \qquad 1 < |z| < 3$$

$$y[n] = \frac{4}{5}\left(\frac{1}{2}\right)^n u[n] - \frac{1}{5}(3)^n u[-n-1] - u[n]$$

(b) ROC includes $z = \infty$ so $h[n]$ is causal. Since both $h[n]$ and $x[n]$ are 0 for $n < 0$, we know that $y[n]$ is also 0 for $n < 0$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}}$$

$$Y(z) - \frac{7}{2}z^{-1}Y(z) + \frac{3}{2}z^{-2}Y(z) = z^{-2}X(z)$$

$$y[n] = x[n-2] + \frac{7}{2}y[n-1] - \frac{3}{2}y[n-2]$$

Since $y[n] = 0$ for $n < 0$, recursion can be done:

$$y[0] = 0, \qquad y[1] = 0, \qquad y[2] = 1$$

(c)

$$H_i(z) = \frac{1}{H(z)} = z^2 - \frac{7}{2}z + \frac{3}{2}, \qquad \text{ROC: entire z-plane}$$

$$h_i[n] = \delta[n+2] - \frac{7}{2}\delta[n+1] + \frac{3}{2}\delta[n]$$

5.33. Appears in: Fall05 PS2, Spring05 PS2, Spring04 PS2, Fall03 PS2, Spring03 PS2.

Problem

$H(z)$ is the system function for a stable LTI system and is given by:

$$H(z) = \frac{(1 - 2z^{-1})(1 - 0.75z^{-1})}{z^{-1}(1 - 0.5z^{-1})}.$$

(a) $H(z)$ can be represented as a cascade of a minimum phase system $H_1(z)$ and a unity-gain all-pass system $H_A(z)$, *i.e.*

$$H(z) = H_1(z)H_A(z).$$

Determine a choice for $H_1(z)$ and $H_A(z)$ and specify whether or not they are unique up to a scale factor.

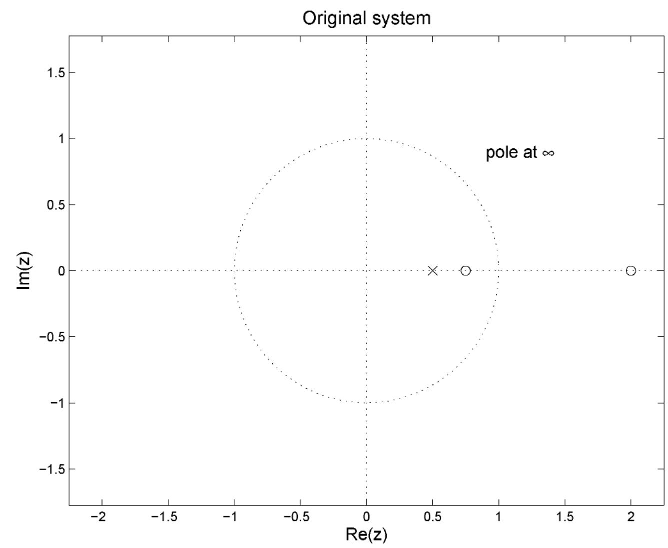
(b) $H(z)$ can be expressed as a cascade of a minimum-phase system $H_2(z)$ and a generalized linear phase FIR system $H_L(z)$:

$$H(z) = H_2(z)H_L(z).$$

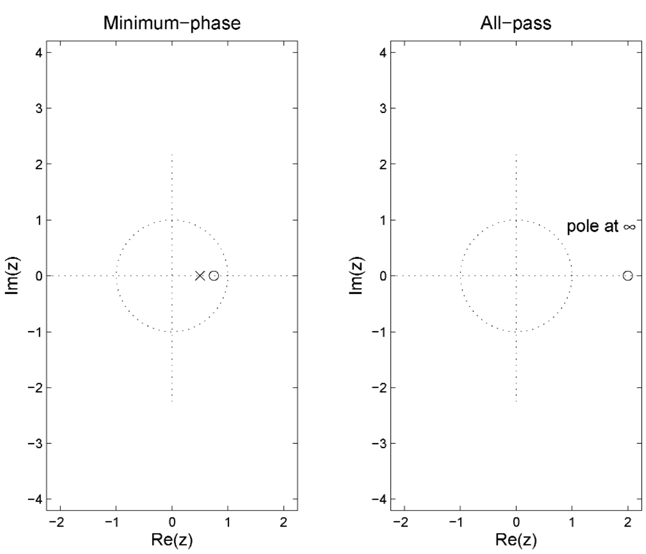
Determine a choice for $H_2(z)$ and $H_L(z)$ and specify whether or not these are unique up to a scale factor.

Solution from Fall05 PS2

The pole-zero diagram for the original system is as follows:



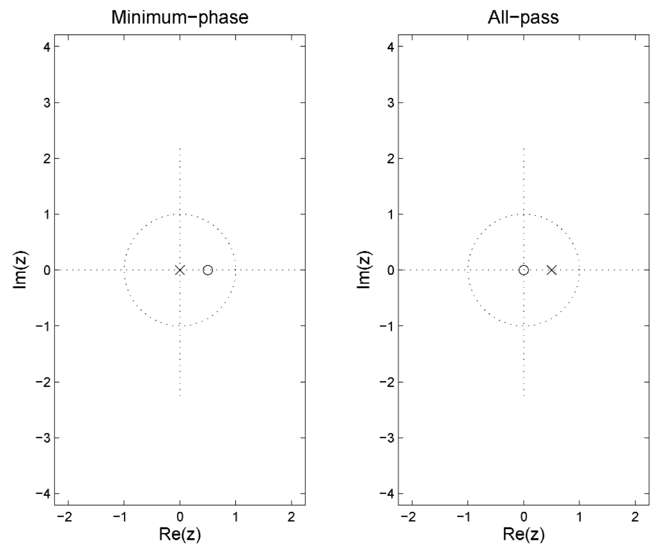
(a) One way to carry out the minimum-phase and all-pass decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the all-pass system.



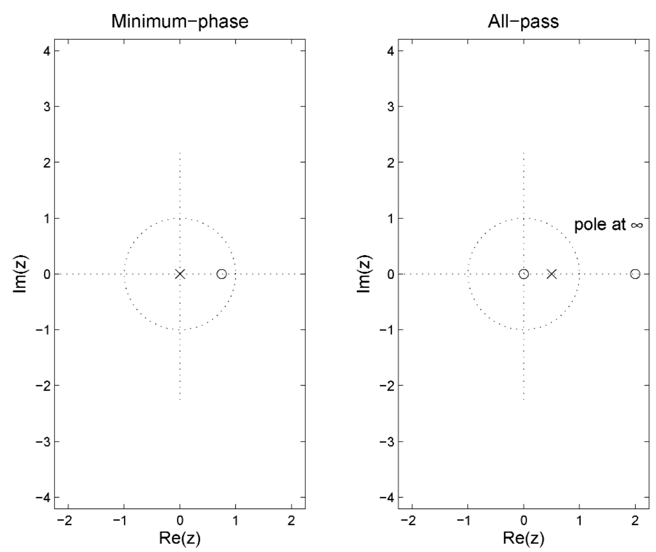
Next, we need to modify the first stage because we need to make sure that the all-pass

system really is all-pass, so add a pole at $z = 1/2$ and a zero at $z = 0$.

To preserve the original system, we can cancel these newcomers by placing a zero at $z = 1/2$ and a pole at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and all-pass systems are as shown below.



In the minimum-phase system, the pole at $z = 1/2$ from the first stage has been cancelled by the zero added in the second stage. Another way to look at that is that for this

particular system, we started with an all-pass pair (a pole at $z = 1/2$ and a zero at $z = 2$, so we could have put these into the all-pass system initially.

The minimum-phase system function is:

$$\begin{aligned} H_{M1}(z) &= \frac{z - \frac{3}{4}}{z} \\ &= 1 - \frac{3}{4}z^{-1} \end{aligned}$$

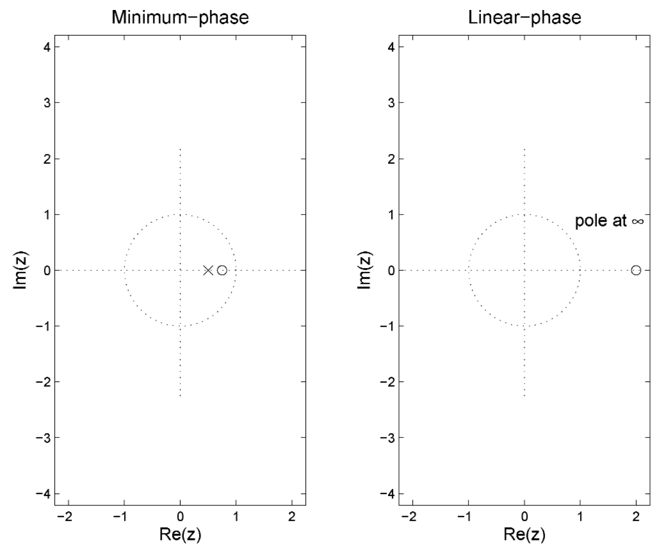
The all-pass system function is:

$$\begin{aligned} H_{ap}(z) &= \frac{z(z-2)}{z - \frac{1}{2}} \\ &= \frac{1 - 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) z^{-1}} \end{aligned}$$

In constructing these systems, we didn't come across any decision where we could have chosen different routes. If we wanted to change one of the systems, we would have to add the same number of poles and zeros to it, and these would have to be cancelled by zeros and poles in the other system to preserve the original system.

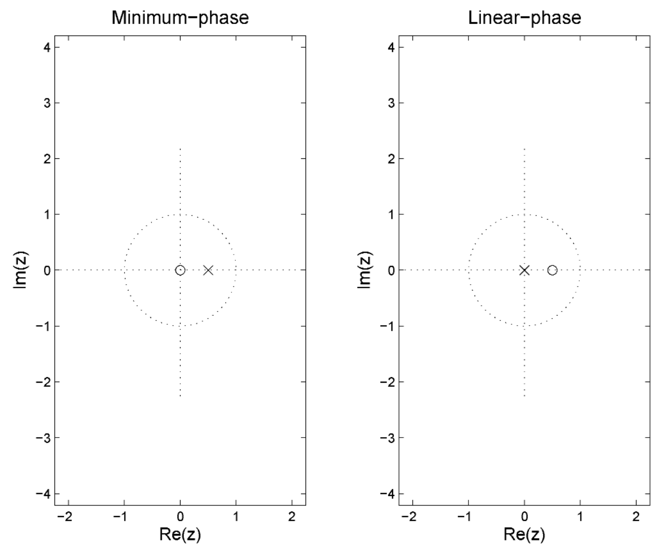
We can't add poles or zeros to the minimum phase system, because if we did, then when we added the cancelling zeros or poles to the all-pass system, they would have to be reflected outside the unit circle to keep the latter system all-pass. These items outside the unit circle could not be cancelled in the minimum phase system. Finally, we cannot change the all-pass system because if we added a zero and a pole, then to keep the system all-pass, we would have to reflect a pole or zero to the other side of the unit circle, and the items outside the unit circle could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique up to a scale factor.

(b) One way to carry out the minimum-phase and FIR linear-phase decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the linear-phase system.

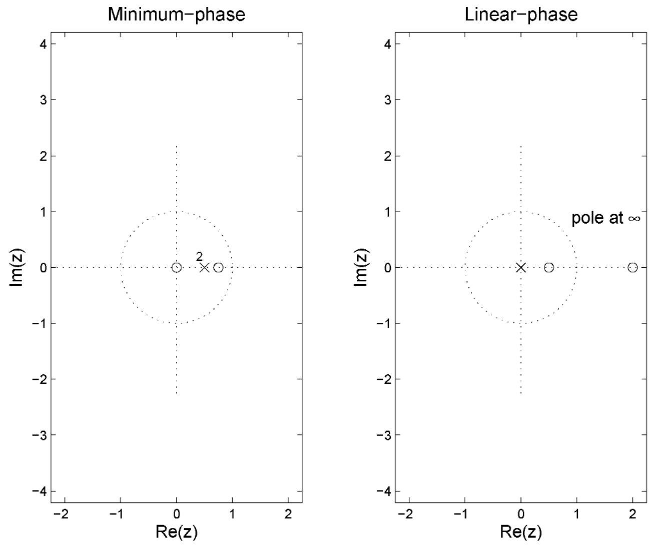


Next, we need to modify the first stage because we need to make sure that the linear-phase FIR system really is linear-phase FIR, so add a zero at $z = 1/2$. Since the system has to have the same number of zeros and poles, we also need to add a pole. For an FIR system, the pole must be at $z = 0$ or at $z = \infty$. We choose to add the pole at $z = 0$ because we will have to cancel the pole by a zero in the minimum-phase system.

To preserve the original system, we can cancel these newcomers by placing a pole at $z = 1/2$ and a zero at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and FIR linear-phase systems are as shown below.



The minimum phase system function is:

$$
\begin{aligned}
H_{M2}(z) &= \frac{z\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{2}\right)^2} \\
&= \frac{1-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}
\end{aligned}
$$

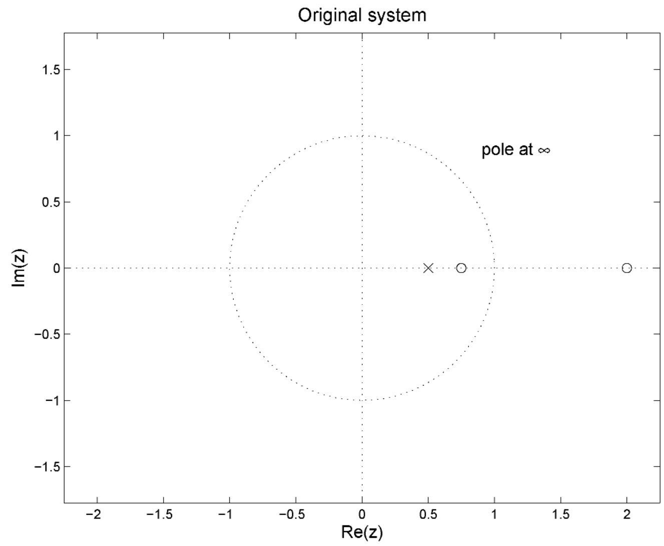
The FIR generalized linear-phase system function is:

$$
\begin{aligned}
H_L(z) &= \frac{\left(z-\frac{1}{2}\right)(z-2)}{z} \\
&= z\left[\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\right] \\
&= z-2.5+z^{-1}
\end{aligned}
$$

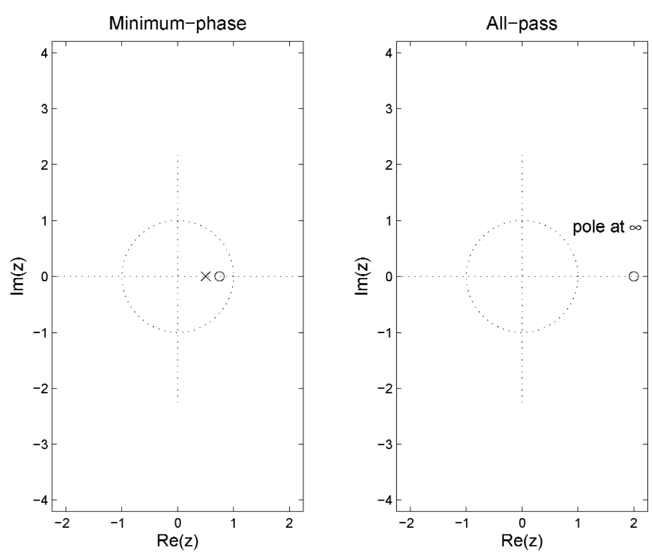
Since this expression for $H_L(z)$ has even symmetry and an odd number of taps, we would not necessarily expect a zero at $z=1$ or at $z=-1$, and this is consistent with the pole-zero diagram above. In constructing these systems, we didn't come across any decisions where we could have chosen different routes. Furthermore, we cannot change the minimum phase system. If we tried adding a pole and zero to it, these would have to be cancelled in the FIR linear phase system. But the zero in the linear-phase system would have to be reflected outside the unit circle to maintain linear-phase, and this could not be compensated for in the minimum-phase system. Similarly, we cannot add a pole and zero to the FIR linear-phase system because if we did, then to keep it linear-phase, we would have to reflect the zero outside the unit circle, and this could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

Solution from Spring05 PS2

The pole-zero diagram for the original system is as follows:



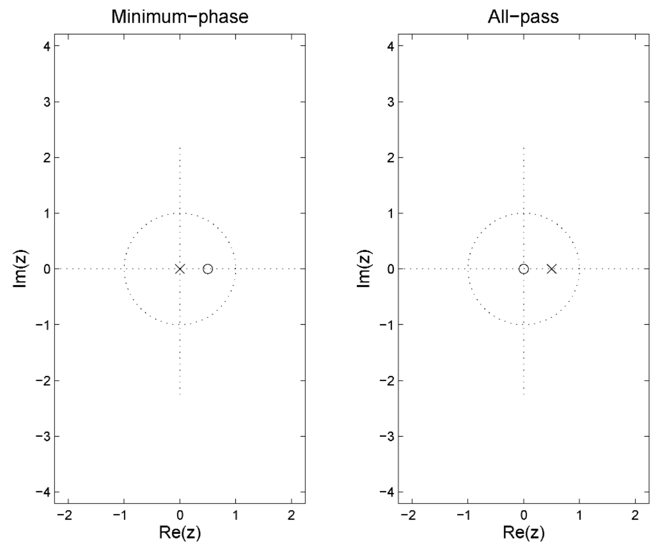
(a) One way to carry out the minimum-phase and all-pass decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the all-pass system.



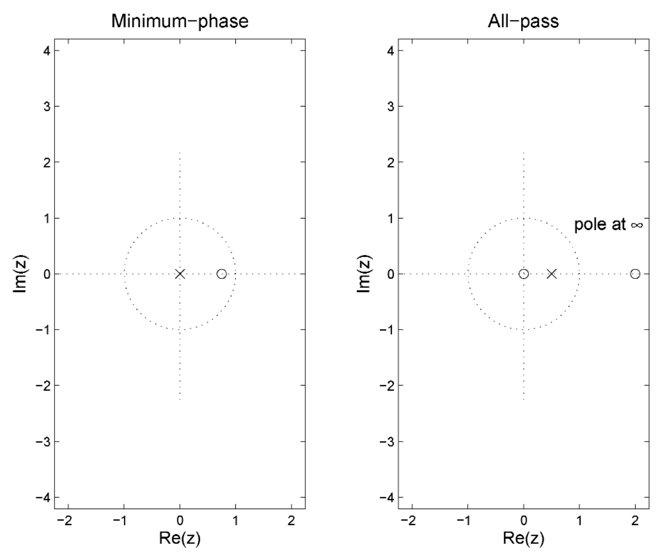
Next, we need to modify the first stage because we need to make sure that the all-pass

system really is all-pass, so add a pole at $z = 1/2$ and a zero at $z = 0$.

To preserve the original system, we can cancel these newcomers by placing a zero at $z = 1/2$ and a pole at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and all-pass systems are as shown below.



In the minimum-phase system, the pole at $z = 1/2$ from the first stage has been cancelled by the zero added in the second stage. Another way to look at that is that for this

particular system, we started with an all-pass pair (a pole at $z = 1/2$ and a zero at $z = 2$, so we could have put these into the all-pass system initially.

The minimum-phase system function is:

$$\begin{aligned} H_{M1}(z) &= \frac{z - \frac{3}{4}}{z} \\ &= 1 - \frac{3}{4}z^{-1} \end{aligned}$$

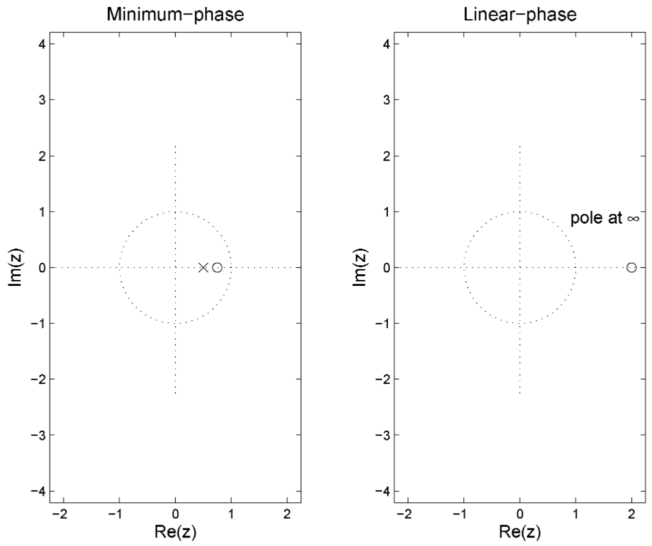
The all-pass system function is:

$$\begin{aligned} H_{ap}(z) &= \frac{z(z-2)}{z - \frac{1}{2}} \\ &= \frac{1 - 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) z^{-1}} \end{aligned}$$

In constructing these systems, we didn't come across any decision where we could have chosen different routes. If we wanted to change one of the systems, we would have to add the same number of poles and zeros to it, and these would have to be cancelled by zeros and poles in the other system to preserve the original system.

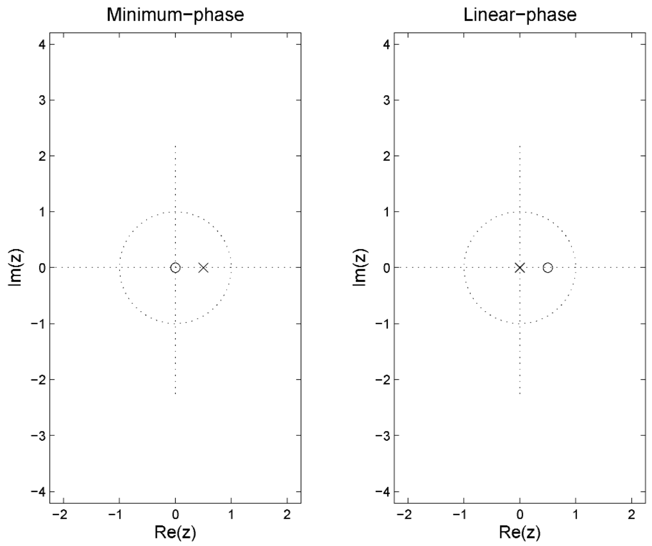
We can't add poles or zeros to the minimum phase system, because if we did, then when we added the cancelling zeros or poles to the all-pass system, they would have to be reflected outside the unit circle to keep the latter system all-pass. These items outside the unit circle could not be cancelled in the minimum phase system. Finally, we cannot change the all-pass system because if we added a zero and a pole, then to keep the system all-pass, we would have to reflect a pole or zero to the other side of the unit circle, and the items outside the unit circle could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

(b) One way to carry out the minimum-phase and FIR linear-phase decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the linear-phase system.

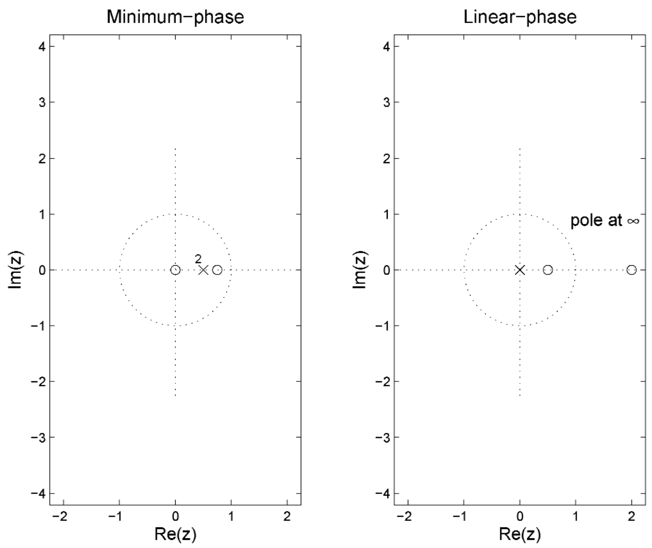


Next, we need to modify the first stage because we need to make sure that the linear-phase FIR system really is linear-phase FIR, so add a zero at $z = 1/2$. Since the system has to have the same number of zeros and poles, we also need to add a pole. For an FIR system, the pole must be at $z = 0$ or at $z = \infty$. We choose to add the pole at $z = 0$ because we will have to cancel the pole by a zero in the minimum-phase system.

To preserve the original system, we can cancel these newcomers by placing a pole at $z = 1/2$ and a zero at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and FIR linear-phase systems are as shown below.



The minimum phase system function is:

$$
\begin{aligned}
H_{M2}(z) &= \frac{z\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{2}\right)^2} \\
&= \frac{1-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}
\end{aligned}
$$

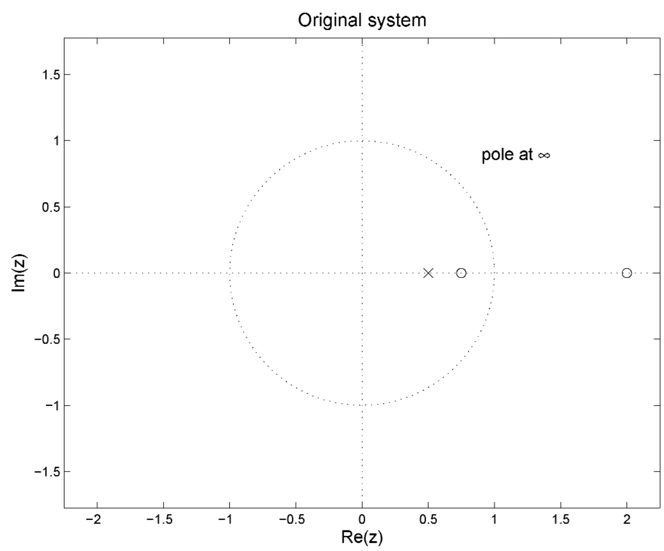
The FIR generalized linear-phase system function is:

$$
\begin{aligned}
H_L(z) &= \frac{\left(z-\frac{1}{2}\right)(z-2)}{z} \\
&= z\left[\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\right] \\
&= z-2.5+z^{-1}
\end{aligned}
$$

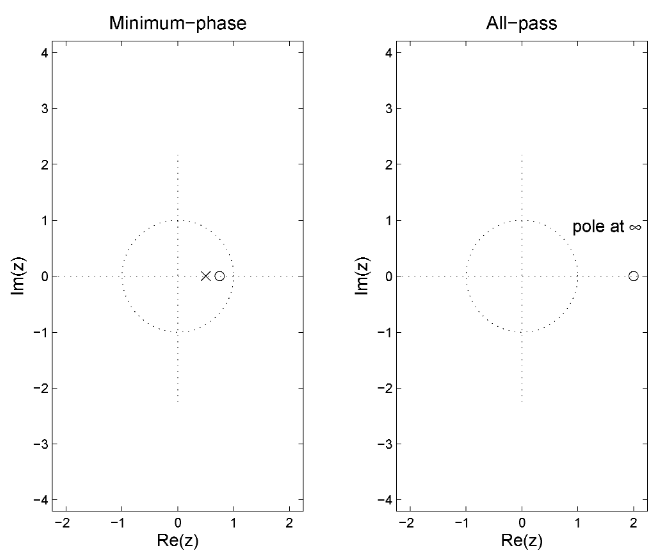
Since this expression for $H_L(z)$ has even symmetry and an odd number of taps, we would not necessarily expect a zero at $z=0$ or at $z=\pi$, and this is consistent with the pole-zero diagram above. In constructing these systems, we didn't come across any decisions where we could have chosen different routes. Furthermore, we cannot change the minimum phase system. If we tried adding a pole and zero to it, these would have to be cancelled in the FIR linear phase system. But the zero in the linear-phase system would have to be reflected outside the unit circle to maintain linear-phase, and this could not be compensated for in the minimum-phase system. Similarly, we cannot add a pole and zero to the FIR linear-phase system because if we did, then to keep it linear-phase, we would have to reflect the zero outside the unit circle, and this could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

Solution from Spring04 PS2

The pole-zero diagram for the original system is as follows:



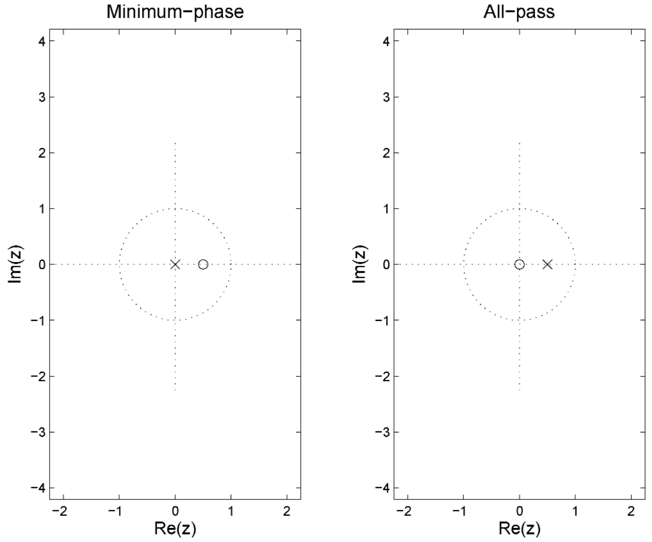
(a) One way to carry out the minimum-phase and all-pass decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the all-pass system.



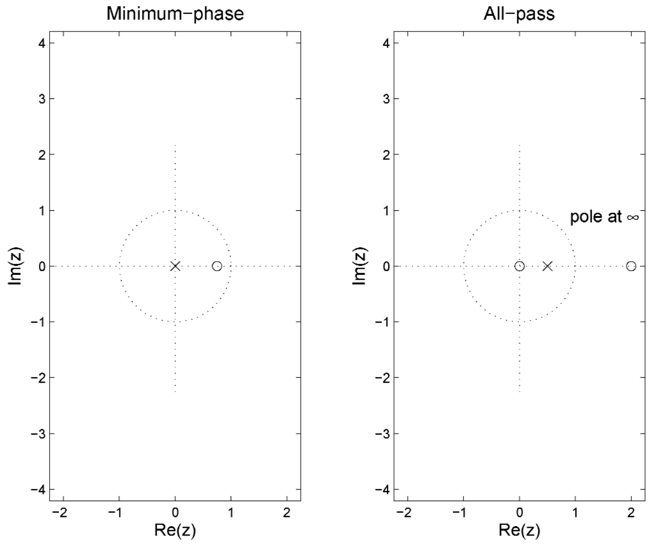
Next, we need to modify the first stage because we need to make sure that the all-pass

system really is all-pass, so add a pole at $z = 1/2$ and a zero at $z = 0$.

To preserve the original system, we can cancel these newcomers by placing a zero at $z = 1/2$ and a pole at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and all-pass systems are as shown below.



In the minimum-phase system, the pole at $z = 1/2$ from the first stage has been cancelled by the zero added in the second stage. Another way to look at that is that for this

particular system, we started with an all-pass pair (a pole at $z = 1/2$ and a zero at $z = 2$, so we could have put these into the all-pass system initially.

The minimum-phase system function is:

$$\begin{aligned} H_{M1}(z) &= \frac{z - \frac{3}{4}}{z} \\ &= 1 - \frac{3}{4}z^{-1} \end{aligned}$$

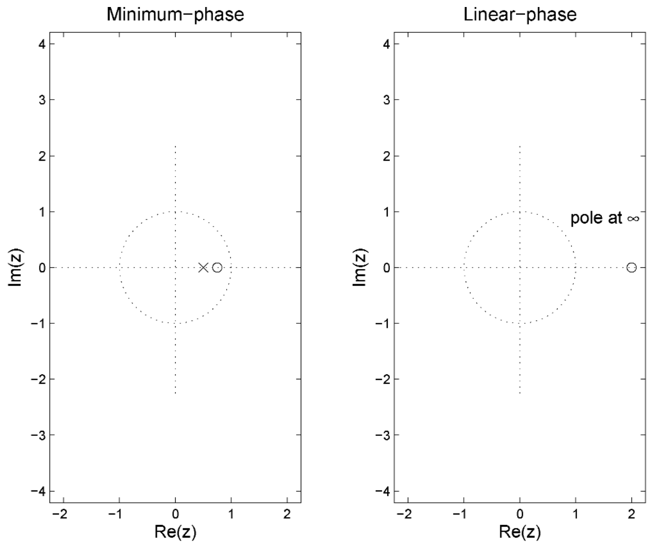
The all-pass system function is:

$$\begin{aligned} H_{ap}(z) &= \frac{z(z-2)}{z - \frac{1}{2}} \\ &= \frac{1 - 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) z^{-1}} \end{aligned}$$

In constructing these systems, we didn't come across any decision where we could have chosen different routes. If we wanted to change one of the systems, we would have to add the same number of poles and zeros to it, and these would have to be cancelled by zeros and poles in the other system to preserve the original system.

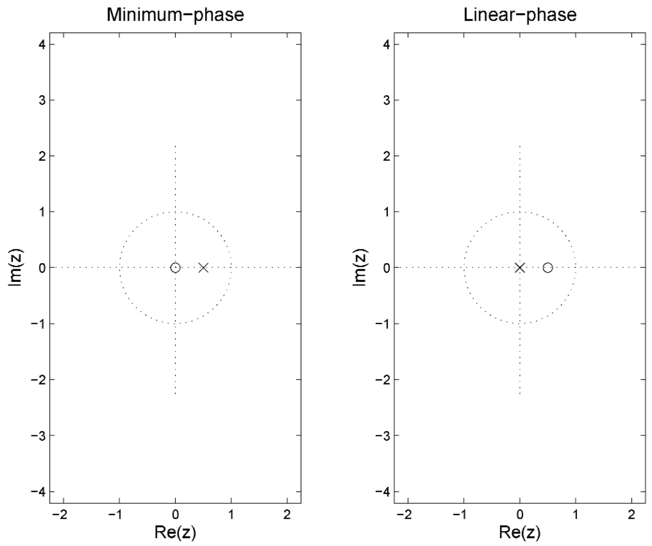
We can't add poles or zeros to the minimum phase system, because if we did, then when we added the cancelling zeros or poles to the all-pass system, they would have to be reflected outside the unit circle to keep the latter system all-pass. These items outside the unit circle could not be cancelled in the minimum phase system. Finally, we cannot change the all-pass system because if we added a zero and a pole, then to keep the system all-pass, we would have to reflect a pole or zero to the other side of the unit circle, and the items outside the unit circle could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

(b) One way to carry out the minimum-phase and FIR linear-phase decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the linear-phase system.

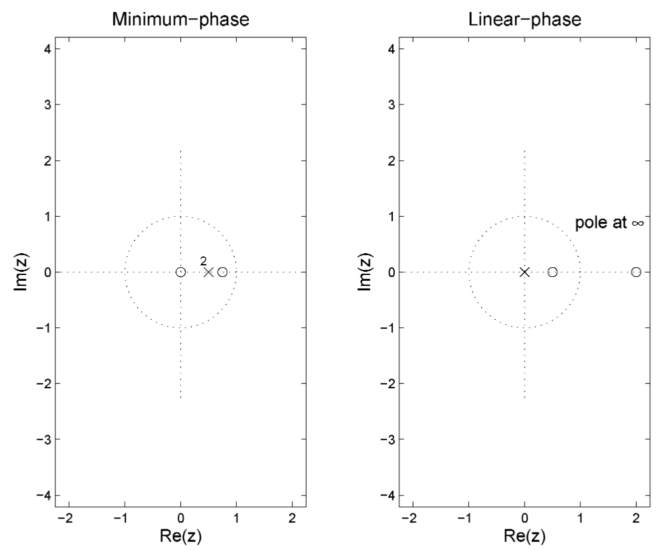


Next, we need to modify the first stage because we need to make sure that the linear-phase FIR system really is linear-phase FIR, so add a zero at $z = 1/2$. Since the system has to have the same number of zeros and poles, we also need to add a pole. For an FIR system, the pole must be at $z = 0$ or at $z = \infty$. We choose to add the pole at $z = 0$ because we will have to cancel the pole by a zero in the minimum-phase system.

To preserve the original system, we can cancel these newcomers by placing a pole at $z = 1/2$ and a zero at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and FIR linear-phase systems are as shown below.



The minimum phase system function is:

$$
\begin{aligned}
H_{M2}(z) &= \frac{z\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{2}\right)^2} \\
&= \frac{1-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}
\end{aligned}
$$

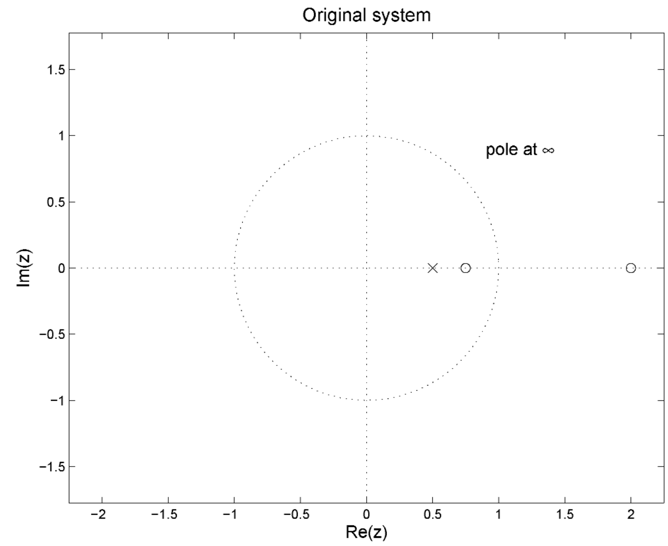
The FIR generalized linear-phase system function is:

$$
\begin{aligned}
H_L(z) &= \frac{\left(z-\frac{1}{2}\right)(z-2)}{z} \\
&= z\left[\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\right] \\
&= z-2.5+z^{-1}
\end{aligned}
$$

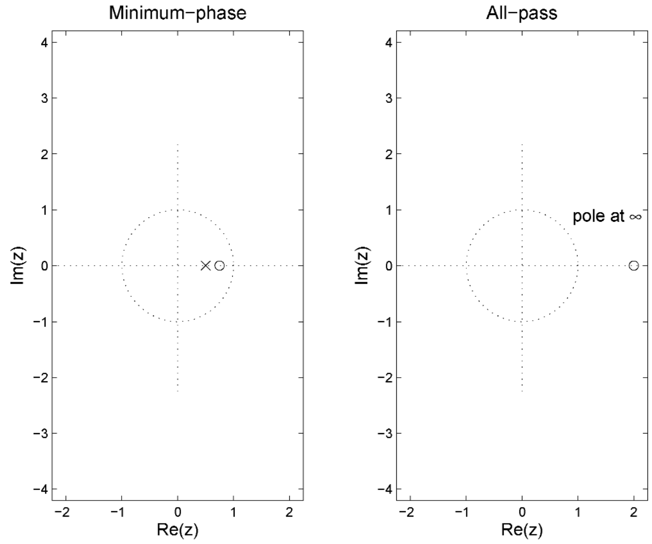
Since this expression for $H_L(z)$ has even symmetry and an odd number of taps, we would not necessarily expect a zero at $z=0$ or at $z=\pi$, and this is consistent with the pole-zero diagram above. In constructing these systems, we didn't come across any decisions where we could have chosen different routes. Furthermore, we cannot change the minimum phase system. If we tried adding a pole and zero to it, these would have to be cancelled in the FIR linear phase system. But the zero in the linear-phase system would have to be reflected outside the unit circle to maintain linear-phase, and this could not be compensated for in the minimum-phase system. Similarly, we cannot add a pole and zero to the FIR linear-phase system because if we did, then to keep it linear-phase, we would have to reflect the zero outside the unit circle, and this could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

Solution from Fall03 PS2

The pole-zero diagram for the original system is as follows:



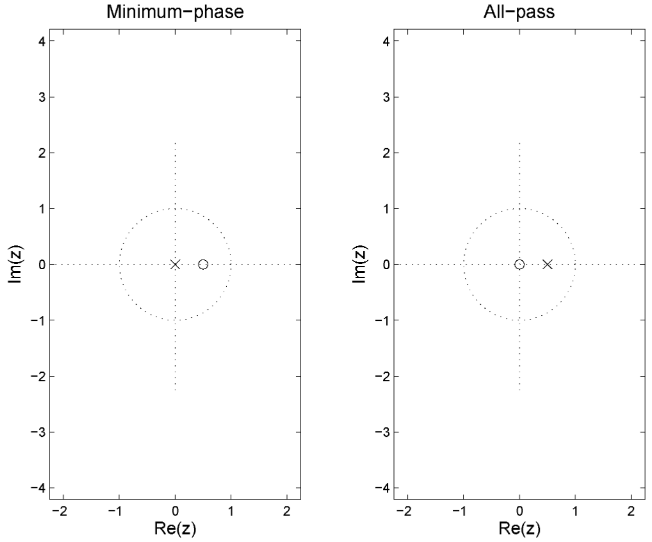
(a) One way to carry out the minimum-phase and all-pass decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the all-pass system.



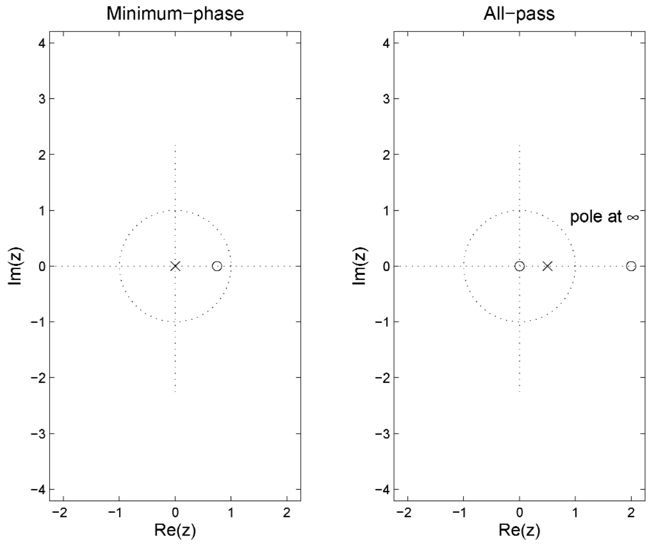
Next, we need to modify the first stage because we need to make sure that the all-pass

system really is all-pass, so add a pole at $z = 1/2$ and a zero at $z = 0$.

To preserve the original system, we can cancel these newcomers by placing a zero at $z = 1/2$ and a pole at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and all-pass systems are as shown below.



In the minimum-phase system, the pole at $z = 1/2$ from the first stage has been cancelled by the zero added in the second stage. Another way to look at that is that for this

particular system, we started with an all-pass pair (a pole at $z = 1/2$ and a zero at $z = 2$, so we could have put these into the all-pass system initially.

The minimum-phase system function is:

$$\begin{aligned} H_{M1}(z) &= \frac{z - \frac{3}{4}}{z} \\ &= 1 - \frac{3}{4}z^{-1} \end{aligned}$$

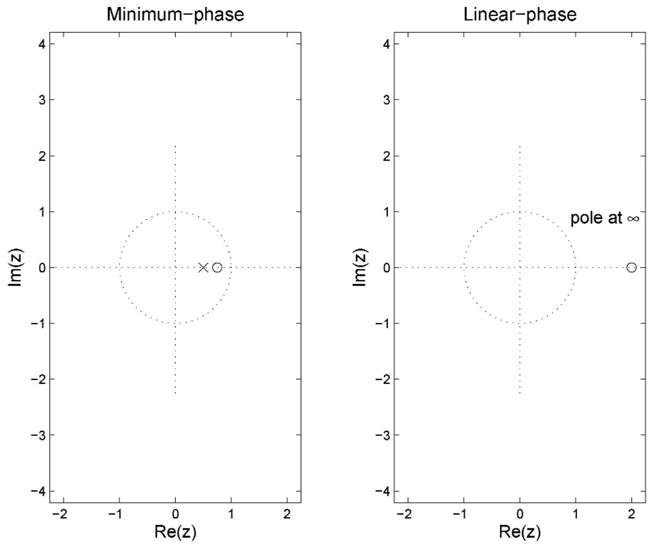
The all-pass system function is:

$$\begin{aligned} H_{ap}(z) &= \frac{z(z-2)}{z - \frac{1}{2}} \\ &= \frac{1 - 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) z^{-1}} \end{aligned}$$

In constructing these systems, we didn't come across any decision where we could have chosen different routes. If we wanted to change one of the systems, we would have to add the same number of poles and zeros to it, and these would have to be cancelled by zeros and poles in the other system to preserve the original system.

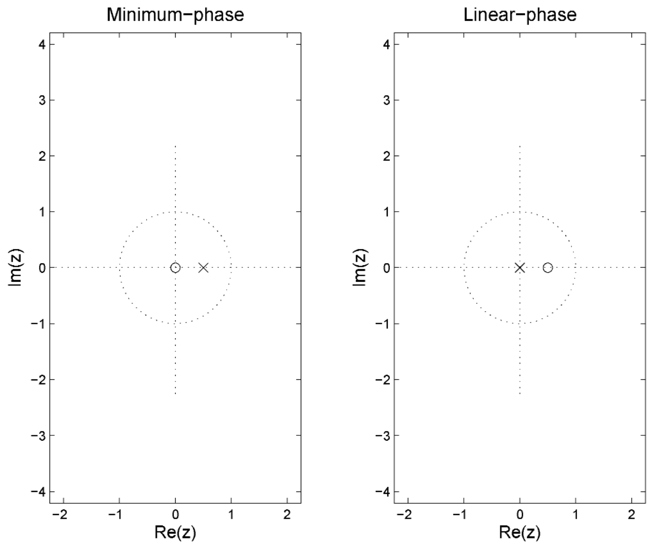
We can't add poles or zeros to the minimum phase system, because if we did, then when we added the cancelling zeros or poles to the all-pass system, they would have to be reflected outside the unit circle to keep the latter system all-pass. These items outside the unit circle could not be cancelled in the minimum phase system. Finally, we cannot change the all-pass system because if we added a zero and a pole, then to keep the system all-pass, we would have to reflect a pole or zero to the other side of the unit circle, and the items outside the unit circle could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

(b) One way to carry out the minimum-phase and FIR linear-phase decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the linear-phase system.

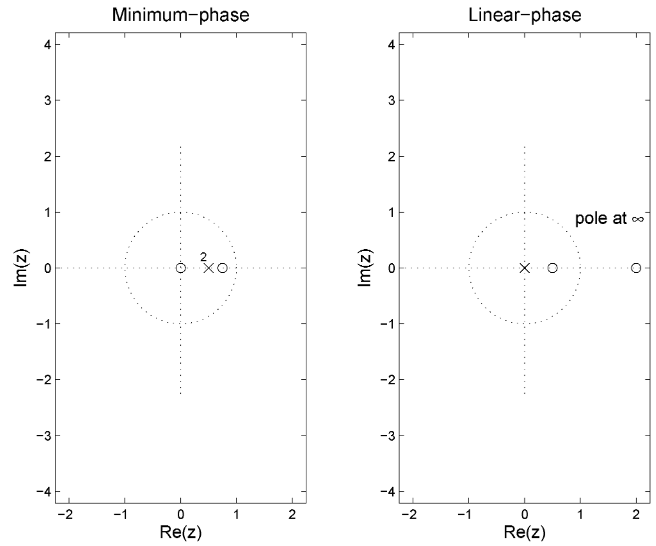


Next, we need to modify the first stage because we need to make sure that the linear-phase FIR system really is linear-phase FIR, so add a zero at $z = 1/2$. Since the system has to have the same number of zeros and poles, we also need to add a pole. For an FIR system, the pole must be at $z = 0$ or at $z = \infty$. We choose to add the pole at $z = 0$ because we will have to cancel the pole by a zero in the minimum-phase system.

To preserve the original system, we can cancel these newcomers by placing a pole at $z = 1/2$ and a zero at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and FIR linear-phase systems are as shown below.



The minimum phase system function is:

$$\begin{aligned} H_{M2}(z) &= \frac{z\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{2}\right)^2} \\ &= \frac{1-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \end{aligned}$$

The FIR generalized linear-phase system function is:

$$\begin{aligned} H_L(z) &= \frac{\left(z-\frac{1}{2}\right)(z-2)}{z} \\ &= z\left[\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\right] \\ &= z-2.5+z^{-1} \end{aligned}$$

Since this expression for $H_L(z)$ has even symmetry and an odd number of taps, we would not necessarily expect a zero at $z=0$ or at $z=\pi$, and this is consistent with the pole-zero diagram above. In constructing these systems, we didn't come across any decisions where we could have chosen different routes. Furthermore, we cannot change the minimum phase system. If we tried adding a pole and zero to it, these would have to be cancelled in the FIR linear phase system. But the zero in the linear-phase system would have to be reflected outside the unit circle to maintain linear-phase, and this could not be compensated for in the minimum-phase system. Similarly, we cannot add a pole and zero to the FIR linear-phase system because if we did, then to keep it linear-phase, we would have to reflect the zero outside the unit circle, and this could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

Solution from Spring03 PS2

(a) All-pass: Poles at $z = 1/2$ and ∞, and zeros at $z = 0$ and 2.
Min-Phase: Pole at $z = 0$, and zero at $z = 3/4$.

$$H_A(z) = \frac{A(1 - 2z^{-1})}{z^{-1}(1 - 0.5z^{-1})} \qquad H_1(z) = B(1 - 0.75z^{-1})$$

$H_A(z)$ and $H_1(z)$ are unique up to a scale factor, since we can't add poles and zeros to the all-pass sytem, because one of them would be outside the unit circle; which the min-phase system can't cancel.

(b) FIR: Poles at $z = 0$ and ∞, and zeros at $z = 1/2$ and 2.
Min-Phase: Second order pole at $z = 1/2$, and zeros at $z = 0$ and $3/4$.

$$H_L(z) = Cz(1 - 2z^{-1})(1 - 0.5z^{-1}) \qquad H_2(z) = \frac{D(1 - 0.75z^{-1})}{(1 - 0.5z^{-1})^2}$$

Due to similar reasoning with part (a), $H_L(z)$ and $H_2(z)$ are unique up to a scale factor.

5.34. Appears in: Fall05 PS1, Spring05 PS1, Fall04 PS1, Fall02 PS1, Spring01 PS2. Note: Spring01 PS2 uses different plots than Fall04 and Fall02. The problem statement in Spring01 has also been modified for Fall02 and Fall04. The Spring01 version of the problem is included after the Fall04 and Fall02 version.

Problem

Filter A is a discrete-time LTI system with input $x[n]$ and output $y[n]$.

The frequency response magnitude and group delay functions for Filter A are shown in Figure 1. The signal $x[n]$, also shown in Figure 1, is the sum of three narrowband pulses. In particular, Figure 1 contains the following plots:

- $x[n]$.
- $|X(e^{j\omega})|$, the Fourier transform magnitude of $x[n]$.
- Frequency response magnitude plot for filter A.
- Group delay plot for filter A.

In Figure 2 you are given 4 possible output signals, $y_i[n]$ $i = 1, 2, \ldots, 4$. Determine which one of the possible output signals is the output of filter A when the input is $x[n]$. Provide a justification for your choice.

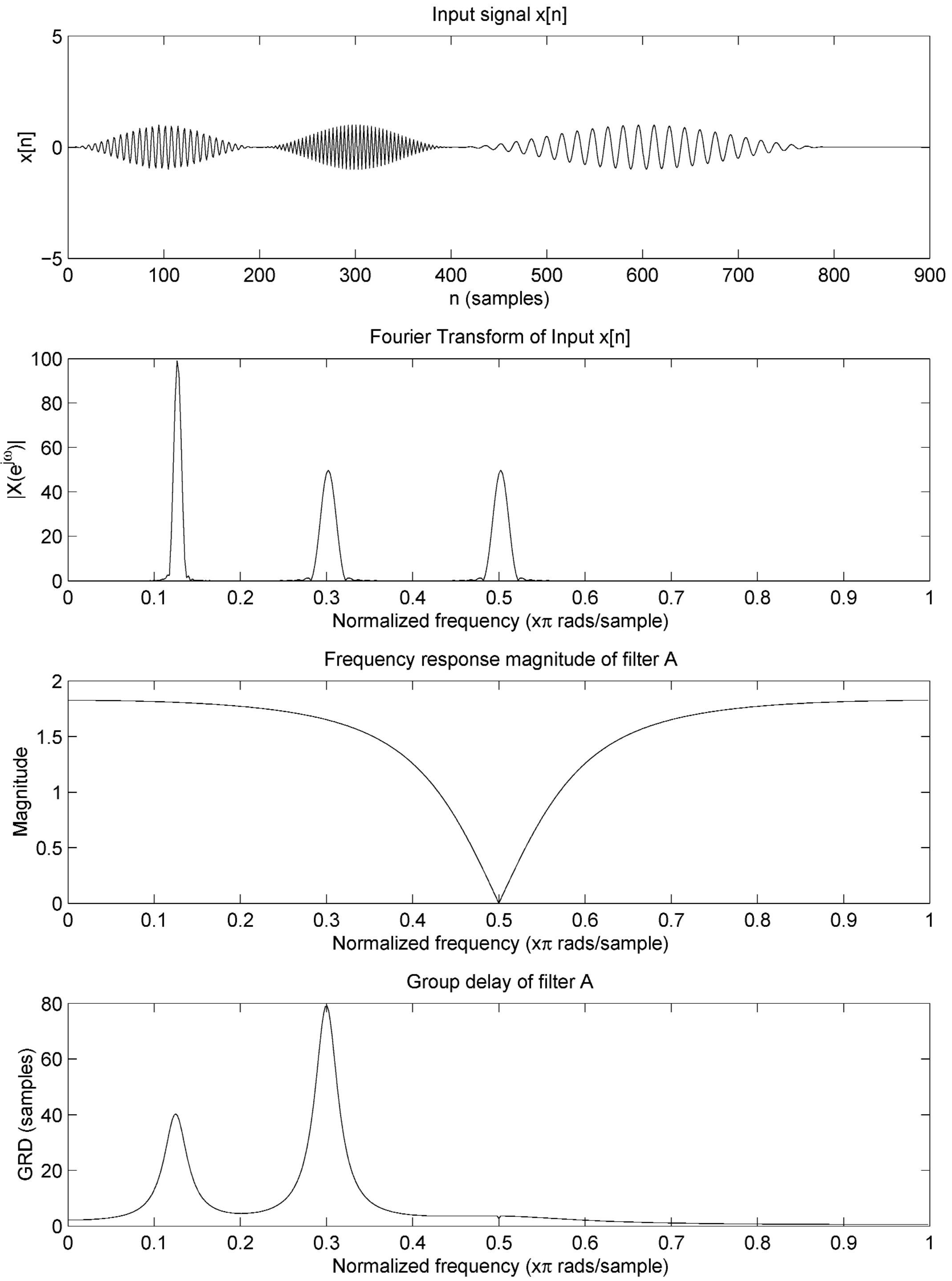


Figure 1: The signal and the filter for Problem

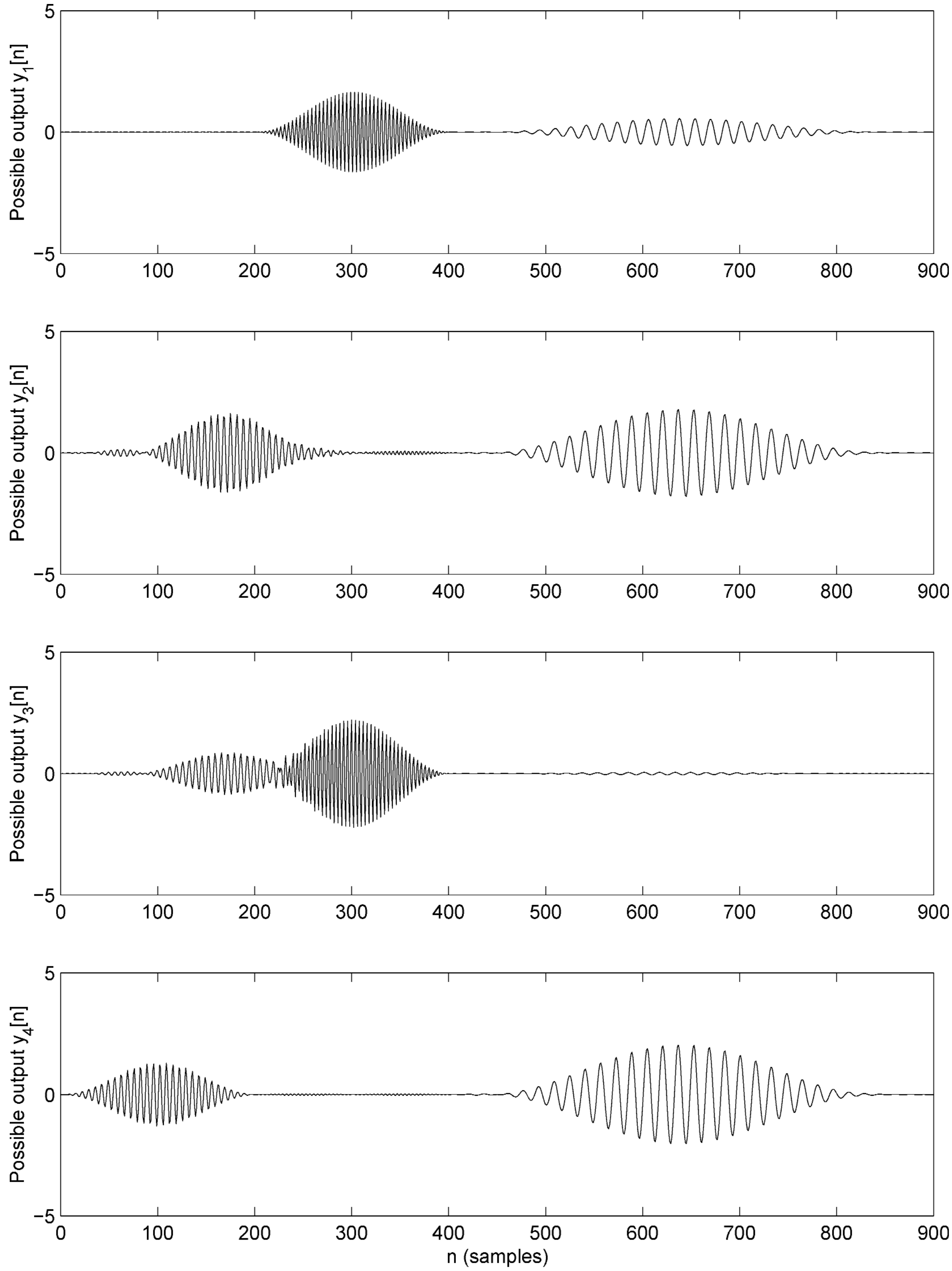


Figure 2: Possible output signals for Problem.

Spring01 Version of Problem Filter A is a discrete-time LTI system. Its frequency response magnitude and group delay functions are shown in Figure 2.6a. A signal, $x[n]$, also shown in Figure 2.6a, is the sum of three narrowband pulses which do not overlap in time. In Figures 2.6b and 2.6c you are given 8 possible output signals, $y_i[n]$ $i = 1, 2, \ldots, 8$. Determine which of the possible output signals is the output of filter A when the input is $x[n]$. **Clearly state your reasoning.**

Figure 2.6a contains the following plots:

- $x[n]$.
- $|X(e^{j\omega})|$, the Fourier transform magnitude of $x[n]$.
- Group delay plot for filter A.
- Frequency response magnitude plot for filter A.

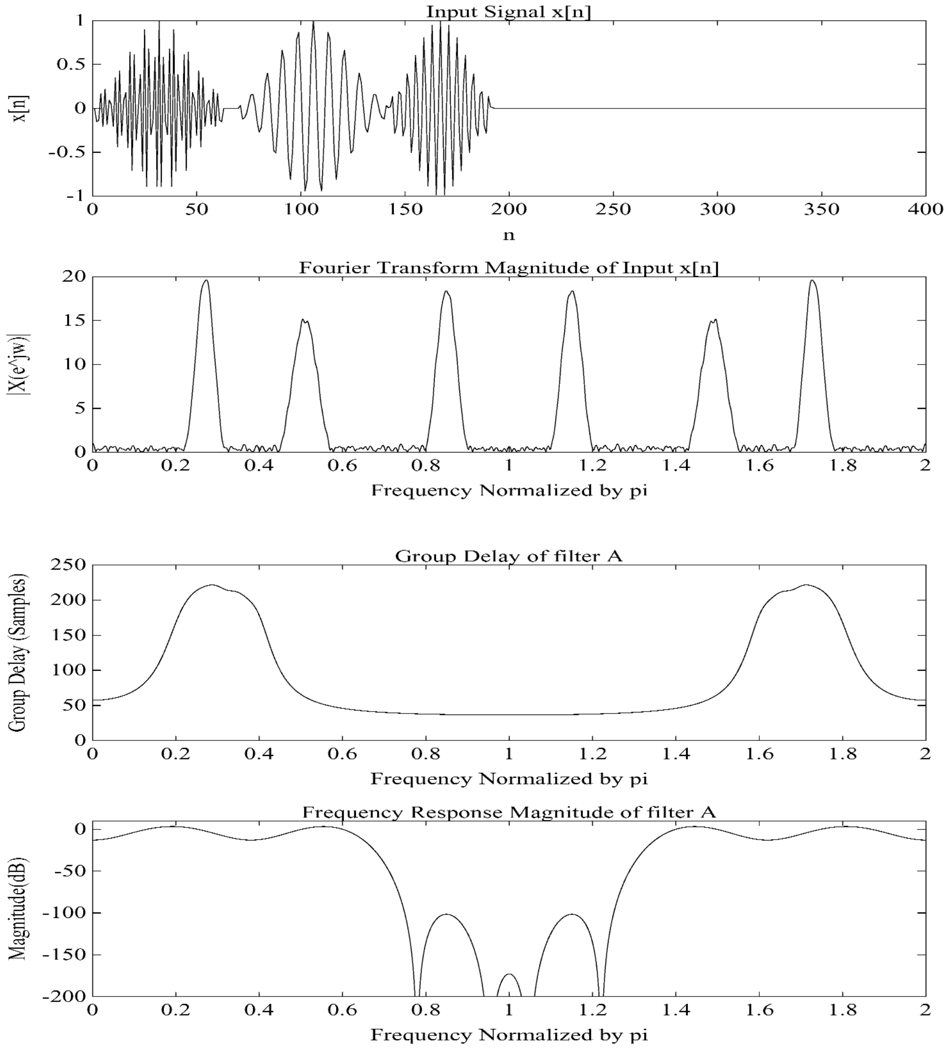


Figure 2.6a

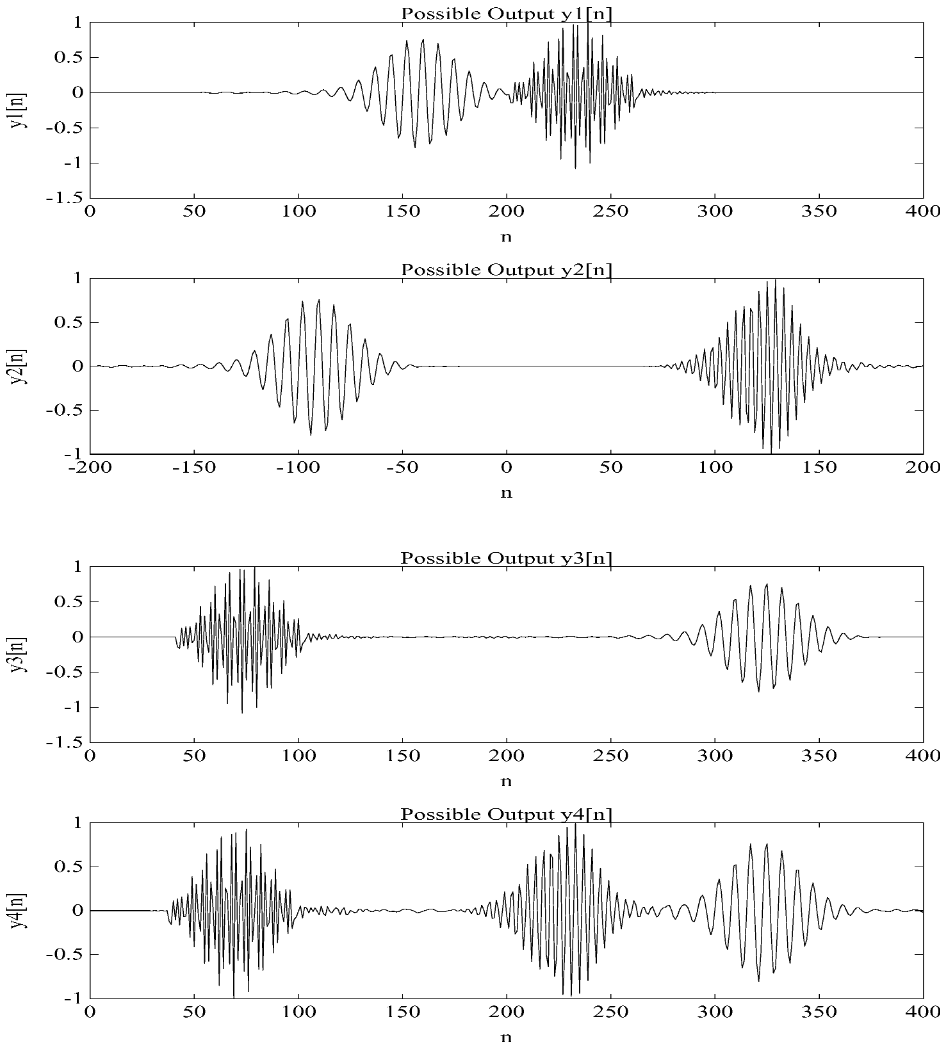

Possible Output y1[n]
y1[n]
n
Possible Output y2[n]
y2[n]
n
Possible Output y3[n]
y3[n]
n
Possible Output y4[n]
y4[n]
n

Figure 2.6b

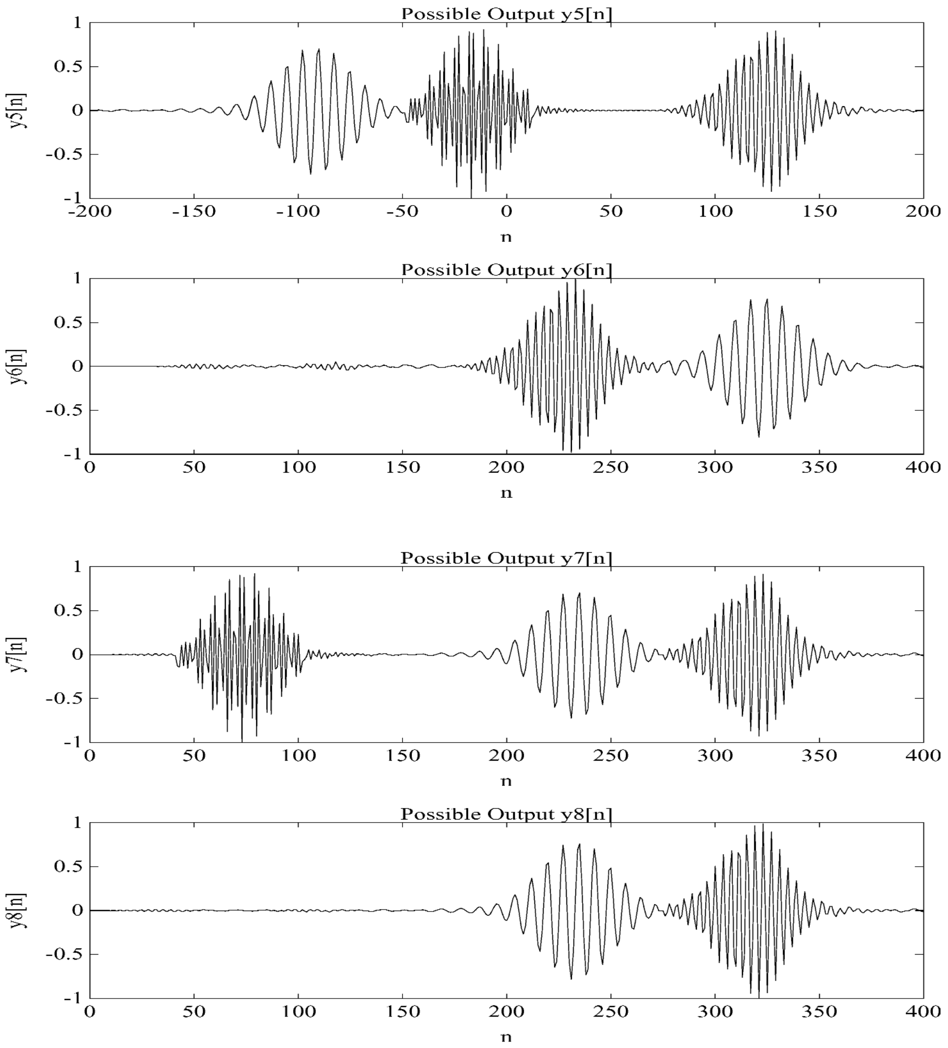

Possible Output y5[n]
y5[n]
n
Possible Output y6[n]
y6[n]
n
Possible Output y7[n]
y7[n]
n
Possible Output y8[n]
y8[n]
n

Figure 2.6c

Solution from Fall05 PS1

$y[n] = y_2[n]$

Justification:

The input signal $x[n]$ is made up of three narrow-band pulses: pulse-1 is a low-frequency pulse (whose peak is around 0.12π radians), pulse-2 is a higher-frequency pulse (0.3π radians), and pulse-3 is the highest-frequency pulse (0.5π radians).

Let $H(e^{j\omega})$ be the frequency response of Filter A. We read off the following values from the frequency response magnitude and group delay plots:

$$\begin{aligned}
|H(e^{j(0.12\pi)})| &\approx 1.8 \\
|H(e^{j(0.3\pi)})| &\approx 1.7 \\
|H(e^{j(0.5\pi)})| &\approx 0 \\
\tau_g(0.12\pi) &\approx 40 \text{ samples} \\
\tau_g(0.3\pi) &\approx 80 \text{ samples}
\end{aligned}$$

From these values, we would expect pulse-3 to be totally absent from the output signal $y[n]$. Pulse-1 will be scaled up by a factor of 1.8 and its envelope delayed by about 40 samples. Pulse-2 will be scaled up by a factor of 1.7 and its envelope delayed by about 80 samples. The correct output is thus $y_2[n]$.

Solution from Spring05 PS1

$y[n] = y_2[n]$

Justification:

We see that the input signal $x[n]$ is made up of three narrow-band pulses; pulse-1 is a low-frequency pulse (whose peak is at $.12\pi$ radians) pulse-2 is a higher-frequency pulse (whose peak is at $.3\pi$ radians), and pulse-3 is the highest-frequency pulse (whose peak is at $.5\pi$ radians).

From the given figure, we can also read off the following values of the filters frequency response magnitude and group delay. Call $H(e^{j\omega})$ the frequency response magnitude of Filter A. Then

$$\begin{aligned}
|H(e^{j(.12\pi)})| &\approx 1.8 \\
|H(e^{j(.3\pi)})| &\approx 1.75 \\
|H(e^{j(.5\pi)})| &\approx 0 \\
\tau_g(.12\pi) &\approx 40 \text{ samples} \\
\tau_g(.3\pi) &\approx 80 \text{ samples}
\end{aligned}$$

From these values, we would expect pulse-3 to be totally absent from the output signal $y[n]$. We would expect pulse-1 to be scaled up by a factor of 1.8, and its envelope delayed by about 40 samples. Pulse-2 will be scaled up by a factor of 1.75, with its envelope delayed by about 80 samples. The output which corresponds to this is $y_2[n]$.

Solution from Fall04 PS1

$y[n] = y_2[n]$

Justification:

We see that the input signal $x[n]$ is made up of three narrow-band pulses; pulse-1 is a low-frequency pulse of frequency $.12\pi$ radians, pulse-2 is a higher-frequency pulse of frequency $.3\pi$ radians, and pulse-3 is the highest-frequency pulse of frequency $.5\pi$ radians.

From the given figure, we can also read off the following values of the filters frequency response magnitude and group delay. Call $H(e^{j\omega})$ the frequency response magnitude of Filter A. Then

$$\begin{aligned} |H(e^{j(.12\pi)})| &\approx 1.8 \\ |H(e^{j(.3\pi)})| &\approx 1.75 \\ |H(e^{j(.5\pi)})| &\approx 0 \\ \tau_g(.12\pi) &\approx 40 \text{ samples} \\ \tau_g(.3\pi) &\approx 80 \text{ samples} \end{aligned}$$

From these values, we would expect pulse-3 to be totally absent from the output signal $y[n]$. We would expect pulse-1 to be scaled up by a factor of 1.8, and its envelope delayed by about 40 samples. Pulse-2 will be scaled up by a factor of 1.75, with its envelope delayed by about 80 samples. The output which corresponds to this is $y_2[n]$.

Solution from Fall02 PS1

$y[n] = y_2[n]$

Justification:

We see that the input signal $x[n]$ is made up of three narrow-band pulses; pulse-1 is a low-frequency pulse of frequency $.12\pi$ radians, pulse-2 is a higher-frequency pulse of frequency $.3\pi$ radians, and pulse-3 is the highest-frequency pulse of frequency $.5\pi$ radians.

From the given figure, we can also read off the following values of the filters frequency response magnitude and group delay. Call $H(e^{j\omega})$ the frequency response magnitude of Filter A. Then

$$\begin{aligned} |H(e^{j(.12\pi)})| &\approx 1.8 \\ |H(e^{j(.3\pi)})| &\approx 1.75 \\ |H(e^{j(.5\pi)})| &\approx 0 \\ \tau_g(.12\pi) &\approx 40 \text{ samples} \\ \tau_g(.3\pi) &\approx 80 \text{ samples} \end{aligned}$$

From these values, we would expect pulse-3 to be totally absent from the output signal $y[n]$. We would expect pulse-1 to be scaled up by a factor of 1.8, and its envelope delayed by about 40 samples. Pulse-2 will be scaled up by a factor of 1.75, with its envelope delayed by about 80 samples. The output which corresponds to this is $y_2[n]$.

Solution from Spring01 PS2

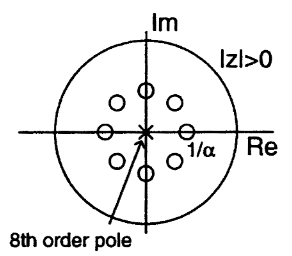
N/A

5.35. (a)

$$X(z) = S(z)(1 - e^{-8\alpha}z^{-8})$$

$$H_1(z) = 1 - e^{-8\alpha}z^{-8}$$

There are 8 zeros at $z = e^{-\alpha}e^{j\frac{\pi}{4}k}$ for $k = 0, \dots, 7$ and 8 poles at the origin.



(b)

$$Y(z) = H_2(z)X(z) = H_2(z)H_1(z)S(z)$$

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1}{1 - e^{-8\alpha}z^{-8}}$$

$|z| > e^{-\alpha}$ stable and causal, $|z| < e^{-\alpha}$ not causal or stable

(c) Only the causal $h_2[n]$ is stable, therefore only it can be used to recover $s[n]$.

$$h[n] = \begin{cases} e^{-\alpha n}, & n = 0, 8, 16, \dots \\ 0, & \text{otherwise} \end{cases}$$

(d)

$$s[n] = \delta[n] \Rightarrow x[n] = \delta[n] - e^{-8\alpha}\delta[n-8]$$

$$\begin{aligned} x[n] * h_2[n] &= \delta[n] - e^{-8\alpha}\delta[n-8] \\ &\quad + e^{-8\alpha}(\delta[n-8] - e^{-8\alpha}\delta[n-16]) \\ &\quad + e^{-16\alpha}(\delta[n-16] - e^{-8\alpha}\delta[n-32]) + \cdots \\ &= \delta[n] \end{aligned}$$

5.36.

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$$

(a)

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}} = \frac{2-\frac{5}{6}z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}, \quad |z| > \frac{1}{2}$$

Since $h[n], x[n] = 0$ for $n < 0$ we can assume initial rest conditions.

$$y[n] = \frac{5}{6}y[n-1] - \frac{1}{6}y[n-2] + 2x[n] - \frac{5}{6}x[n-1]$$

(b)

$$h_1[n] = \begin{cases} h[n], & n \le 10^9 \\ 0, & n > 10^9 \end{cases}$$

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{m=0}^{N-1} h[m]z^{-m}, \quad N = 10^9 + 1$$

$$y[n] = \sum_{m=0}^{N-1} h[m]x[n-m]$$

(d) For IIR, we have 4 multiplies and 3 adds per output point. This gives us a total of $4N$ multiplies and $3N$ adds. So, IIR grows with order N. For FIR, we have N multiplies and $N-1$ adds for the n^{th} output point, so this configuration has order N^2.

5.37. Convolving two symmetric sequences yields another symmetric sequence. A symmetric sequence convolved with an antisymmetric sequence gives an antisymmetric sequence. If you convolve two antisymmetric sequences, you will get a symmetric sequence.

$$A: \; h_1[n] * h_2[n] * h_3[n] = (h_1[n] * h_2[n]) * h_3[n]$$

$h_1[n] * h_2[n]$ is symmetric about $n = 3$, $\quad (-1 \leq n \leq 7)$
$(h_1[n] * h_2[n]) * h_3[n]$ is antisymmetric about $n = 3$, $\quad (-3 \leq n \leq 9)$

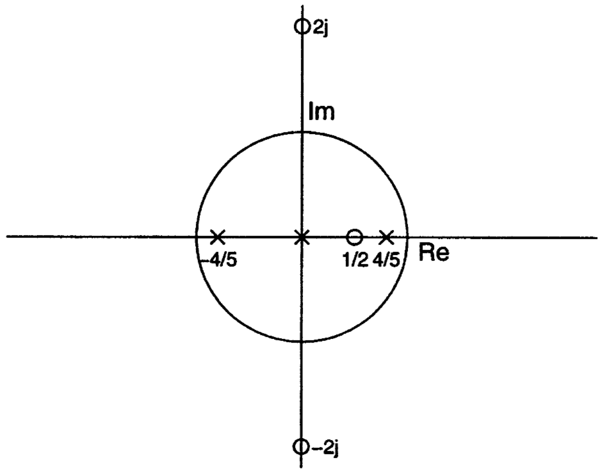
Thus, system A has generalized linear phase

$$B: \; (h_1[n] * h_2[n]) + h_3[n]$$

$h_1[n] * h_2[n]$ is symmetric about $n = 3$, as we noted above. Adding $h_3[n]$ to this sequence will destroy all symmtery, so this does not have generalized linear phase.

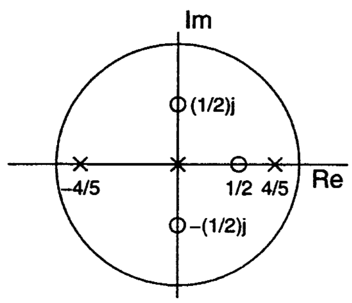
5.38.

$$H(z) = \frac{(1-0.5z^{-1})(1+2jz^{-1})(1-2jz^{-1})}{(1-0.8z^{-1})(1+0.8z^{-1})}$$

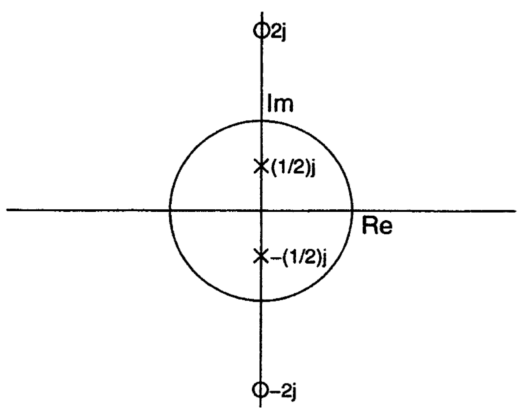


(a) A minimum phase system has all poles and zeros inside $|z| = 1$

$$H_1(z) = \frac{(1-0.5z^{-1})(1+\frac{1}{4}z^{-2})}{(1-0.64z^{-2})}$$

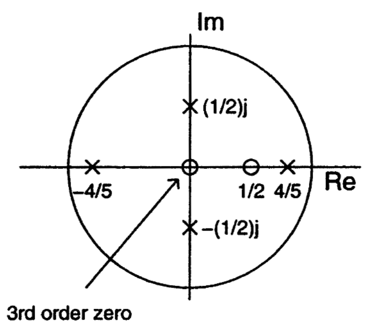


$$H_{ap}(z) = \frac{(1+4z^{-2})}{(1+\frac{1}{4}z^{-2})}$$

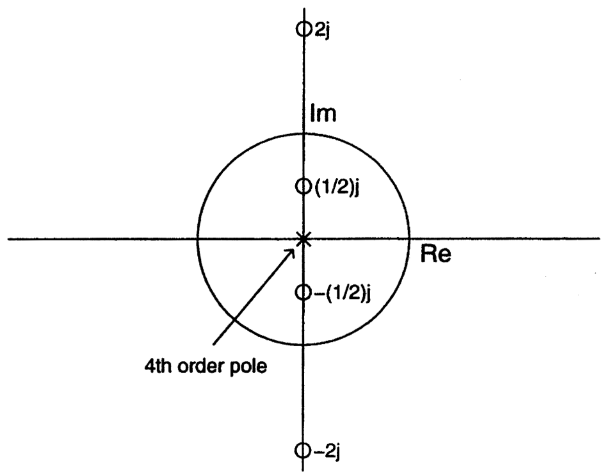


(b) A generalized linear phase system has zeros and poles at $z = 1, -1, 0$ or ∞ or in conjugate reciprocal pairs.

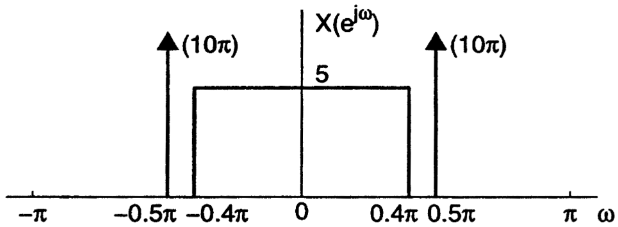
$$H_2(z) = \frac{(1 - 0.5z^{-1})}{(1 - 0.64z^{-2})(1 + \frac{1}{4}z^{-2})}$$



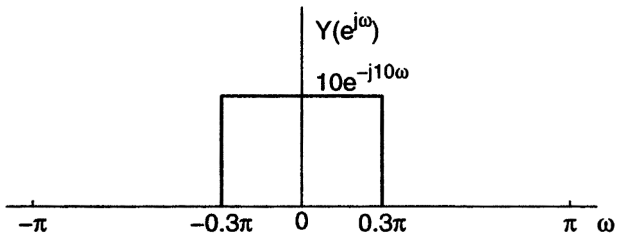
$$H_{lin}(z) = (1 + \tfrac{1}{4}z^{-2})(1 + 4z^{-2})$$



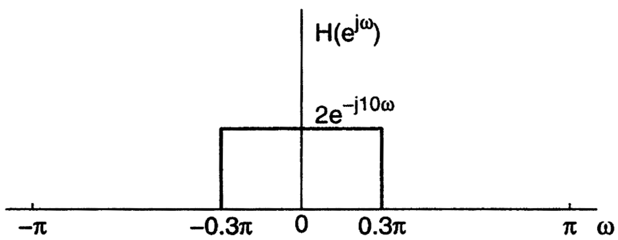
5.39. The input $x[n]$ in the frequency domain looks like



while the corresponding output $y[n]$ looks like



Therefore, the filter must be



In the time domain this is

$$h[n] = \frac{2\sin[0.3\pi(n-10)]}{\pi(n-10)}$$

5.40. Problem 1 in spring2003 midterm exam.

Problem

$H(z)$ is the system function for a stable LTI system and is given by:

$$H(z) = \frac{(1 - 9z^{-2})(1 + \frac{1}{3}z^{-1})}{1 - \frac{1}{3}z^{-1}}$$

(a) $H(z)$ can be represented as a cascade of a min-phase system $H_{min}(z)$ and a unity-gain all-pass system $H_A(z)$. Determine a choice for $H_{min}(z)$ and $H_A(z)$ and specify whether or not they are unique up to a scale factor.

(b) Is the min-phase system, $H_{min}(z)$, an FIR system? Explain.

(c) Is the min-phase system, $H_{min}(z)$, a generalized linear phase system? If not, can $H(z)$ be represented as a cascade of a generalized linear-phase system $H_{lp}(z)$ and an all-pass system $H_{A2}(z)$? If your answer is yes, determine $H_{lp}(z)$ and $H_{A2}(z)$. If your answer is no, explain why such representation does not exist.

Solution from Spring03 midterm

(a) To find the poles and zeros of $H(z)$, rewrite it as

$$H(z) = \frac{\left(z^2 - 9\right)\left(z + \frac{1}{3}\right)}{z^2\left(z - \frac{1}{3}\right)}$$

Zeros: $z = 3,\ -3,\ -1/3$

Poles: $z = 0,\ 0,\ 1/3$

The zeros at 3 and -3 can't be in the minimum phase system, so they must go in the all-pass system. In order to make the latter all-pass, it must also have poles at $1/3$ and $-1/3$. Since these were not part of $H(z)$, they must be cancelled by zeros in the minimum phase system. Inserting the zero at $1/3$ in the minimum phase system cancels the pole that was there.

$$\begin{aligned} H_{min}(z) &= \frac{1}{K}\left(1 + \frac{1}{3}z^{-1}\right)^2 \\ H_A(z) &= K\frac{\left(1 - 3z^{-1}\right)\left(1 + 3z^{-1}\right)}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)} \end{aligned}$$

The product of these two functions is the original $H(z)$ given in the problem. Since we want the all-pass system to have unity gain, $|H_A(z)| = 1$ for any z on the unit circle, e.g. $z = 1$. This yields $|K| = 1/9$.

Decompositions into minimum phase and all-pass systems are unique up to a scale factor.

(b) Yes, $H_{min}(z)$ is FIR. All its poles are at the origin.

(c) The phase of $H_{min}\left(e^{j\omega}\right)$ is

$$-\arctan\left(\frac{-\frac{2}{3}\sin(\omega) - \frac{1}{9}\sin(2\omega)}{1 + \frac{2}{3}\cos(\omega) + \frac{1}{9}\cos(2\omega)}\right)$$

This is not a linear or affine function of ω. However, we can rewrite $H(z)$ as the product of the following two systems:

$$\begin{aligned} H_{lp}(z) &= \left(1 + 3z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right) \\ H_{A2}(z) &= \frac{1 - 3z^{-1}}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$

This is equivalent to $H_{lp}(z) = 1 + (10/3)z^{-1} + z^{-2}$. The impulse response has even symmetry and the system is linear phase.

5.41. (a)

Property	Applies?	Comments
Stable	No	For a stable, causal system, all poles must be inside the unit circle.
IIR	Yes	The system has poles at locations other than $z = 0$ or $z = \infty$.
FIR	No	FIR systems can only have poles at $z = 0$ or $z = \infty$.
Minimum Phase	No	Minimum phase systems have all poles and zeros located inside the unit circle.
Allpass	No	Allpass systems have poles and zeros in conjugate reciprocal pairs.
Generalized Linear Phase	No	The causal generalized linear phase systems presented in this chapter are FIR.
Positive Group Delay for all w	No	This system is not in the appropriate form.

(b)

Property	Applies?	Comments
Stable	Yes	The ROC for this system function, $\|z\| > 0$, contains the unit circle. (Note there is 7th order pole at $z = 0$).
IIR	No	The system has poles only at $z = 0$.
FIR	Yes	The system has poles only at $z = 0$.
Minimum Phase	No	By definition, a minimum phase system must have all its poles and zeros located *inside* the unit circle.
Allpass	No	Note that the zeros on the unit circle will cause the magnitude spectrum to drop zero at certain frequencies. Clearly, this system is not allpass.
Generalized Linear Phase	Yes	This is the pole/zero plot of a type II FIR linear phase system.
Positive Group Delay for all w	Yes	This system is causal and linear phase. Consequently, its group delay is a positive constant.

(c)

Property	Applies?	Comments
Stable	Yes	All poles are inside the unit circle. Since the system is causal, the ROC includes the unit circle.
IIR	Yes	The system has poles at locations other than $z = 0$ or $z = \infty$.
FIR	No	FIR systems can only have poles at $z = 0$ or $z = \infty$.
Minimum Phase	No	Minimum phase systems have all poles and zeros located inside the unit circle.
Allpass	Yes	The poles inside the unit circle have corresponding zeros located at conjugate reciprocal locations.
Generalized Linear Phase	No	The causal generalized linear phase systems presented in this chapter are FIR.
Positive Group Delay for all w	Yes	Stable allpass systems have positive group delay for all w.

5.42. (a) *Yes.* By the region of convergence we know there are no poles at $z = \infty$ and it therefore must be causal. Another way to see this is to use long division to write $H_1(z)$ as

$$H_1(z) = \frac{1 - z^{-5}}{1 - z^{-1}} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}\ ,\ |z| > 0$$

(b) $h_1[n]$ is a causal rectangular pulse of length 5. If we convolve $h_1[n]$ with another causal rectangular pulse of length N we will get a triangular pulse of length $N + 5 - 1 = N + 4$. The triangular pulse is symmetric around its apex and thus has linear phase. To make the triangular pulse $g[n]$ have at least 9 nonzero samples we can choose $N = 5$ or let $h_2[n] = h_1[n]$.

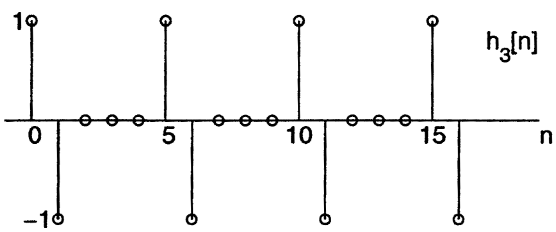
Proof:

$$\begin{aligned} G(e^{j\omega}) &= H_1(e^{j\omega})H_2(e^{j\omega}) = H_1^2(e^{j\omega}) \\ &= \left[\frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}}\right]^2 \\ &= \left[\frac{e^{-j\omega 5/2}\left(e^{j\omega 5/2} - e^{-j\omega 5/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}\right]^2 \\ &= \frac{\sin^2(5\omega/2)}{\sin^2(\omega/2)}\, e^{-j4\omega} \end{aligned}$$

(c) The required values for $h_3[n]$ can intuitively be worked out using the flip and slide idea of convolution. Here is a second way to get the answer. Pick $h_3[n]$ to be the inverse system for $h_1[n]$ and then simplify using the geometric series as follows.

$$\begin{aligned} H_3(z) &= \frac{1 - z^{-1}}{1 - z^{-5}} \\ &= (1 - z^{-1})\left[1 + z^{-5} + z^{-10} + z^{-15} + \cdots\right] \\ &= 1 - z^{-1} + z^{-5} - z^{-6} + z^{-10} - z^{-11} + z^{-15} - z^{-16} + \cdots \end{aligned}$$

This choice for $h_3[n]$ will make $q[n] = \delta[n]$ for all n. However, since we only need equality for $0 \le n \le 19$ truncating the infinite series will give us the desired result. The final answer is shown below.



5.43. (a) To be rational, X(z) must be of the form

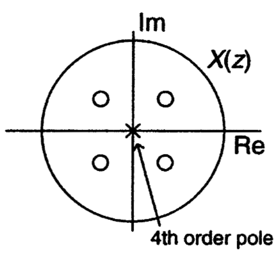
$$X(z) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Because x[n] is real, its zeros must appear in conjugate pairs. Consequently, there are two more zeros, at $z = \frac{1}{2}e^{-j\pi/4}$, and $z = \frac{1}{2}e^{-j3\pi/4}$. Since x[n] is zero outside $0 \le n \le 4$, there are only four zeros (and poles) in the system function. Therefore, the system function can be written as

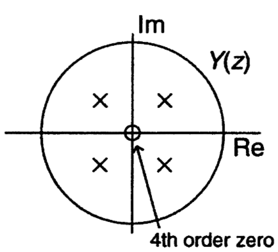
$$X(z) = \left(1 - \frac{1}{2}e^{j\pi/4}z^{-1}\right)\left(1 - \frac{1}{2}e^{j3\pi/4}z^{-1}\right)\left(1 - \frac{1}{2}e^{-j\pi/4}z^{-1}\right)\left(1 - \frac{1}{2}e^{-j3\pi/4}z^{-1}\right)$$

Clearly, X(z) is rational.

(b) A sketch of the pole-zero plot for $X(z)$ is shown below. Note that the ROC for X(z) is $|z| > 0$.



(c) A sketch of the pole-zero plot for $Y(z)$ is shown below. Note that the ROC for Y(z) is $|z| > \frac{1}{2}$.



5.44

A. Given

$$H(z)=\frac{z^{-2}\left(1-2z^{-1}\right)}{2\left(1-\frac{1}{2}z^{-1}\right)}, \quad |z|>\tfrac{1}{2},$$

we have

$$H\left(e^{j\omega}\right)=\frac{e^{-j2\omega}\left(1-2e^{-j\omega}\right)}{2\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$=-e^{-j3\omega}\frac{\left(1-\frac{1}{2}e^{j\omega}\right)}{\left(1-\frac{1}{2}e^{-j\omega}\right)}.$$

Now $1-\frac{1}{2}e^{j\omega}$ and $1-\frac{1}{2}e^{-j\omega}$ are complex conjugates, and therefore have the same magnitude. Further, $\left|-e^{-j3\omega}\right|=1$. We conclude that $H(z)$ is an all-pass system.

B. We can write

$$H(z)=\frac{\frac{1}{2}}{\left(1-\frac{1}{2}z^{-1}\right)}\left(1-2z^{-1}\right)z^{-2}, \quad |z|>\tfrac{1}{2}.$$

Then $H_{\min}(z)=\dfrac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}}$, $|z|>\frac{1}{2}$; $H_{\max}(z)=1-2z^{-1}$; and $H_d(z)=z^{-2}$.

Inverse transforming gives $h_{\min}[n]=\frac{1}{2}\left(\frac{1}{2}\right)^n u[n]$, $h_{\max}[n]=\delta(n)-2\delta[n-1]$, and $h_d[n]=\delta[n-2]$.

(Note that the factor of $\frac{1}{2}$ can be alternatively placed in the maximum-phase term.)

5.45

Assume that each impulse response corresponds to at most one frequency response.

A. $h_1[n]$ is a Type I FIR filter. Frequency reponse C corresponds to a Type I filter, as $\left|H_C\left(e^{j\omega}\right)\right| \neq 0$ for $\omega = 0$ or $\omega = \pi$.

B. $h_2[n]$ is a Type II FIR filter. Frequency response B corresponds to a Type II filter, as $\left|H_B\left(e^{j\omega}\right)\right| = 0$ for $\omega = \pi$.

C. $h_3[n]$ is a Type III filter. The frequency response must be D, as D is the only frequency response for which $\left|H_D\left(e^{j\omega}\right)\right| = 0$ for $\omega = 0$ and $\omega = \pi$.

D. $h_4[n]$ is a Type IV filter. Frequency response A corresponds to a Type IV filter, as $\left|H_A\left(e^{j\omega}\right)\right| = 0$ for $\omega = 0$.

5.46

A. Systems B, C, D, and E are IIR systems. All of these have poles at places other than the origin and infinity.

B. Systems A and F are FIR systems. These have poles only at the origin.

C. A causal LTI system is stable if and only if all of its poles lie inside the unit circle. Systems A, B, C, E, and F (i.e., all but D) are stable.

D. A stable causal system is minimum phase if its inverse system is also stable and causal. This means that all of the zeros as well as all of the poles must lie inside the unit circle. System E is the only minimum-phase system.

E. A system that is causal with a rational frequency response must be an FIR system to have linear phase. Both systems A and F are linear phase systems, as for both of these systems the zeros occur in reciprocal pairs or at $z = \pm 1$.

F. System C is allpass. It is the only system for which poles and zeros occur in conjugate reciprocal pairs.

G. Only System E has a stable and causal inverse. This is the only system having all of its zeros inside the unit circle.

H. System F has the shortest impulse response, with seven nonzero samples. System A has 12 nonzero samples, and the remaining systems are IIR.

I. Systems A and F are lowpass systems. Systems B and D are eliminated as they each have a zero at $\omega = 0$. System C is an allpass system. System E will have a frequency response whose magnitude tends to peak at frequencies near the system poles. None of these poles are near $\omega = 0$ and one of them is near $\omega = \pi$.

J. The “minimum group delay” property is an attribute of a minimum phase system. System E is the only minimum phase system in the given set. (Note that System E has the minimum group delay among systems with the same magnitude response. System E may not have the minimum group delay among the systems shown.)

5.47

A.

$$\begin{aligned} H\left(e^{j\omega}\right) &= H_1\left(e^{j\omega}\right)H_2\left(e^{j\omega}\right) \\ &= jA_1\left(e^{j\omega}\right)A_2\left(e^{j\omega}\right)e^{-j\omega(M_1+M_2)/2}. \end{aligned}$$

B. The overall impulse response has length $M_1 + M_2 + 1$.

C. The delay of the overall system is $(M_1 + M_2)/2$ samples.

D. The overall system is a Type-IV generalized linear-phase system. Note that $M_1 + M_2$ is an odd integer.

5.48. (a)

$$H(z) = \frac{A}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \qquad |z| > \frac{1}{2} \;\; h[n] \text{ causal}$$

$$H(1) = 6 \Rightarrow A = 4$$

(b)

$$\begin{aligned} H(z) &= \frac{4}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \qquad |z| > \frac{1}{2} \\ &= \frac{(\frac{12}{5})}{1-\frac{1}{2}z^{-1}} + \frac{(\frac{8}{5})}{1+\frac{1}{3}z^{-1}} \\ h[n] &= \frac{12}{15}\left(\frac{1}{2}\right)^n u[n] + \frac{8}{5}\left(-\frac{1}{3}\right)^n u[n] \end{aligned}$$

(c) (i)

$$x[n] = u[n] - \frac{1}{2}u[n-1] \Leftrightarrow X(z) = \frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}}, \qquad |z| > 1$$

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= \frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}} \cdot \frac{4}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \qquad |z| > 1 \\ &= \frac{4}{(1-z^{-1})(1+\frac{1}{3}z^{-1})} \\ &= \frac{3}{1-z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \\ y[n] &= 3u[n] + \left(-\frac{1}{3}\right)^n u[n] \end{aligned}$$

(ii)

$$x(t) = 50 + 10\cos(20\pi t) + 30\cos(40\pi t)$$

$$T = \frac{1}{40} \qquad t = nT$$

$$\begin{aligned} x[n] &= 50 + 10\cos\frac{\pi}{2}n + 30\cos\pi n \\ &= 50 + 5e^{j(n\pi/2)} + 5e^{-j(n\pi/2)} + 15e^{jn\pi} + 15e^{-jn\pi} \end{aligned}$$

Using the eigenfunction property:

$$y[n] = 50H(e^{j0}) + 5e^{j(n\pi/2)}H(e^{j(\pi/2)}) + 5e^{-j(n\pi/2)}H(e^{-j(\pi/2)}) + 15e^{jn\pi}H(e^{j\pi}) + 15e^{-jn\pi}H(e^{-j\pi})$$

$$H(e^{j\omega}) = \frac{4}{1-\frac{1}{6}e^{-j\omega}-\frac{1}{6}e^{-j2\omega}}$$

$H(e^{j0}) = 6$, $H(e^{j(\pi/2)}) = 7\left(\frac{12}{25}\right) - j\frac{12}{25}$, $H(e^{-j(\pi/2)}) = 7\left(\frac{12}{25}\right) + j\frac{12}{25}$,
$H(e^{j\pi}) = 4$, $H(e^{-j\pi}) = 4$

$$y[n] = 300 + 24\sqrt{2}\cos\left(\frac{\pi}{2}n - \tan^{-1}\left(\frac{1}{7}\right)\right) + 120\cos\pi n$$

5.49.

$$\begin{aligned} H(z) &= \frac{21}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})(1-4z^{-1})} \\ &= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{28}{1-2z^{-1}} + \frac{48}{1-4z^{-1}} \end{aligned}$$

Since we know the sequence is not stable, the ROC must not include $|z| = 1$, and since it is two-sided, the ROC must be a ring. This leaves only one possible choice: the ROC is $2 < |z| < 4$.

(a)

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - 28(2)^n u[n] - 48(4)^n u[-n-1]$$

(b)

$$H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{28}{1-2z^{-1}}$$

$$H_2(z) = \frac{48}{1-4z^{-1}}$$

5.50. Since $H(e^{jw})$ has a zero on the unit circle, its inverse system will have a pole on the unit circle and thus is not stable.

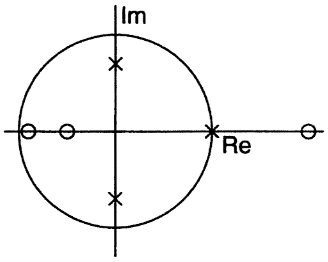
5.51. (a)

$$\begin{aligned} H(z) &= \frac{(1-2z^{-1})(1+\frac{1}{2}z^{-1})(1+0.9z^{-1})}{(1-z^{-1})(1+0.7jz^{-1})(1-0.7jz^{-1})} \\ &= \frac{1-0.6z^{-1}-2.35z^{-2}-0.9z^{-3}}{1-z^{-1}+0.49z^{-2}-0.49z^{-3}} \\ &= \frac{Y(z)}{X(z)} \end{aligned}$$

Cross multiplying and taking the inverse z-transform gives

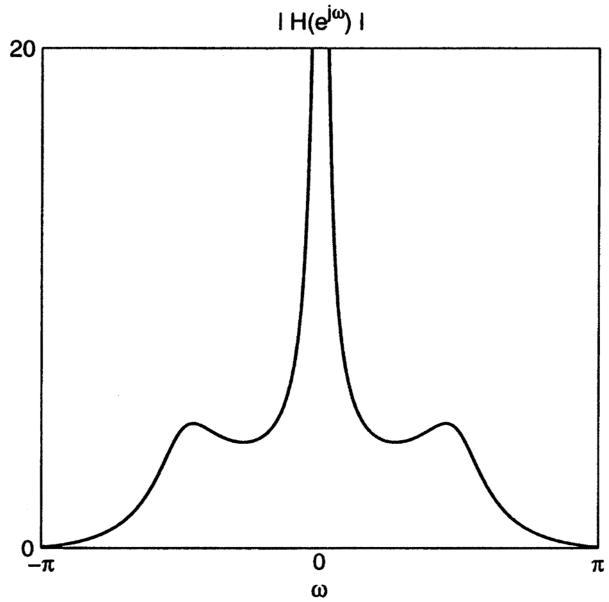
$$y[n]-y[n-1]+0.49y[n-2]-0.49y[n-3] = x[n]-0.6x[n-1]-2.35x[n-2]-0.9x[n-3]$$

(b)

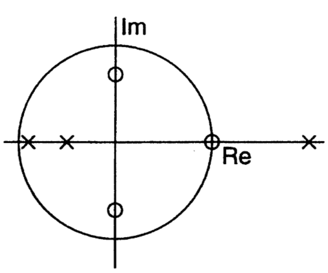


Note that since $h[n]$ is causal, ROC is $|z| > 1$.

(c)



(d) (i) The system is not stable since the ROC does not include $|z| = 1$.
(ii) Because $h[n]$ is not stable, $h[n]$ does not approach a constant as $n \to \infty$.
(iii) We can see peaks at $\omega = \pm\frac{\pi}{2}$ in the graph of $|H(e^{j\omega})|$ shown in part (c), so this is false.
(iv) Swapping poles and zeros gives:



There is a ROC that includes the unit circle $(0.9 < |z| < 2)$. However, this stable system would be two sided, so we must conclude the statement is false.

5.52.

$$X(z) = \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-\frac{1}{5}z)}{(1-\frac{1}{6}z)} = \frac{6}{5}\frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-5z^{-1})}{(1-6z^{-1})}$$

$$\alpha^n x[n] \Leftrightarrow X(\alpha^{-1}z) = \frac{6}{5}\frac{(1-\frac{1}{2}\alpha z^{-1})(1-\frac{1}{4}\alpha z^{-1})(1-5\alpha z^{-1})}{(1-6\alpha z^{-1})}$$

A minimum phase sequence has all poles and zeros inside the unit circle.

$$\begin{aligned}
|\alpha/2| < 1 \quad &\Rightarrow \quad |\alpha| < 2 \\
|\alpha/4| < 1 \quad &\Rightarrow \quad |\alpha| < 4 \\
|5\alpha| < 1 \quad &\Rightarrow \quad |\alpha| < \frac{1}{5} \\
|6\alpha| < 1 \quad &\Rightarrow \quad |\alpha| < \frac{1}{6}
\end{aligned}$$

Therefore, $\alpha^n x[n]$ is real and minimum phase iff α is real and $|\alpha| < \frac{1}{6}$.

5.53. (a) The causal systems have conjugate zero pairs inside or outside the unit circle. Therefore

$$\begin{aligned}
H(z) &= (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1}) \\
H_1(z) &= (0.9)^2(1.25)^2(1-(10/9)e^{j0.6\pi}z^{-1})(1-(10/9)e^{-j0.6\pi}z^{-1}) \\
&\quad (1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1}) \\
H_2(z) &= (0.9)^2(1-(10/9)e^{j0.6\pi}z^{-1})(1-(10/9)e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1}) \\
&\quad (1-1.25e^{-j0.8\pi}z^{-1}) \\
H_3(z) &= (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1}) \\
&\quad (1-0.8e^{-j0.8\pi}z^{-1})
\end{aligned}$$

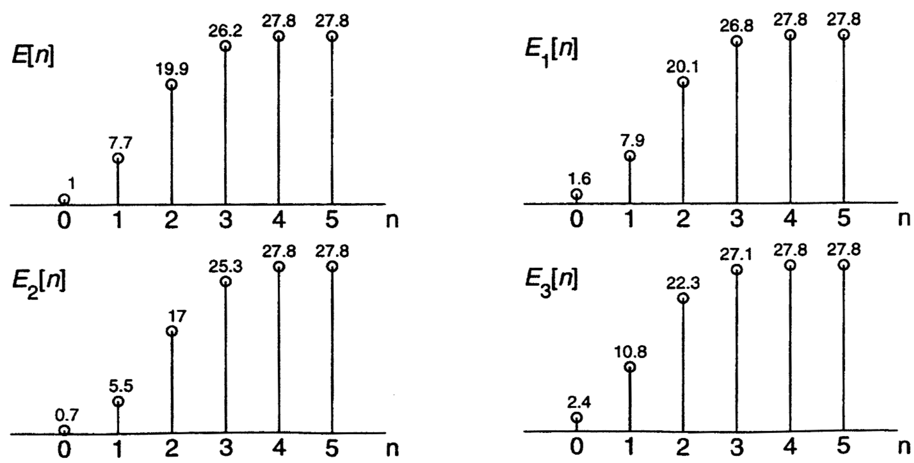
$H_2(z)$ has all its zeros outside the unit circle, and is a maximum phase sequence. $H_3(z)$ has all its zeros inside the unit circle, and thus is a minimum phase sequence.

(b)

$$\begin{aligned}
H(z) &= 1+2.5788z^{-1}+3.4975z^{-2}+2.5074z^{-3}+1.2656z^{-4} \\
h[n] &= \delta[n]+2.5788\delta[n-1]+3.4975\delta[n-2]+2.5074\delta[n-3]+1.2656\delta[n-4] \\
H_1(z) &= 1.2656+2.5074z^{-1}+3.4975z^{-2}+2.5788z^{-3}+z^{-4} \\
h_1[n] &= 1.2656\delta[n]+2.5074\delta[n-1]+3.4975\delta[n-2]+2.5788\delta[n-3]+\delta[n-4] \\
H_2(z) &= 0.81+2.1945z^{-1}+3.3906z^{-2}+2.8917z^{-3}+1.5625z^{-4} \\
h_2[n] &= 0.81\delta[n]+2.1945\delta[n-1]+3.3906\delta[n-2]+2.8917\delta[n-3]+1.5625\delta[n-4] \\
H_3(z) &= 1.5625+2.8917z^{-1}+3.3906z^{-2}+2.1945z^{-3}+0.81z^{-4} \\
h_3[n] &= 1.5625\delta[n]+2.8917\delta[n-1]+3.3906\delta[n-2]+2.1945\delta[n-3]+0.81\delta[n-4]
\end{aligned}$$

(c)

n	$E(n)$	$E_1(n)$	$E_2(n)$	$E_3(n)$
0	1.0	1.6	0.7	2.4
1	7.7	7.9	5.5	10.8
2	19.9	20.1	17.0	22.3
3	26.2	26.8	25.3	27.1
4	27.8	27.8	27.8	27.8
5	27.8	27.8	27.8	27.8



The plot of $E_3[n]$ corresponds to the minimum phase sequence.

5.54. All zeros inside the unit circle means the sequence is minimum phase. Since

$$\sum_{n=0}^{M} |h_{min}[n]|^2 \geq \sum_{n=0}^{M} |h[n]|^2$$

is true for all M, we can use $M = 0$ and just compute $h^2[0]$. The largest result will be the minimum phase sequence.

A	B	C	D	E	F	G	H
44.5	28.4	1.8	2.8	1.8	177.7	113.8	7.1

The answer is F.

5.55.

(i) A zero phase sequence has all its poles and zeros in conjugate reciprocal pairs. Generalized linear phase systems are zero phase systems with additional poles or zeros at $z = 0, \infty, 1$ or -1.

(ii) A stable system's ROC includes the unit circle.

(a) The poles are not in conjugate reciprocal pairs, so this does not have zero or generalized linear phase. $H_i(z)$ has a pole at $z = 0$ and perhaps $z = \infty$. Therefore, the ROC is $0 < |z| < \infty$, which means the inverse is stable. If the ROC includes $z = \infty$, the inverse will also be causal.

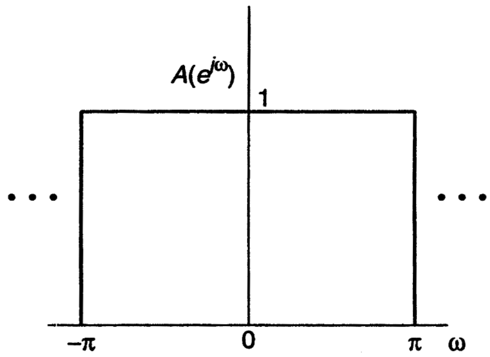
(b) Since the poles are not conjugate reciprocal pairs, this does not have zero or generalized linear phase either. $H_i(z)$ has poles inside the unit circle, so ROC is $|z| > \frac{2}{3}$ to match the ROC of $H(z)$. Therefore, the inverse is both stable and causal.

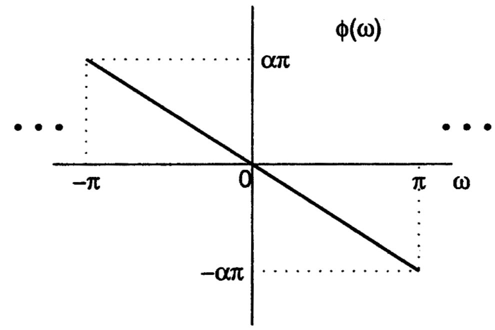
(c) The zeros occur in conjugate reciprocal pairs, so this is a zero phase system. The inverse has poles both inside and outside the unit circle. Therefore, a stable non-causal inverse exists.

(d) The zeros occur in conjugate reciprocal pairs, so this is a zero phase system. Since the poles of the inverse system are on the unit circle a stable inverse does not exist.

5.56. (a)

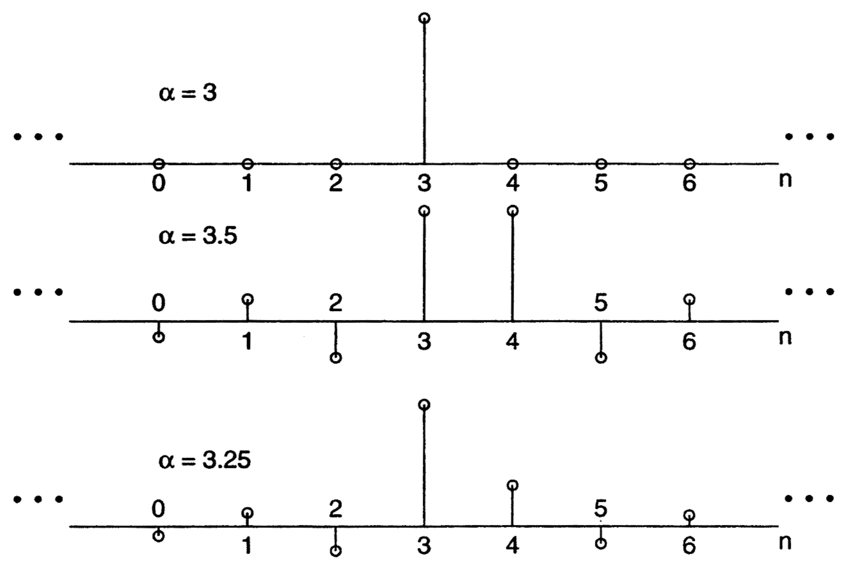
$$A(e^{j\omega}) = 1, \quad |\omega| < \pi$$
$$\phi(\omega) = -\alpha\omega, \quad |\omega| < \pi$$





(b)

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\alpha\omega}e^{j\omega n}d\omega = \frac{\sin\pi(n-\alpha)}{\pi(n-\alpha)}$$



(c) If α is an integer, then $h[n]$ is symmetric about about the point $n = \alpha$. If $\alpha = \frac{M}{2}$, where M is odd, then $h[n]$ is symmetric about $\frac{M}{2}$, which is not a point of the sequence. For α in general, $h[n]$ will not be symmetric.

5.57. Type I: Symmetric, M Even, Odd Length

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{n=(M+2)/2}^{M} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega(M/2)} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-2)/2} h[M-m]e^{-j\omega(M-m)} + h[M/2]e^{-j\omega(M/2)} \\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-2)/2} h[m]e^{j\omega((M/2)-m)} + \sum_{m=0}^{(M-2)/2} h[m]e^{-j\omega((M/2)-m)} + h[M/2]\right) \\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-2)/2} 2h[m]\cos\omega((M/2)-m) + h[M/2]\right) \\
&= e^{-j\omega(M/2)}\left(\sum_{n=1}^{M/2} 2h[(M/2)-n]\cos\omega n + h[M/2]\right)
\end{aligned}$$

Let

$$a[n] = \begin{cases} h[M/2], & n = 0 \\ 2h[(M/2)-n], & n = 1, \ldots, M/2 \end{cases}$$

Then

$$H(e^{j\omega}) = e^{-j\omega(M/2)} \sum_{n=0}^{M/2} a[n]\cos\omega n$$

and we have

$$A(\omega) = \sum_{n=0}^{M/2} a[n]\cos(\omega n), \quad \alpha = \frac{M}{2}, \quad \beta = 0$$

Type II: Symmetric, M Odd, Even Length

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-1)/2} h[M-m]e^{-j\omega(M-m)} \\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-1)/2} h[m]e^{j\omega((M/2)-m)} + \sum_{m=0}^{(M-1)/2} h[m]e^{-j\omega((M/2)-m)}\right) \\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-1)/2} 2h[m]\cos\omega((M/2)-m)\right) \\
&= e^{-j\omega(M/2)}\left(\sum_{n=1}^{(M+1)/2} 2h[(M+1)/2-n]\cos\omega(n-(1/2))\right)
\end{aligned}$$

Let

$$b[n] = 2h[(M+1)/2-n], \quad n = 1, \ldots, (M+1)/2$$

Then

$$H(e^{j\omega}) = e^{-j\omega(M/2)} \sum_{n=1}^{(M+1)/2} b[n]\cos\omega(n-(1/2))$$

and we have

$$A(\omega) = \sum_{n=1}^{(M+1)/2} b[n] \cos\omega(n-(1/2)), \quad \alpha = \frac{M}{2}, \quad \beta = 0$$

Type III: Antisymmetric, M Even, Odd Length

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + 0 + \sum_{(M+2)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-2)/2} h[M-m]e^{-j\omega(M-m)} \\
&= e^{-j\omega(M/2)} \left(\sum_{m=0}^{(M-2)/2} h[m]e^{j\omega((M/2)-m)} - \sum_{m=0}^{(M-2)/2} h[m]e^{-j\omega((M/2)-m)} \right) \\
&= e^{-j\omega(M/2)} \left(j \sum_{m=0}^{(M-2)/2} 2h[m] \sin\omega((M/2)-m) \right) \\
&= e^{-j\omega(M/2)} e^{j(\pi/2)} \left(\sum_{n=1}^{M/2} 2h[(M/2)-n] \sin\omega n \right)
\end{aligned}$$

Let

$$c[n] = h[(M/2)-n], \quad n = 1, \ldots, M/2$$

Then

$$H(e^{j\omega}) = e^{-j\omega(M/2)} e^{j(\pi/2)} \sum_{n=1}^{M/2} c[n] \sin\omega n$$

and we have

$$A(\omega) = \sum_{n=1}^{M/2} c[n] \sin(\omega n), \quad \alpha = \frac{M}{2}, \quad \beta = \frac{\pi}{2}$$

Type IV: Antisymmetric, M Odd, Even Length

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-1)/2} h[M-m]e^{-j\omega(M-m)} \\
&= e^{-j\omega(M/2)} \left(\sum_{m=0}^{(M-1)/2} h[m]e^{j\omega((M/2)-m)} - \sum_{m=0}^{(M-1)/2} h[m]e^{-j\omega((M/2)-m)} \right) \\
&= e^{-j\omega(M/2)} \left(j \sum_{m=0}^{(M-1)/2} 2h[m] \sin\omega((M/2)-m) \right) \\
&= e^{-j\omega(M/2)} e^{j(\pi/2)} \sum_{n=1}^{(M+1)/2} 2h[(M+1)/2-n] \sin\omega(n-(1/2))
\end{aligned}$$

Let

$$d[n] = 2h[(M+1)/2 - n], \quad n = 1, \ldots, (M+1)/2$$

Then

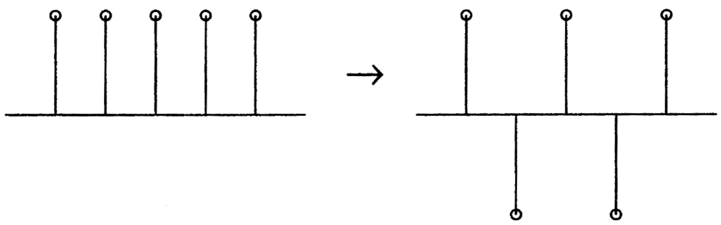
$$H(e^{j\omega}) = e^{-j\omega(M/2)} e^{j(\pi/2)} \sum_{n=1}^{(M+1)/2} d[n] \sin \omega(n - (1/2))$$

and we have

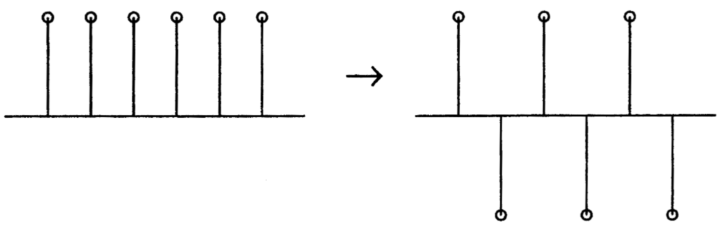
$$A(\omega) = \sum_{n=1}^{(M+1)/2} d[n] \sin \omega(n - (1/2)), \quad \alpha = \frac{M}{2}, \quad \beta = \frac{\pi}{2}$$

5.58. Filter Types II and III cannot be highpass filters since they both must have a zero at $z = 1$.

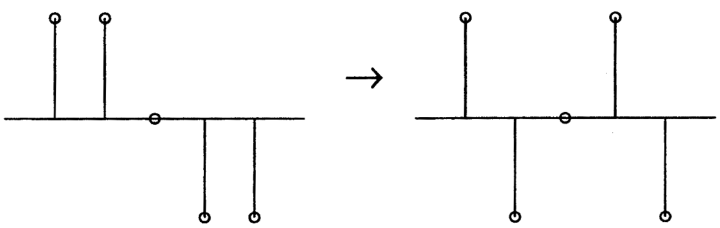
Type I $\rightarrow$ Type I could be highpass:



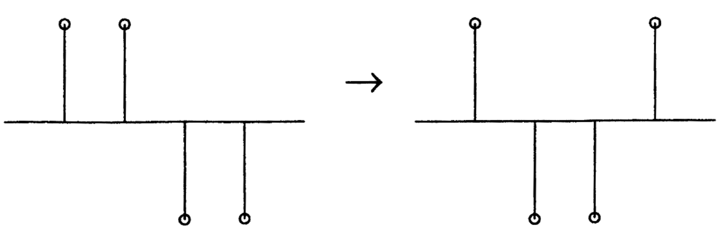
Type II $\rightarrow$ Type IV can be highpass:



Type III $\rightarrow$ Type III cannot be highpass:



Type IV $\rightarrow$ Type II cannot be highpass:



5.59. (a) Minimum phase systems have all poles and zeros inside $|z| = 1$. Allpass systems have pole-zero pairs at conjugate reciprocal locations. Generalized linear phase systems have pole pairs and zero pairs in conjugate reciprocal locations and at $z = 0, 1, -1$ and ∞. This implies that all the poles and zeros of $H_{min}(z)$ are second-order. When the allpass filter flips a pole or zero outside the unit circle, one is left in the conjugate reciprocal location, giving us linear phase.

(b) We know that $h[n]$ is length 8 and therefore has 7 zeros. Since it is an even length generalized linear phase filter with real coefficients and odd symmetry it must be a Type IV filter. It therefore has the property that its zeros come in conjugate reciprocal pairs stated mathematically as $H(z) = H(1/z^*)$. The zero at $z = -2$ implies a zero at $z = -\frac{1}{2}$, while the zero at $z = 0.8e^{j(\pi/4)}$ implies zeros at $z = 0.8e^{-j(\pi/4)}$, $z = 1.25e^{j(\pi/4)}$ and $z = 1.25e^{-j(\pi/4)}$. Because it is a IV filter, it also must have a zero at $z = 1$. Putting all this together gives us

$$\begin{aligned} H(z) \quad = \quad & (1 + 2z^{-1})(1 + 0.5z^{-1})(1 - 0.8e^{j(\pi/4)}z^{-1})(1 - 0.8e^{-j(\pi/4)}z^{-1}) \\ & (1 - 1.25e^{j(\pi/4)}z^{-1})(1 - 1.25e^{-j(\pi/4)}z^{-1})(1 - z^{-1}) \end{aligned}$$

5.60. ***False.*** **Let $h[n]$ equal**

$$h[n] = \frac{\sin \omega_c(n-4.3)}{\pi(n-4.3)} \longleftrightarrow H(e^{jw}) = \begin{cases} e^{-4.3\omega}, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

***Proof*: Although the group delay is constant (grd $\left[H(e^{jw})\right] = 4.3$) the resulting M is not an integer.**

$$\begin{aligned} h[n] &= \pm h[M-n] \\ H(e^{j\omega}) &= \pm e^{jM\omega} H(e^{-j\omega}) \\ e^{-j4.3\omega} &= \pm e^{j(M+4.3)\omega}, \qquad |\omega| < \omega_c \\ M &= -8.6 \end{aligned}$$

5.61. The type II FIR system $H_{II}(z)$ has generalized linear phase. Therefore, it can be written in the form

$$H_{II}(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$

where M is an odd integer and $A_e(e^{j\omega})$ is a real, even, periodic function of ω. Note that the system $G(z) = (1 - z^{-1})$ is a type IV generalized linear phase system.

$$\begin{aligned}
G(e^{j\omega}) &= 1 - e^{-j\omega} \\
&= e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2}) \\
&= e^{-j\omega/2}(2j\sin(\omega/2)) \\
&= 2\sin(\omega/2)e^{-j\omega/2+j\pi/2} \\
&= A_o(e^{j\omega})e^{-j\omega/2+j\pi/2} \\
A_o(e^{j\omega}) &= 2\sin(\omega/2) \\
\angle G(e^{j\omega}) &= -\frac{\omega}{2} + \frac{\pi}{2}
\end{aligned}$$

The cascade of $H_{II}(z)$ with $G(z)$ results in a generalized linear phase system $H(z)$.

$$\begin{aligned}
H(e^{j\omega}) &= A_e(e^{j\omega})A_o(e^{j\omega})e^{-j\omega M/2}e^{-j\omega/2+j\pi/2} \\
&= A'_o(e^{j\omega})e^{j\omega M'/2+j\pi/2}
\end{aligned}$$

where $A'_o(e^{j\omega})$ is a real, odd, periodic function of ω and M' is an even integer.

Thus, the resulting system $H(e^{j\omega})$ has the form of a type III FIR generalized linear phase system. It is antisymmetric, has odd length (M is even), and has generalized linear phase.

5.62. (a) The LTI system S_2 is characterized as a lowpass filter.
The z-transform of $h_1[n]$ is found below.

$$\begin{aligned}
y[n] - y[n-1] + \frac{1}{4}y[n-2] &= x[n] \\
Y(z) - Y(z)z^{-1} + \frac{1}{4}Y(z)z^{-2} &= X(z) \\
Y(z)\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right) &= X(z)
\end{aligned}$$

$$H_1(z) = \frac{1}{(1 - z^{-1} + \frac{1}{4}z^{-2})} = \frac{1}{(1 - \frac{1}{2}z^{-1})^2}$$

This system function has a second order pole at $z = \frac{1}{2}$. (There is also a second order zero at $z = 0$). Evaluating this pole-zero plot on the unit circle yields a low pass filter, as the second order pole boosts the low frequencies.
Since

$$\begin{aligned}
H_2(e^{j\omega}) &= H_1(-e^{j\omega}) \\
H_2(z) &= H_1(-z)
\end{aligned}$$

If we replace all references to z in $H_1(z)$ with $-z$, we will get $H_2(z)$.

$$H_2(z) = \frac{1}{(1 + \frac{1}{2}z^{-1})^2}$$

Consequently, $H_2(z)$ has two poles at $z = -\frac{1}{2}$. (There is also a second order zero at $z = 0$). Evaluating this pole-zero plot on the unit circle yields a high pass filter, as the second order pole now boosts the high frequencies.

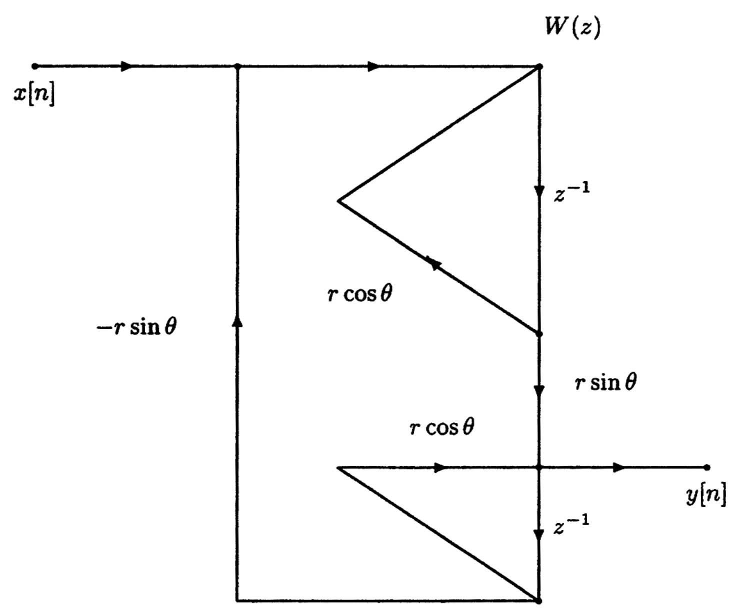
6.1. We proceed by obtaining the transfer functions for each of the networks. For network 1,

$$Y(z) = 2r\cos\theta z^{-1}Y(z) - r^2 z^{-2}Y(z) + X(z)$$

or

$$H_1(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

For network 2, define $W(z)$ as in the figure below:



then

$$W(z) = X(z) - r\sin\theta z^{-1}Y(z) + r\cos\theta z^{-1}W(z)$$

and

$$Y(z) = r\sin\theta z^{-1}W(z) + r\cos\theta z^{-1}Y(z)$$

Eliminate $W(z)$ to get

$$H_2(z) = \frac{Y(z)}{X(z)} = \frac{r\sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

Hence the two networks have the same poles.

6.2. The only input to the $y[n]$ node is a unity branch connection from the $x[n]$ node. The rest of the network does not affect the input-output relationship. The difference equation is $y[n] = x[n]$.

6.3.

$$H(z) = \frac{2 + \frac{1}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}$$

System (d) is recognizable as a transposed direct form II implementation of $H(z)$.

6.4. (a) From the flow graph, we have:

$$Y(z) = 2X(z) + (\frac{1}{4}X(z) - \frac{1}{4}Y(z) + \frac{3}{8}Y(z)z^{-1})z^{-1}$$

That is:

$$Y(z)(1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}) = X(z)(2 + \frac{1}{4}z^{-1}).$$

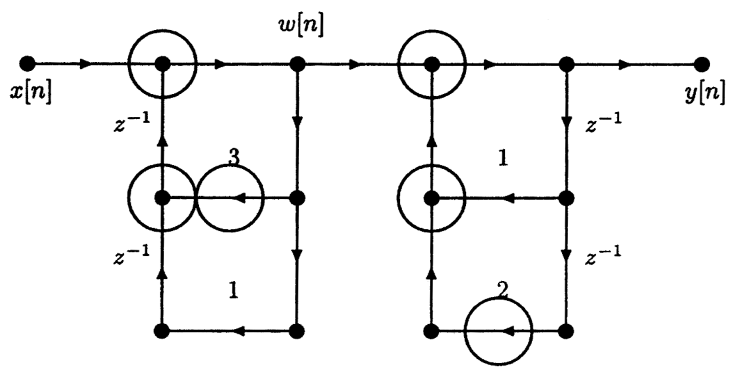
The system function is thus given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + \frac{1}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

(b) To get the difference equation, we just inverse $Z-$transform the equation in a. We get:

$$y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = 2x[n] + \frac{1}{4}x[n-1].$$

6.5. The flow graph for this system is drawn below.



(a)

$$w[n] = x[n] + 3w[n-1] + w[n-2]$$

$$y[n] = w[n] + y[n-1] + 2y[n-2]$$

(b)

$$W(z) = X(z) + 3z^{-1}W(z) + z^{-2}W(z)$$

$$Y(z) = W(z) + z^{-1}Y(z) + 2z^{-2}Y(z)$$

So

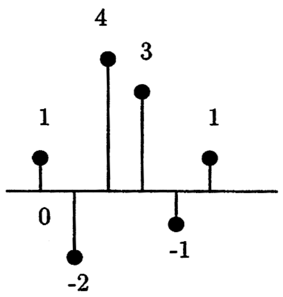
$$\begin{aligned} \frac{Y(z)}{X(z)} &= H(z) \\ &= \frac{1}{(1 - z^{-1} - 2z^{-2})(1 - 3z^{-1} - z^{-2})} \\ &= \frac{1}{1 - 4z^{-1} + 7z^{-3} + 2z^{-4}}. \end{aligned}$$

(c) Adds and multiplies are circled above: 4 real adds and 2 real multiplies per output point.

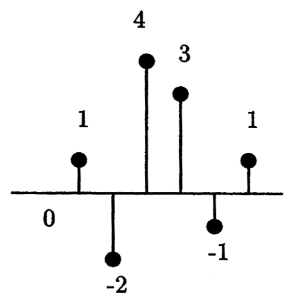
(d) It is not possible to reduce the number of storage registers. Note that implementing $H(z)$ above in the canonical direct form II (minimum storage registers) also requires 4 registers.

6.6. The impulse responses of each system are shown below.

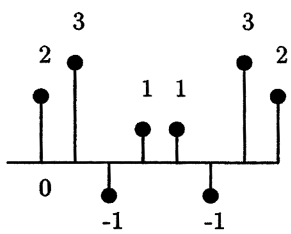
(a)



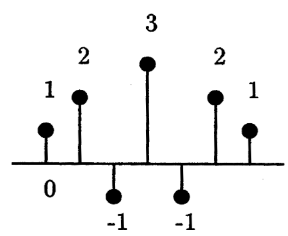
(b)



(c)



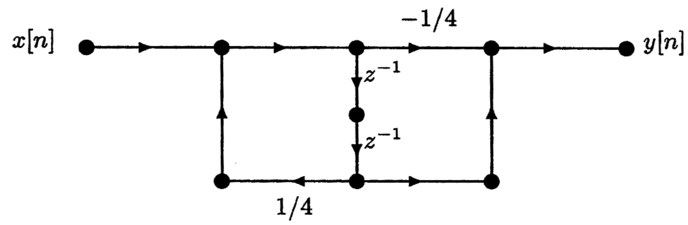
(d)



6.7. We have

$$H(z) = \frac{-\frac{1}{4} + z^{-2}}{1 - \frac{1}{4}z^{-2}}.$$

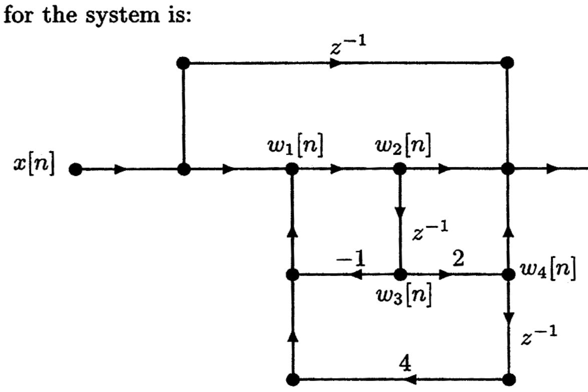
Therefore the direct form II is given by:



6.8. By looking at the graph, we get:

$$y[n] = 2y[n-2] + 3x[n-1] + x[n-2].$$

6.9. The signal flow graph for the system is:



(a) First we need to determine the transfer function. We have

$$\begin{aligned}
w_1[n] &= x[n] - w_3[n] + 4w_4[n-1] \\
w_2[n] &= w_1[n] \\
w_3[n] &= w_2[n-1] \\
w_4[n] &= 2w_3[n] \\
y[n] &= w_2[n] + x[n-1] + w_4[n].
\end{aligned}$$

Taking the Z−transform of the above equations, rearranging and substituting terms, we get:

$$H(z) = \frac{1 + 3z^{-1} + z^{-2} - 8z^{-3}}{1 + z^{-1} - 8z^{-2}}.$$

The difference equation is thus given by:

$$y[n] + y[n-1] - 8y[n-2] = x[n] + 3x[n-1] + x[n-2] - 8x[n-3].$$

The impulse response is the response to an impulse, therefore:

$$h[n] + h[n-1] - 8h[n-2] = \delta[n] + 3\delta[n-1] + \delta[n-2] - 8\delta[n-3].$$

From the above equation, we have:

$$\begin{aligned}
h[0] &= 1 \\
h[1] &= 3 - h[0] = 2.
\end{aligned}$$

(b) From part (a) we have:

$$y[n] + y[n-1] - 8y[n-2] = x[n] + 3x[n-1] + x[n-2] - 8x[n-3].$$

6.10. (a)

$$\begin{aligned} w[n] &= \frac{1}{2}y[n] + x[n] \\ v[n] &= \frac{1}{2}y[n] + 2x[n] + w[n-1] \\ y[n] &= v[n-1] + x[n]. \end{aligned}$$

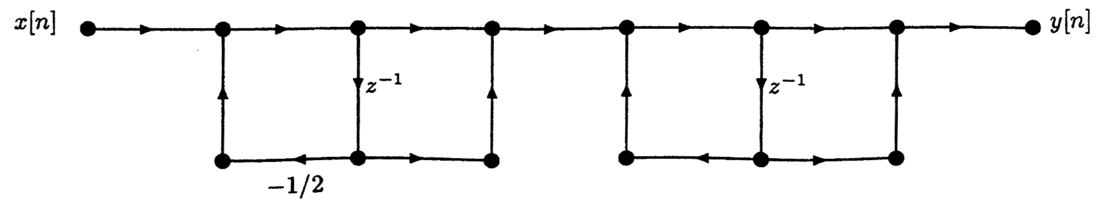
(b) Using the Z−transform of the difference equations in part (a), we get the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}.$$

We can rewrite it as :

$$H(z) = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 + \frac{1}{2}z^{-1})(1 - z^{-1})}.$$

We thus get the following cascade form:

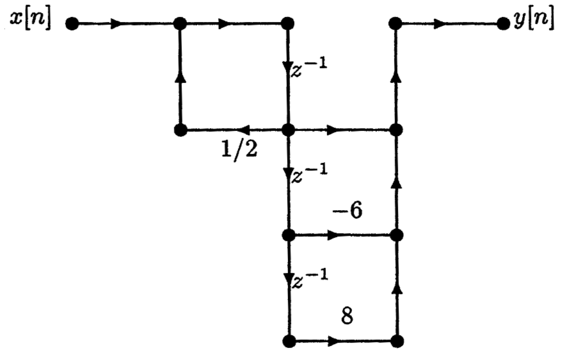


(c) The system function has poles at $z = -\frac{1}{2}$ and $z = 1$. Since the second pole is on the unit circle, the system is not stable.

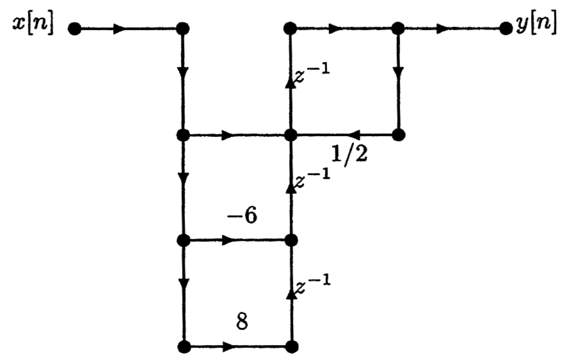
6.11. (a) $H(z)$ can be rewritten as:

$$H(z) = \frac{z^{-1} - 6z^{-2} + 8z^{-3}}{1 - \frac{1}{2}z^{-1}}.$$

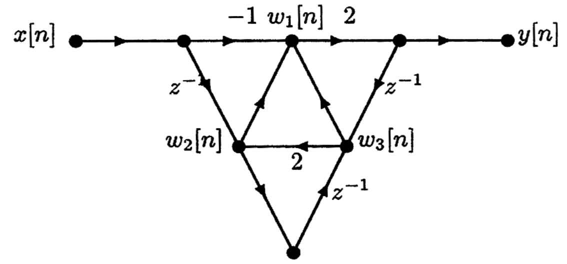
We thus get the following direct from II flow graph :



(b) To get the transposed form, we just reverse the arrows and exchange the input and the ouput. The graph can then be redrawn as:



6.12. We define the intermediate variables $w_1[n]$, $w_2[n]$ and $w_3[n]$ as follows:



We thus have the following relationships:

$$\begin{aligned} w_1[n] &= -x[n] + w_2[n] + w_3[n] \\ w_2[n] &= x[n-1] + 2w_3[n] \\ w_3[n] &= w_2[n-1] + y[n-1] \\ y[n] &= 2w_1[n]. \end{aligned}$$

$Z-$transforming the above equations and rearranging and grouping terms, we get:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-2 + 6z^{-1} + 2z^{-2}}{1 - 8z^{-1}}.$$

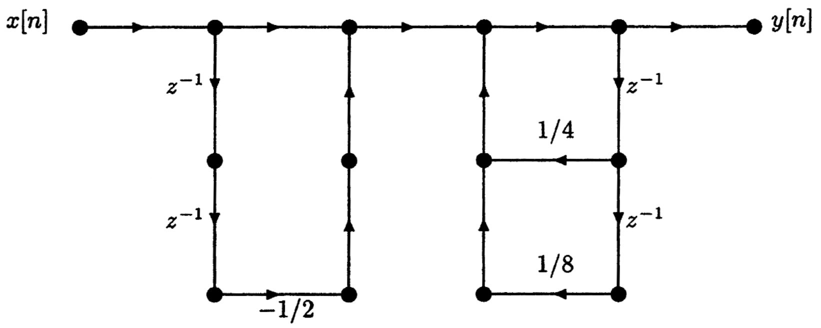
Taking the inverse $Z-$transform, we get the following difference equation:

$$y[n] - 8y[n-1] = -2x[n] + 6x[n-1] + 2x[n-2].$$

6.13.

$$H(z) = \frac{1 - \frac{1}{2}z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}.$$

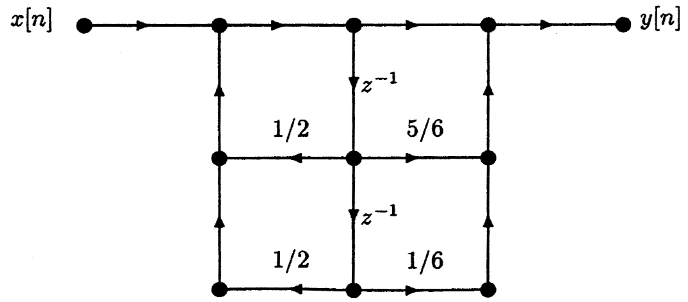
The direct form I implementation is:



6.14.

$$H(z) = \frac{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}.$$

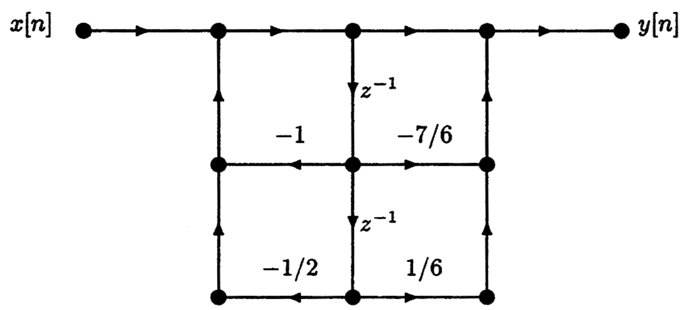
The direct form II implementation is:



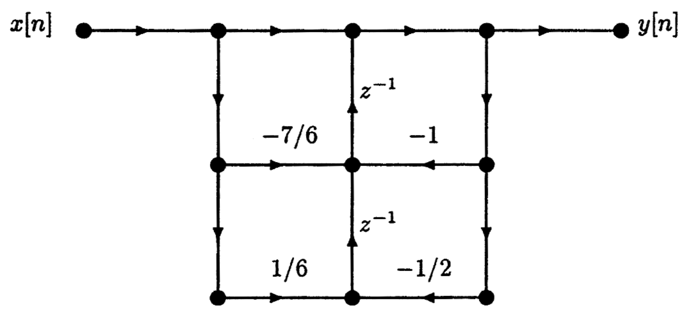
6.15.

$$H(z) = \frac{1 - \frac{7}{6}z^{-1} + \frac{1}{6}z^{-2}}{1 + z^{-1} + \frac{1}{2}z^{-2}}.$$

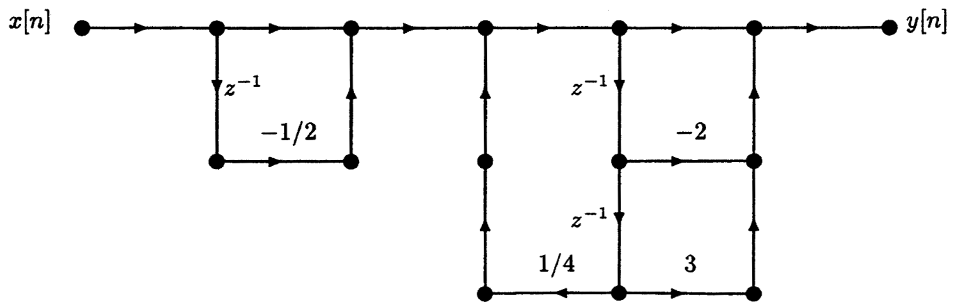
To get the transposed direct form II implementation, we first get the direct form II:



Now, we reverse the arrows and exchange the role of the input and the ouput to get the transposed direct form II:



6.16. (a) We just reverse the arrows and reverse the role of the input and the output, we get:



(b) The original system is the cascade of two transposed direct form II structures, therefore the system function is given by:

$$H(z) = \left(\frac{1 - 2z^{-1} + 3z^{-2}}{1 - \frac{1}{4}z^{-2}}\right)\left(1 - \frac{1}{2}z^{-1}\right).$$

The transposed graph, on the other hand, is the cascade of two direct form II structures, therefore the system function is given by:

$$H(z) = \left(1 - \frac{1}{2}z^{-1}\right)\left(\frac{1 - 2z^{-1} + 3z^{-2}}{1 - \frac{1}{4}z^{-2}}\right).$$

This confirms that both graphs have the same system function $H(z)$.

6.18. The flow graph is just a cascade of two transposed direct form II structures, the system functio thus given by:

$$H(z) = \left(\frac{1 + \frac{4}{3}z^{-1} - \frac{4}{3}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}\right)\left(\frac{1}{1 - az^{-1}}\right).$$

Which can be rewritten as:

$$H(z) = \frac{(1 + 2z^{-1})(1 - \frac{2}{3}z^{-1})}{(1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2})(1 - az^{-1})}.$$

In order to implement this system function with a second-order direct form II signal flow grapl pole-zero cancellation has to occur, this happens if $a = \frac{2}{3}$, $a = -2$ or $a = 0$. If $a = \frac{2}{3}$, the overall sys function is:

$$H(z) = \frac{1 + 2z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

If $a = -2$, the overall system function is:

$$H(z) = \frac{1 - \frac{2}{3}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

And finally if $a = 0$, the overall system function is:

$$H(z) = \frac{(1 + 2z^{-1})(1 - \frac{2}{3}z^{-1})}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

6.18. The flow graph is just a cascade of two transposed direct form II structures, the system function is thus given by:

$$H(z) = \left(\frac{1 + \frac{4}{3}z^{-1} - \frac{4}{3}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}\right)\left(\frac{1}{1 - az^{-1}}\right).$$

Which can be rewritten as:

$$H(z) = \frac{(1 + 2z^{-1})(1 - \frac{2}{3}z^{-1})}{(1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2})(1 - az^{-1})}.$$

In order to implement this system function with a second-order direct form II signal flow graph, a pole-zero cancellation has to occur, this happens if $a = \frac{2}{3}$, $a = -2$ or $a = 0$. If $a = \frac{2}{3}$, the overall system function is:

$$H(z) = \frac{1 + 2z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

If $a = -2$, the overall system function is:

$$H(z) = \frac{1 - \frac{2}{3}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

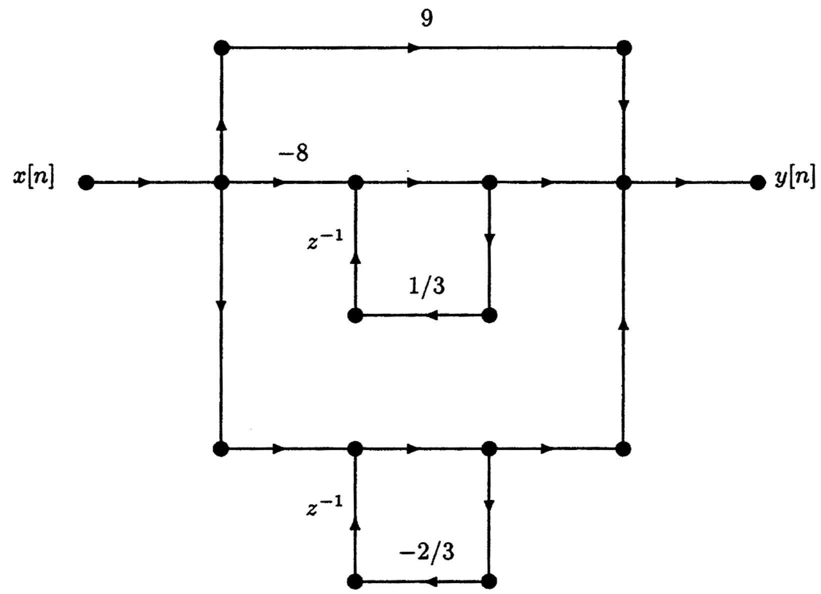
And finally if $a = 0$, the overall system function is:

$$H(z) = \frac{(1 + 2z^{-1})(1 - \frac{2}{3}z^{-1})}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

6.19. Using partial fraction expansion, the system function can be rewritten as:

$$H(z) = \frac{-8}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 + \frac{2}{3}z^{-1}} + 9.$$

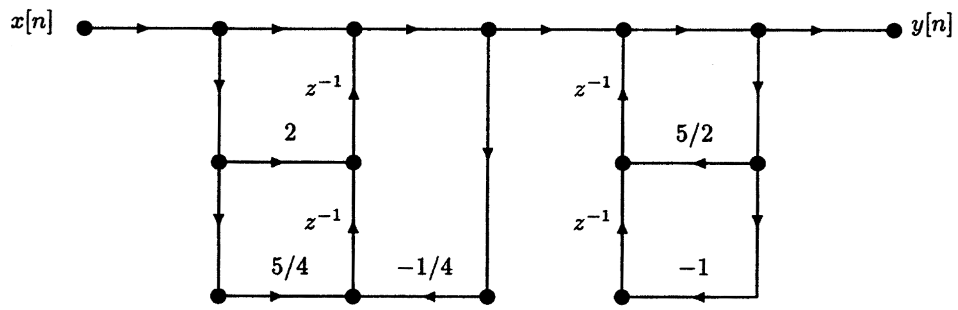
Now we can draw the flow graph that implements this system as a parallel combination of first-order transposed direct form II sections:



6.20. The transfer function can be rewritten as:

$$H(z) = \frac{(1 + 2z^{-1} + \frac{5}{4}z^{-2})}{(1 + \frac{1}{4}z^{-2})(1 - \frac{5}{2}z^{-1} + z^{-2})}$$

which can be implemented as the following cascade of second-order transposed direct form II sections:

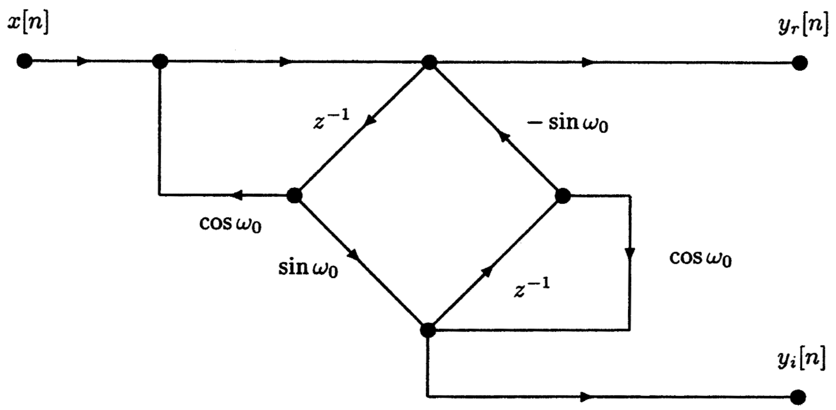


6.21.

$$h[n] = e^{j\omega_0 n}u[n] \longleftrightarrow H(z) = \frac{1}{1 - e^{j\omega_0}z^{-1}} = \frac{Y(z)}{X(z)}.$$

So $y[n] = e^{j\omega_0}y[n-1] + x[n]$. Let $y[n] = y_r[n] + jy_i[n]$. Then $y_r[n] + jy_i[n] = (\cos\omega_0 + j\sin\omega_0)(y_r[n-1] + jy_i[n-1]) + x[n]$. Separate the real and imaginary parts:

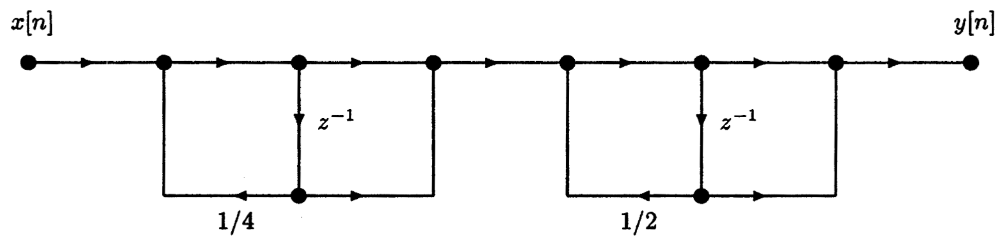
$$\begin{aligned} y_r[n] &= x[n] + \cos\omega_0 y_r[n-1] - \sin\omega_0 y_i[n-1] \\ y_i[n] &= \sin\omega_0 y_r[n-1] + \cos\omega_0 y_i[n-1]. \end{aligned}$$



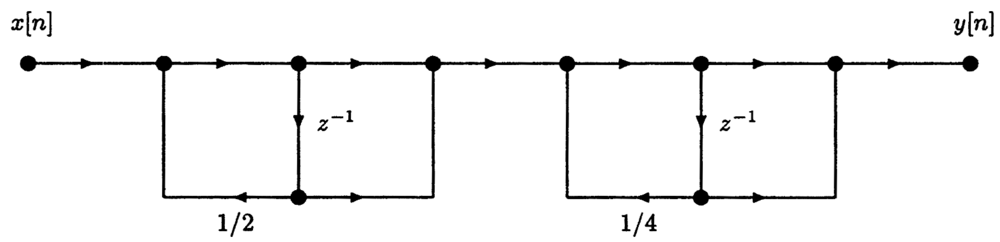
6.22.

$$H(z) = \frac{(1+z^{-1})^2}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}.$$

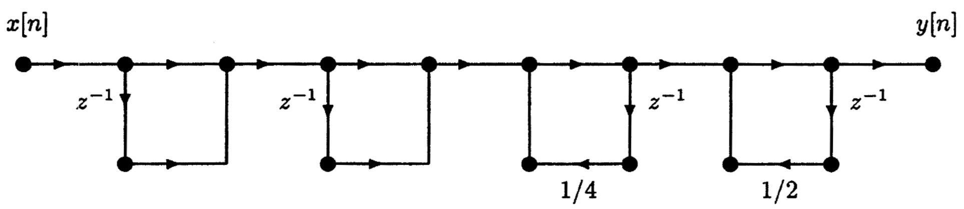
$$H(z) = \left(\frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}\right)\left(\frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}\right).$$



$$H(z) = \left(\frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}\right).$$



Plus 12 systems of this form:



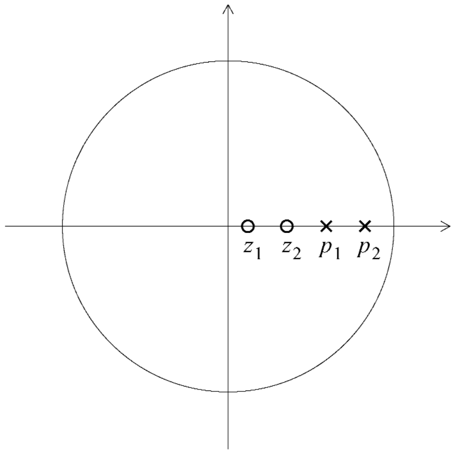
with the three types of 1st-order systems taken in various orders.

6.23. Problem 1 in Fall 2003 Midterm exam Appears in: Spring04 PS3.
Note: The Fall2003 Midterm version additionally includes a part (d):

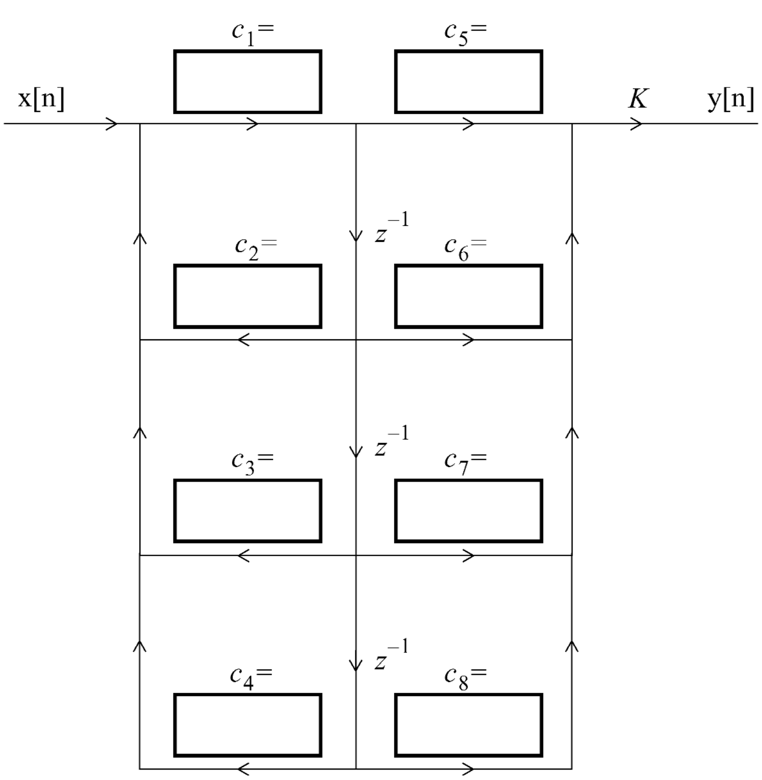
(d) (5%) For the most accurate placement of the **zeros**, which form(s) would you chose: direct, cascade, or parallel? Explain briefly.

Problem

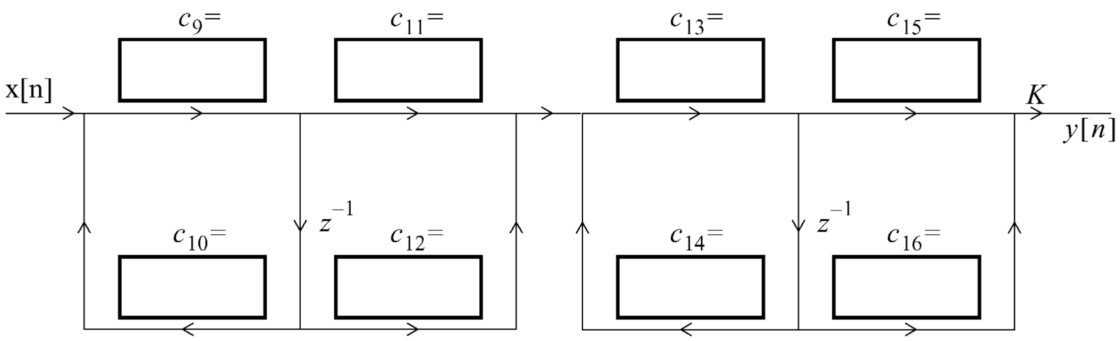
We want to implement a causal system $H(z)$ with the pole-zero diagram shown below. For all parts of this problem, z_1, z_2, p_1, and p_2 are real, and a gain constant that is independent of frequency can be absorbed into the K term in each flowgraph.



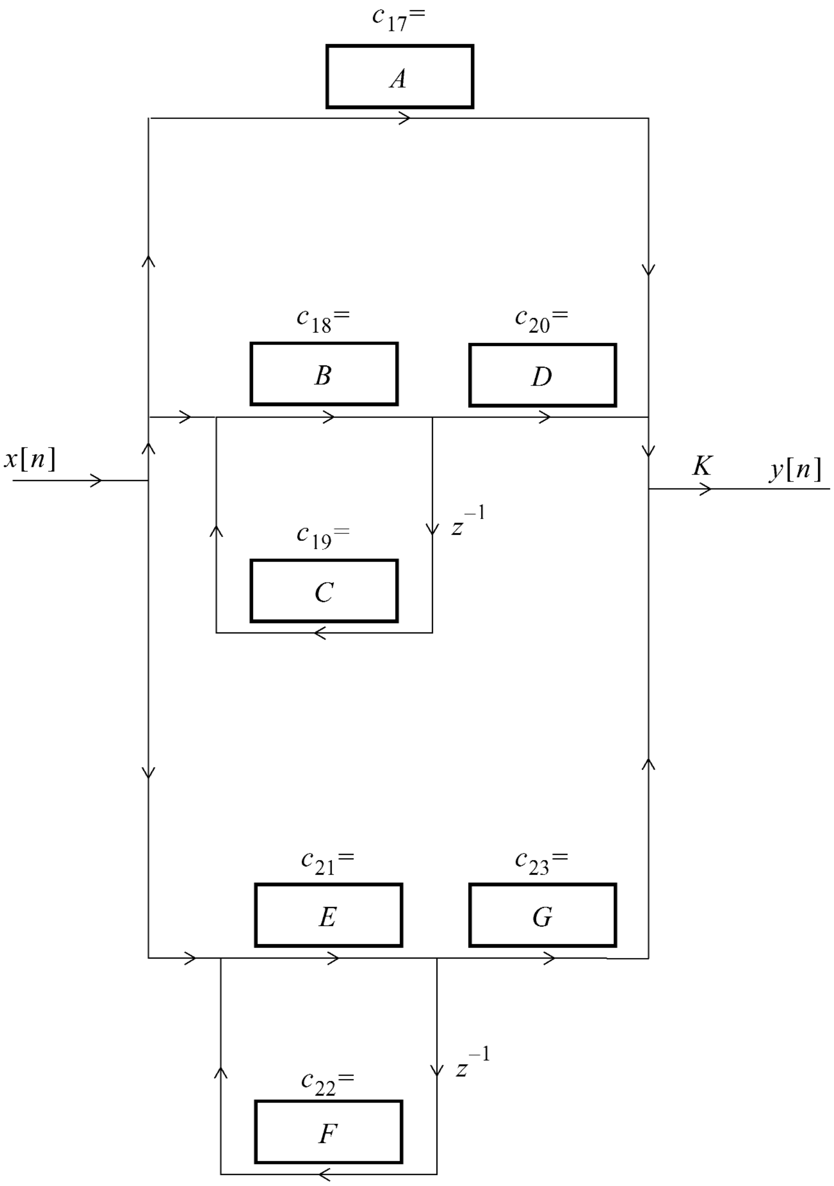
(a) Fill in the following flowgraph, in terms of the variables z_1, z_2, p_1, and p_2.



(b) Fill in the following flowgraph, in terms of the variables z_1, z_2, p_1, and p_2.



(c) Write down the system of linear equations for the variables $A, B, \ldots, G$ in terms of the variables z_1, z_2, p_1, and p_2.



Solution from Spring04 PS3

The system function is proportional to:

$$\begin{aligned}
&\frac{\left(1-z_1z^{-1}\right)\left(1-z_2z^{-1}\right)}{\left(1-p_1z^{-1}\right)\left(1-p_2z^{-1}\right)} \\
=\ &\frac{1-(z_1+z_2)z^{-1}+z_1z_2z^{-2}}{1-(p_1+p_2)z^{-1}+p_1p_2z^{-2}}
\end{aligned}$$

(a) $c_1 = 1,\quad c_2 = p_1 + p_2,\quad c_3 = -p_1p_2,\quad c_4 = 0$

$c_5 = 1,\quad c_6 = -(z_1 + z_2),\quad c_7 = z_1z_2,\quad c_8 = 0$

(b) $c_9 = 1,\quad c_{10} = p_1,\quad c_{11} = 1,\quad c_{12} = -z_1$

$c_{13} = 1,\quad c_{14} = p_2,\quad c_{15} = 1,\quad c_{16} = -z_2$

(c) $B = 1,\quad C = p_1,\quad E = 1,\quad F = p_2$

From a partial fraction expansion,

$$1-(z_1+z_2)z^{-1}+z_1z_2z^{-2} = A(1-p_1z^{-1})(1-p_2z^{-1}) + D(1-p_2z^{-1}) + G(1-p_1z^{-1})$$

Therefore A, D, and G can be found by solving the following system of equations:

$$\begin{aligned}
1 &= A + D + G \\
-(z_1+z_2) &= -A(p_1+p_2) - Dp_2 - Gp_1 \\
z_1z_2 &= Ap_1p_2
\end{aligned}$$

$$A = \frac{z_1z_2}{p_1p_2},\quad D = \frac{(z_1-p_1)(z_2-p_1)}{p_1(p_1-p_2)},\quad G = \frac{(z_1-p_2)(z_2-p_2)}{p_2(p_2-p_1)}$$

Solution from Fall03 Midterm

Problem

The system function is proportional to:

$$\begin{aligned}&\frac{\left(1-z_1z^{-1}\right)\left(1-z_2z^{-1}\right)}{\left(1-p_1z^{-1}\right)\left(1-p_2z^{-1}\right)}\\ =\;&\frac{1-(z_1+z_2)z^{-1}+z_1z_2z^{-2}}{1-(p_1+p_2)z^{-1}+p_1p_2z^{-2}}\end{aligned}$$

(a) $c_1=1,\quad c_2=p_1+p_2,\quad c_3=-p_1p_2,\quad c_4=0$
$c_5=1,\quad c_6=-(z_1+z_2),\quad c_7=z_1z_2,\quad c_8=0$

(b) $c_9=1,\quad c_{10}=p_1,\quad c_{11}=1,\quad c_{12}=-z_1$
$c_{13}=1,\quad c_{14}=p_2,\quad c_{15}=1,\quad c_{16}=-z_2$

(c) $B=1,\quad C=p_1$
$E=1,\quad F=p_2$

From a partial fraction expansion,

$$1-(z_1+z_2)z^{-1}+z_1z_2z^{-2}=A(1-p_1z^{-1})(1-p_2z^{-1})+D(1-p_2z^{-1})+G(1-p_1z^{-1})$$

Therefore A, D, and G can be found by solving the following system of equations:

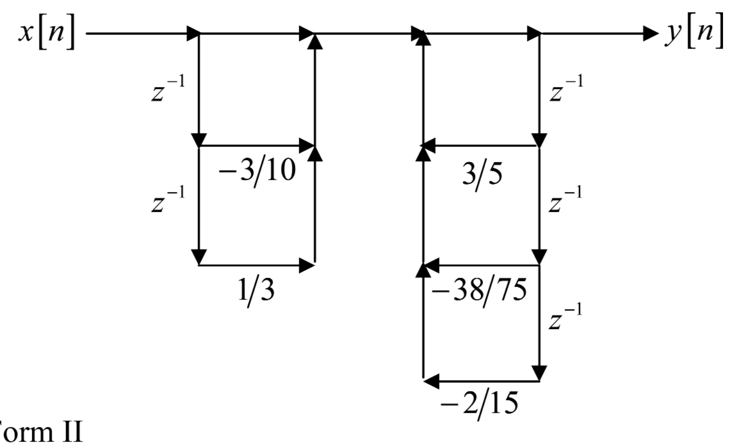
$$\begin{aligned}1&=A+D+G\\-(z_1+z_2)&=-A(p_1+p_2)-Dp_2-Gp_1\\z_1z_2&=Ap_1p_2\end{aligned}$$

6.24

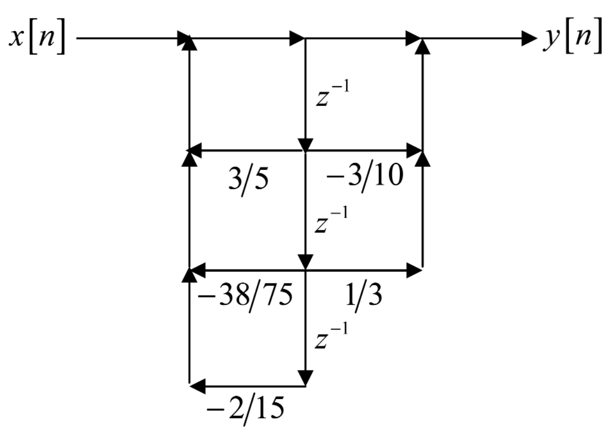
$$H(z) = \frac{1 - \frac{3}{10}z^{-1} + \frac{1}{3}z^{-2}}{\left(1 - \frac{4}{5}z^{-1} + \frac{2}{3}z^{-2}\right)\left(1 + \frac{1}{5}z^{-1}\right)}$$

$$= \frac{1 - \frac{3}{10}z^{-1} + \frac{1}{3}z^{-2}}{1 - \frac{3}{5}z^{-1} + \frac{38}{75}z^{-2} + \frac{2}{15}z^{-3}}.$$

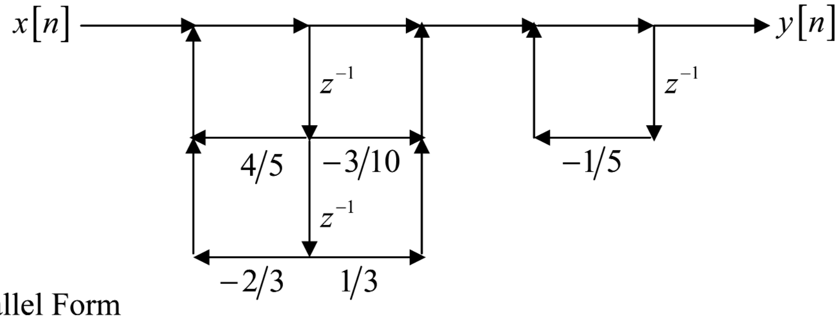
1. i) Direct Form I



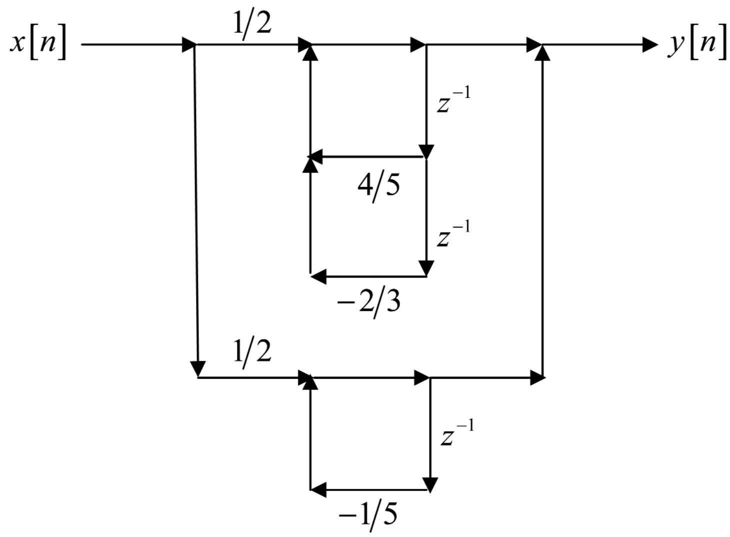
ii) Direct Form II



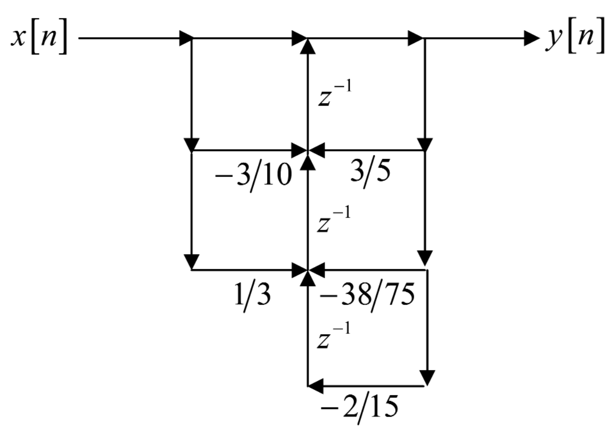
iii) Cascade Form



iv) Parallel Form



v) Transposed Direct Form II



2. Label the interior nodes of the transposed direct form II structure $v_0[n], v_1[n], v_2[n], v_3[n]$ counting from the top down. Then we have

$$\begin{aligned}
y[n] &= v_0[n] \\
v_0[n] &= x[n] + v_1[n-1] \\
v_1[n] &= \tfrac{3}{5}y[n] - \tfrac{3}{10}x[n] + v_2[n-1] \\
v_2[n] &= -\tfrac{38}{75}y[n] + \tfrac{1}{3}x[n] + v_3[n-1] \\
v_3[n] &= -\tfrac{2}{15}y[n].
\end{aligned}$$

Taking the z-transform of these equations gives

$$\begin{aligned}
Y(z) &= V_0(z) \\
V_0(z) &= X(z) + z^{-1}V_1(z) \\
V_1(z) &= \tfrac{3}{5}Y(z) - \tfrac{3}{10}X(z) + z^{-1}V_2(z) \\
V_2(z) &= -\tfrac{38}{75}Y(z) + \tfrac{1}{3}X(z) + z^{-1}V_3(z) \\
V_3(z) &= -\tfrac{2}{15}Y(z).
\end{aligned}$$

Substituting Eq. (5) into Eq. (4) gives

$$\begin{aligned}
V_2(z) &= -\tfrac{38}{75}Y(z) + \tfrac{1}{3}X(z) - \tfrac{2}{15}z^{-1}Y(z) \\
&= -\left(\tfrac{38}{75} + \tfrac{2}{15}z^{-1}\right)Y(z) + \tfrac{1}{3}X(z).
\end{aligned}$$

Substituting into Eq. (3) gives

$$\begin{aligned}
V_1(z) &= \tfrac{3}{5}Y(z) - \tfrac{3}{10}X(z) + z^{-1}\left(-\left(\tfrac{38}{75} + \tfrac{2}{15}z^{-1}\right)Y(z) + \tfrac{1}{3}X(z)\right) \\
&= \left(\tfrac{3}{5} - \tfrac{38}{75}z^{-1} - \tfrac{2}{15}z^{-2}\right)Y(z) + \left(-\tfrac{3}{10} + \tfrac{1}{3}z^{-1}\right)X(z).
\end{aligned}$$

Now substitute into Eq. (2):

$$\begin{aligned}
V_0(z) &= X(z) + z^{-1}\left\{\left(\tfrac{3}{5} - \tfrac{38}{75}z^{-1} - \tfrac{2}{15}z^{-2}\right)Y(z) + \left(-\tfrac{3}{10} + \tfrac{1}{3}z^{-1}\right)X(z)\right\} \\
&= \left(1 - \tfrac{3}{10}z^{-1} + \tfrac{1}{3}z^{-2}\right)X(z) + \left(\tfrac{3}{5}z^{-1} - \tfrac{38}{75}z^{-2} - \tfrac{2}{15}z^{-3}\right)Y(z).
\end{aligned}$$

Finally, substitute into Eq. (1):

$$Y(z) = \left(1 - \tfrac{3}{10}z^{-1} + \tfrac{1}{3}z^{-2}\right)X(z) + \left(\tfrac{3}{5}z^{-1} - \tfrac{38}{75}z^{-2} - \tfrac{2}{15}z^{-3}\right)Y(z)$$

$$\left(1 - \tfrac{3}{5}z^{-1} + \tfrac{38}{75}z^{-2} + \tfrac{2}{15}z^{-3}\right)Y(z) = \left(1 - \tfrac{3}{10}z^{-1} + \tfrac{1}{3}z^{-2}\right)X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(1 - \tfrac{3}{10}z^{-1} + \tfrac{1}{3}z^{-2}\right)}{\left(1 - \tfrac{3}{5}z^{-1} + \tfrac{38}{75}z^{-2} + \tfrac{2}{15}z^{-3}\right)}.$$

This final expression is the correct system function.

6.25

A.

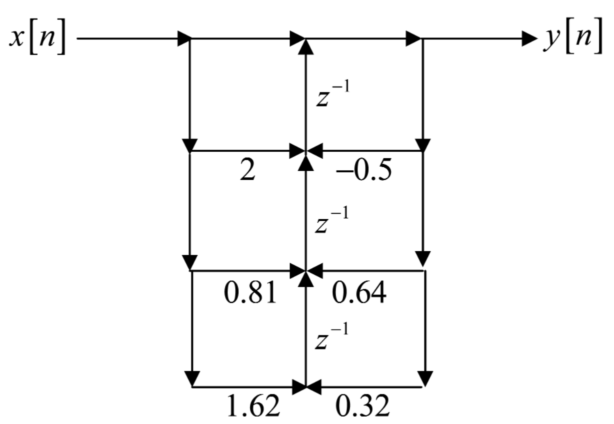
$$H(z)=\frac{1+0.81z^{-2}}{1-0.3z^{-1}-0.4z^{-2}}\times\frac{1+2z^{-1}}{1+0.8z^{-1}}$$
$$=\frac{\left(1+j0.9z^{-1}\right)\left(1-j0.9z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-0.8z^{-1}\right)\left(1+0.5z^{-1}\right)\left(1+0.8z^{-1}\right)}.$$

B. Yes, the overall system is stable. All of the poles are inside the unit circle, which guarantees stability for a causal system.

C. No, the system is not minimum-phase. There is a zero outside the unit circle at $z=-2$.

D.

$$H(z)=\frac{1+2z^{-1}+0.81z^{-2}+1.62z^{-3}}{1+0.5z^{-1}-0.64z^{-2}-0.32z^{-3}}.$$



6.26. Causal LTI system with system function:

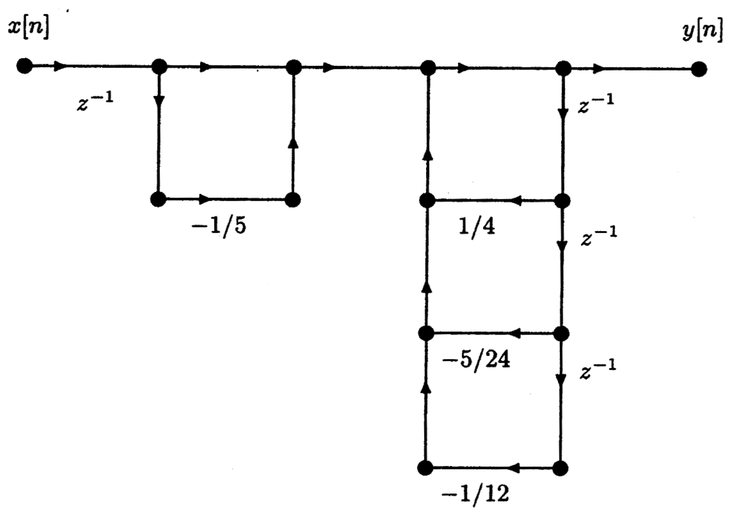
$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{(1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2})(1 + \frac{1}{4}z^{-1})}.$$

(a) (i) Direct form I.

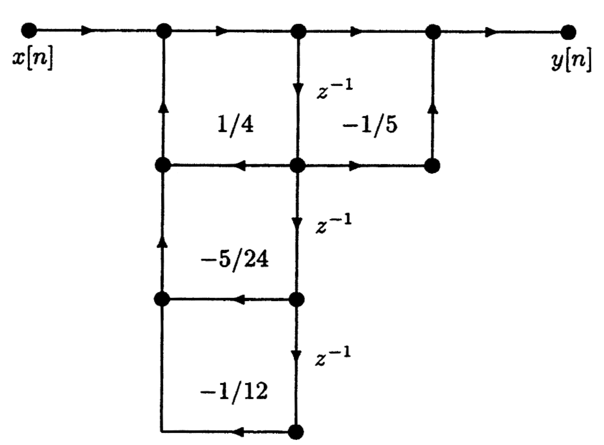
$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z(-3)}$$

so

$$b_0 = 1\ ,\ b_1 = -\frac{1}{5} \text{ and } a_1 = \frac{1}{4}\ ,\ a_2 = -\frac{5}{24}\ ,\ a_3 = -\frac{1}{12}.$$



(ii) Direct form II.

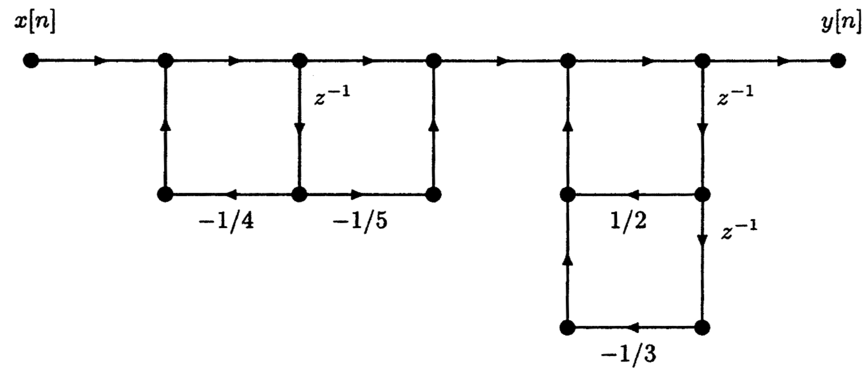


(iii) Cascade form using first and second order direct form II sections.

$$H(z) = \left(\frac{1-\frac{1}{5}z^{-1}}{1+\frac{1}{4}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}\right).$$

So

$$b_{01}=1\ ,\ b_{11}=-\tfrac{1}{5}\ ,\ b_{21}=0\ ,$$
$$b_{02}=1\ ,\ b_{12}=0\ ,\ b_{22}=0 \text{ and}$$
$$a_{11}=-\tfrac{1}{4}\ ,\ a_{21}=0\ ,\ a_{12}=\tfrac{1}{2}\ ,\ a_{22}=-\tfrac{1}{3}.$$

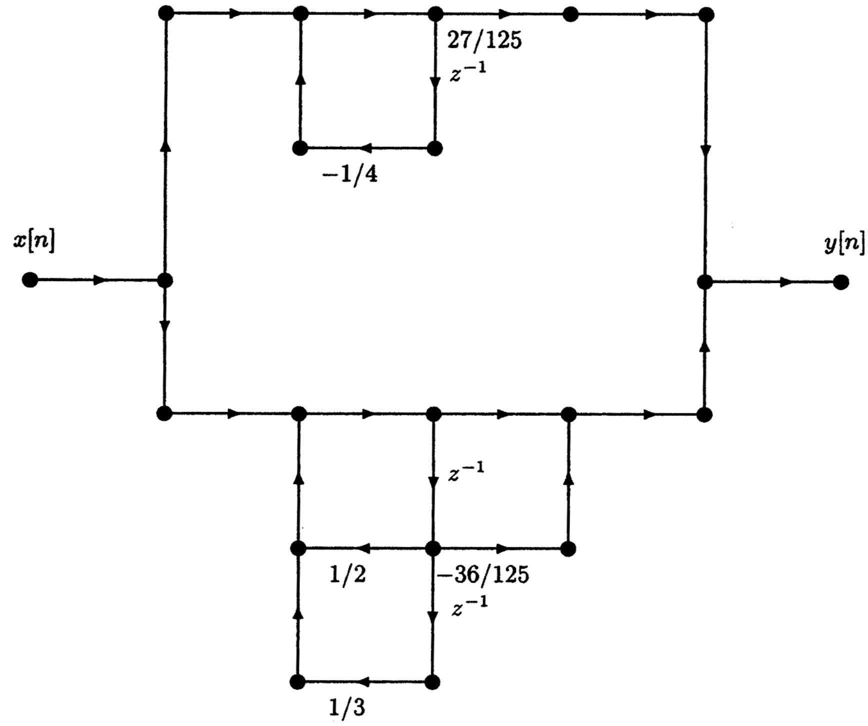


(iv) Parallel form using first and second order direct form II sections.
We can rewrite the transfer function as:

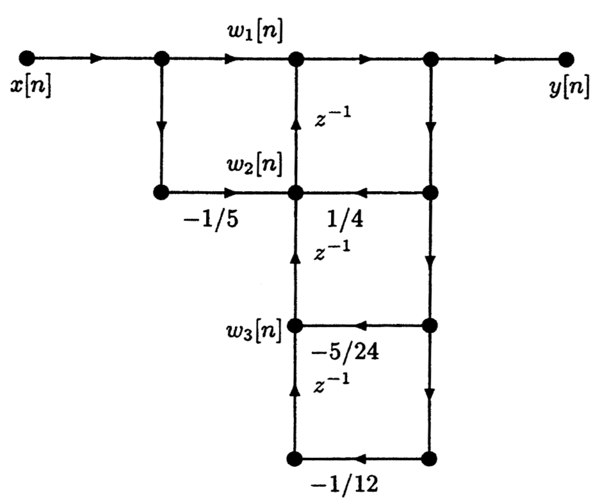
$$H(z) = \frac{\frac{27}{125}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{98}{125}-\frac{36}{125}z^{-1}}{1-\frac{1}{2}z^{-1}-\frac{1}{3}z^{-2}}.$$

So

$$e_{01}=\tfrac{27}{125}\ ,\ e_{11}=0\ ,$$
$$e_{02}=\tfrac{98}{125}\ ,\ e_{12}=-\tfrac{36}{125}\ ,\ \text{and}$$
$$a_{11}=-\tfrac{1}{4}\ ,\ a_{21}=0\ ,\ a_{12}=\tfrac{1}{2}\ ,\ a_{22}=-\tfrac{1}{3}.$$



(v) Transposed direct form II
We take the direct form II derived in part (ii) and reverse the arrows as well as exchange the input and output. Then redrawing the flow graph, we get:



(b) To get the difference equation for the flow graph of part (v) in (a), we first define the intermediate variables: $w_1[n]$, $w_2[n]$ and $w_3[n]$. We have:

$$\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= \frac{1}{4}y[n] + w_3[n-1] - \frac{1}{5}x[n] \\
(3)\ w_3[n] &= -\frac{5}{24}y[n] - \frac{1}{12}y[n-1] \\
(4)\quad y[n] &= w_1[n].
\end{aligned}$$

Combining the above equations, we get:

$$y[n] - \frac{1}{4}y[n-1] + \frac{5}{24}y[n-2] + \frac{1}{12}y[n-3] = x[n] - \frac{1}{5}x[n-1].$$

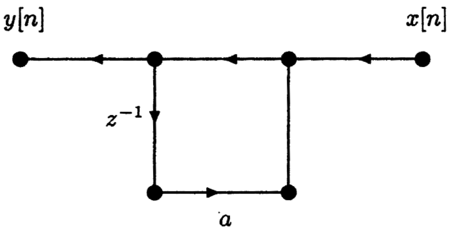
Taking the Z-transform of this equation and combining terms, we get the following transfer function:

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z^{-3}}$$

which is equal to the initial transfer function.

6.27. **(a)**

$$H(z) = \frac{1}{1 - az^{-1}}$$

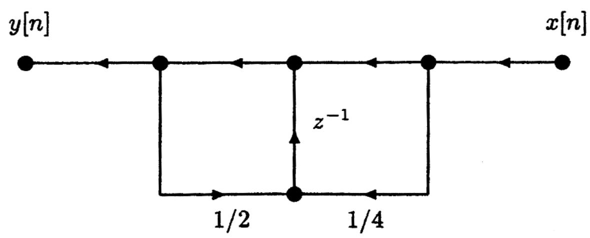


$$y[n] = x[n] + ay[n-1]$$

$$H_T(z) = \frac{1}{1 - az^{-1}} = H(z)$$

(b)

$$H(z) = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

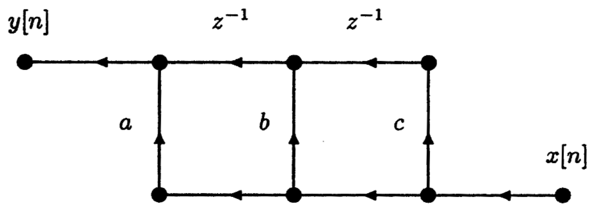


$$y[n] = x[n] + \frac{1}{4}x[n-1] + \frac{1}{2}y[n-1]$$

$$H_T(z) = \frac{1 + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}} = H(z)$$

(c)

$$H(z) = a + bz^{-1} + cz^{-2}$$

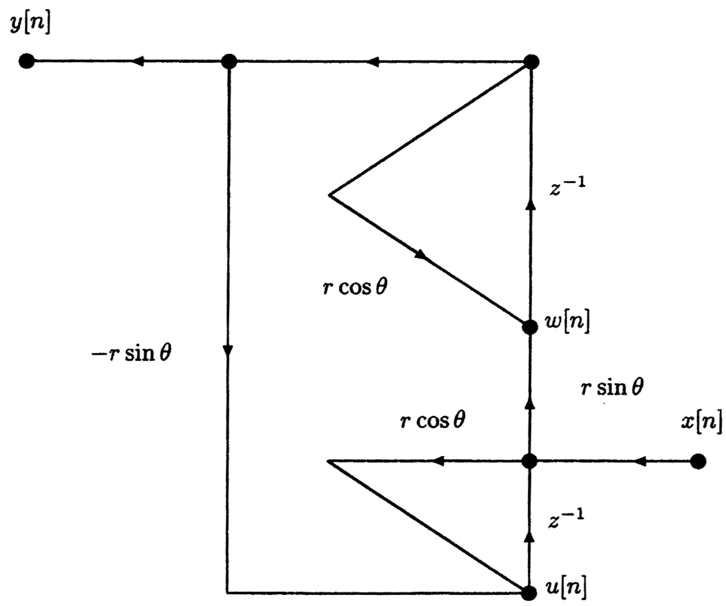


$$y[n] = ax[n] + bx[n-1] + cx[n-2]$$

$$H_T(z) = a + bz^{-1} + cz^{-2} = H(z)$$

(d)

$$H(z) = \frac{r\sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$



$$\begin{aligned}
V &= X + z^{-1}U \\
U &= r\cos\theta V - r\sin\theta Y \\
W &= r\sin\theta V + r\cos\theta z^{-1} W \\
Y &= z^{-1}W \\
\Longrightarrow \frac{Y}{X} &= H_T(z) \\
&= \frac{r\sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}} \\
&= H(z).
\end{aligned}$$

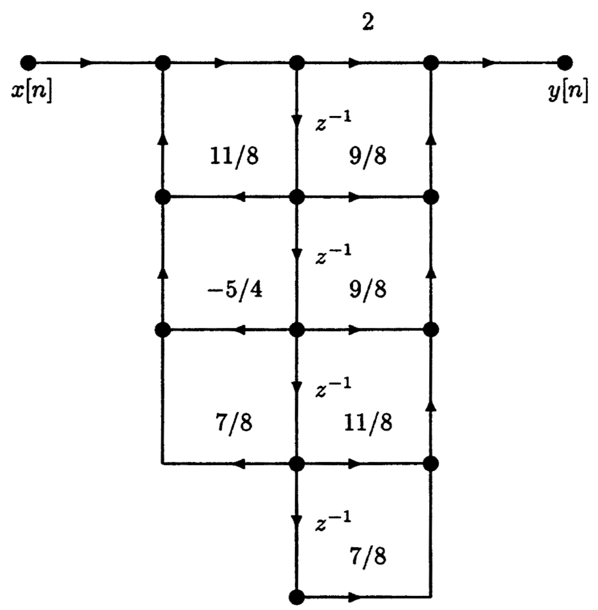
6.28. (a)

$$
\begin{aligned}
H(z) &= \frac{1}{1-z^{-1}}\left[\frac{1-\frac{1}{2}z^{-1}}{1-\frac{3}{8}z^{-1}+\frac{7}{8}z^{-2}}+1+2z^{-1}+z^{-2}\right] \\
&= \frac{2+\frac{9}{8}z^{-1}+\frac{9}{8}z^{-2}+\frac{11}{8}z^{-3}+\frac{7}{8}z^{-4}}{1-\frac{11}{8}z^{-1}+\frac{5}{4}z^{-2}-\frac{7}{8}z^{-3}}.
\end{aligned}
$$

(b)

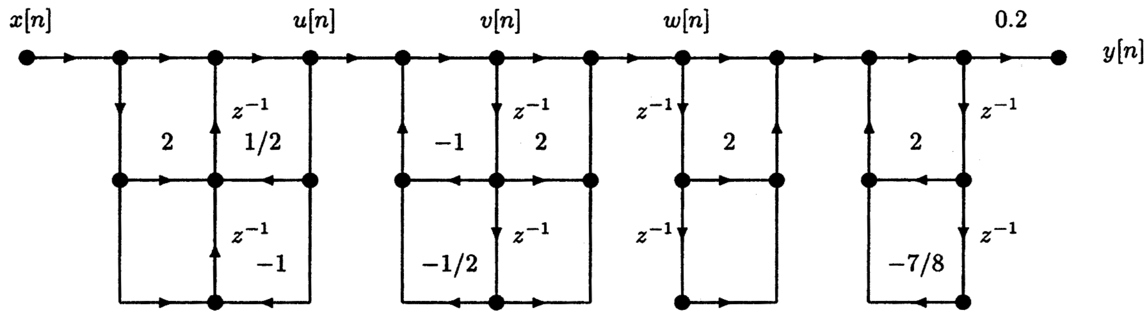
$$
\begin{aligned}
y[n] &= 2x[n]+\frac{9}{8}x[n-1]+\frac{9}{8}x[n-2]+\frac{11}{8}x[n-3]+\frac{7}{8}x[n-4] \\
&+ \frac{11}{8}y[n-1]-\frac{5}{4}y[n-2]+\frac{7}{8}y[n-3].
\end{aligned}
$$

(c) Use Direct Form II:



6.29. (a) We can rearrange $H(z)$ this way:

$$H(z) = \frac{(1+z^{-1})^2}{1-\frac{1}{2}z^{-1}+z^{-2}} \cdot \frac{(1+z^{-1})^2}{1+z^{-1}+\frac{1}{2}z^{-2}} \cdot (1+z^{-1})^2 \cdot \frac{1}{1-2z^{-1}+\frac{7}{8}z^{-2}} \cdot 0.2$$



The solution is not unique; the order of the denominator 2nd-order sections may be rearranged.

(b)

$$\begin{aligned}
u[n] &= x[n] + 2x[n-1] + x[n-2] + \frac{1}{2}u[n-1] - u[n-2] \\
v[n] &= u[n] - v[n-1] - \frac{1}{2}v[n-2] \\
w[n] &= v[n] + 2v[n-1] + v[n-2] \\
y[n] &= w[n] + 2w[n-1] + w[n-2] + 2y[n-1] - \frac{7}{8}y[n-2].
\end{aligned}$$

6.30. Appears in: Fall05 PS4.

Problem

Determine and draw the lattice filter implementation of the following causal all-pole system function:

$$H(z) = \frac{1}{1 + \frac{3}{2}z^{-1} - z^{-2} + \frac{3}{4}z^{-3} + 2z^{-4}}$$

Is the system stable?

Solution from Fall05 PS4

The all-pole filter is fourth-order with coefficients:

$$a_1^{(4)} = -\frac{3}{2}, \quad a_2^{(4)} = 1, \quad a_3^{(4)} = -\frac{3}{4}, \quad a_4^{(4)} = -2.$$

We know immediately that $k_4 = a_4^{(4)} = -2$. To find the remaining reflection coefficients, we need to run the recursion in reverse and find the coefficients for successively lower order filters.

Letting $M = 4$ in equation (11) of the lattice filter notes,

$$\begin{aligned} a_1^{(3)} &= \frac{a_1^{(4)} + k_4 a_3^{(4)}}{1 - k_4^2} = 0 \\ a_2^{(3)} &= \frac{a_2^{(4)} + k_4 a_2^{(4)}}{1 - k_4^2} = \frac{1}{3} \\ a_3^{(3)} &= \frac{a_3^{(4)} + k_4 a_1^{(4)}}{1 - k_4^2} = -\frac{3}{4} \end{aligned}$$

We identify $k_3 = a_3^{(3)} = -\frac{3}{4}$ and proceed to $M = 3$:

$$\begin{aligned} a_1^{(2)} &= \frac{a_1^{(3)} + k_3 a_2^{(3)}}{1 - k_3^2} = -\frac{4}{7} \\ a_2^{(2)} &= \frac{a_2^{(3)} + k_3 a_1^{(3)}}{1 - k_3^2} = \frac{16}{21} \end{aligned}$$

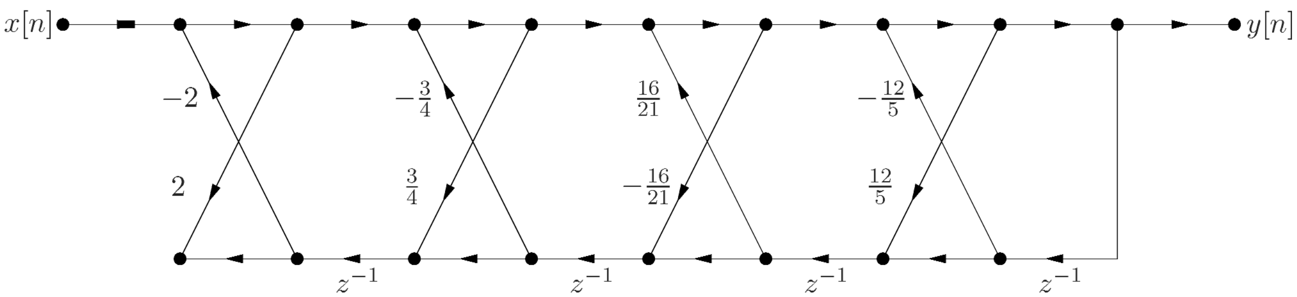
Thus $k_2 = a_2^{(2)} = \frac{16}{21}$. Finally,

$$a_1^{(1)} = \frac{(1 + k_2)a_1^{(2)}}{1 - k_2^2} = \frac{a_1^{(2)}}{1 - k_2} = -\frac{12}{5},$$

and $k_1 = -\frac{12}{5}$.

The lattice structure for $H(z)$ is shown below:

Since $|k_1| > 1$ and $|k_4| > 1$, the all-pole filter cannot be stable.

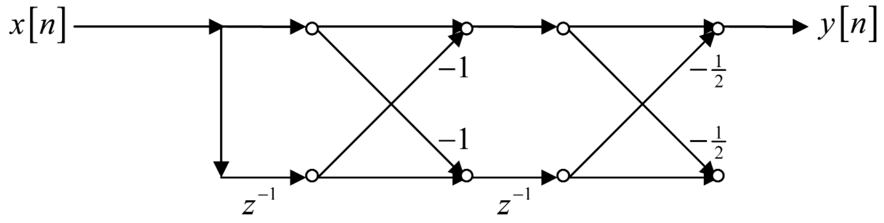


6.31 (AVO)

(a) To determine $y[1]$, sum the gains of all paths with a single delay to the output. This gives

$$y[1] = 1 + (-1)\left(\tfrac{1}{2}\right) = \tfrac{1}{2}.$$

(b) The flow graph for the inverse filter will be a cascade of FIR stages with the *k*-coefficients in the reverse order.



(c) When the FIR lattice of part (b) is driven by an impulse, the response is seen to be

$$\begin{aligned} h_{FIR}[n] &= \delta[n] + \left(-1 + (-1)\left(-\tfrac{1}{2}\right)\right)\delta[n-1] - \tfrac{1}{2}\delta[n-2] \\ &= \delta[n] - \tfrac{1}{2}\delta[n-1] - \tfrac{1}{2}\delta[n-2]. \end{aligned}$$

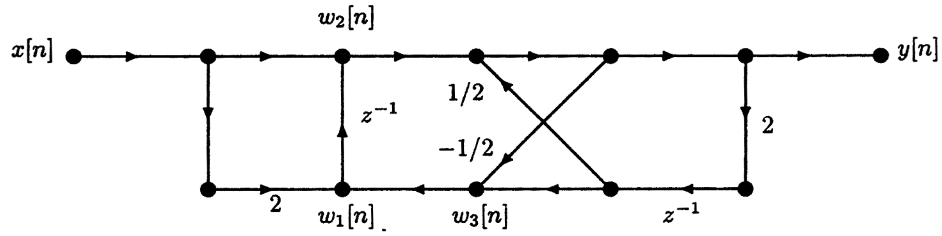
The transfer function is

$$H_{FIR}(z) = 1 - \tfrac{1}{2}z^{-1} - \tfrac{1}{2}z^{-2}.$$

This is the transfer function for the inverse filter. The transfer function for the given lattice is then

$$H(z) = \frac{1}{1 - \tfrac{1}{2}z^{-1} - \tfrac{1}{2}z^{-2}}.$$

6.32. (a) Transpose = reverse arrows direction and reverse the input/output, we get:



(b) From part (a), we have:

$$(1)\ w_1[n] = 2x[n] + w_3[n]$$
$$(2)\ w_2[n] = x[n] + w_1[n-1]$$
$$(3)\ w_3[n] = -\frac{1}{2}y[n] + 2y[n-1]$$
$$(4)\ y[n] = w_2[n] + y[n-1]$$

Taking the Z-transform of the above equations, substituting and rearranging terms, we get:

$$(1 - \frac{1}{2}z^{-1} - 2z^{-2})Y(z) = (2z^{-1} + 1)X(z).$$

Finally, inverse Z-transforming, we get the following difference equation:

$$y[n] - \frac{1}{2}y[n-1] - 2y[n-2] = x[n] + 2x[n-1].$$

(c) From part (b), the system function is given by:

$$H(z) = \frac{1 + 2z^{-1}}{1 - \frac{1}{2}z^{-1} - 2z^{-2}}.$$

It has poles at

$$z = -\frac{8}{1 - \sqrt{33}} \text{ and } z = -\frac{8}{1 + \sqrt{33}}$$

which are outside the unit circle, therefore the system is NOT BIBO stable.

(d)

$$y[2] = x[2] + 2x[1] + \tfrac{1}{2}y[1] + 2y[0]$$
$$y[0] = x[0] = 1$$
$$y[1] = x[1] + 2x[0] + \tfrac{1}{2}y[0] = \tfrac{1}{2} + 2 + \tfrac{1}{2} = 3$$

Therefore,

$$y[2] = \frac{1}{4} + 1 + \frac{3}{2} + 2 = \frac{19}{4}.$$

6.33. Problem 3 in midterm exam.

Problem

For this problem you may find the information on page ?? useful.

Consider the LTI system represented by the FIR lattice structure in Figure 1.

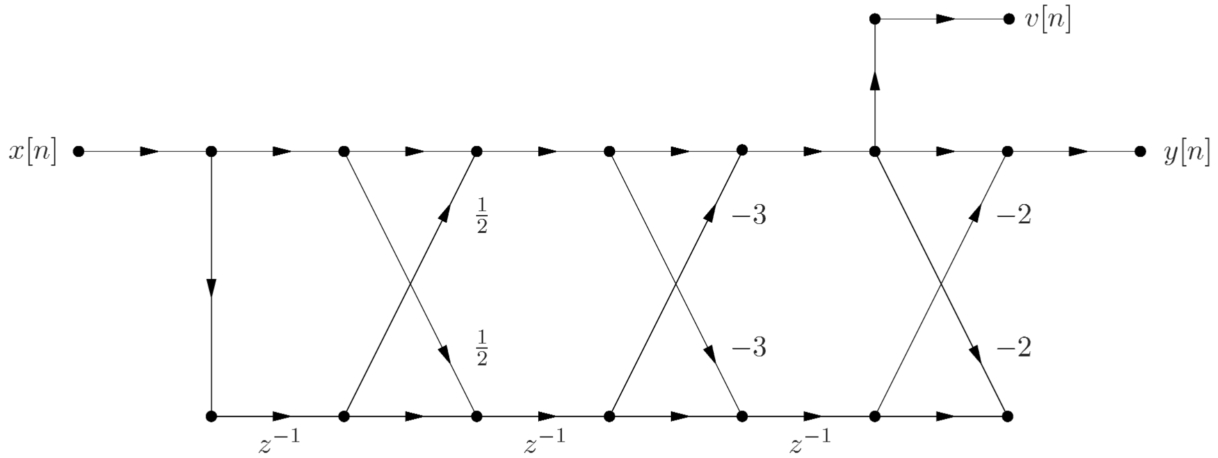


Figure 1:

(a) Determine the system function from the input $x[n]$ to the output $v[n]$ (NOT $y[n]$).

(b) Let $H(z)$ be the system function from the input $x[n]$ to the output $y[n]$, and let $g[n]$ be the result of expanding the associated impulse response $h[n]$ by 2:

The impulse response $g[n]$ defines a new system with system function $G(z)$.

We would like to implement $G(z)$ using an FIR lattice structure as defined by the figure on page **??**. Determine the k-parameters necessary for an FIR lattice implementation of $G(z)$.

Note: You should think carefully before diving into a long calculation.

Solution from Fall05 Midterm

The output $v[n]$ is taken after two stages, so we perform the lattice recursion up to order $p = 2$.

$$k_1 = -\frac{1}{2}, \quad k_2 = 3, \quad k_3 = 2$$

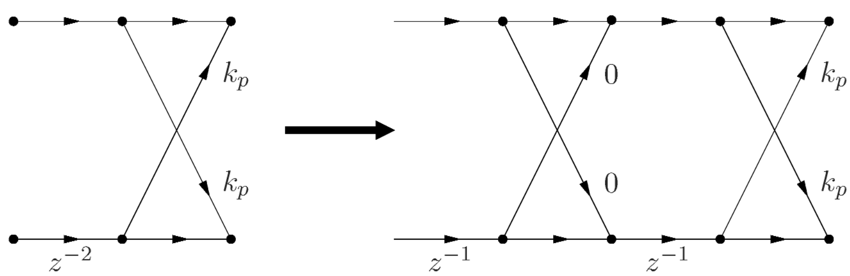
$$a_1^{(1)} = k_1 = -\frac{1}{2}$$

$$\begin{bmatrix} a_1^{(2)} \\ a_2^{(2)} \end{bmatrix} = \begin{bmatrix} a_1^{(1)} \\ 0 \end{bmatrix} - k_2 \begin{bmatrix} a_1^{(1)} \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\frac{V(z)}{X(z)} = 1 - z^{-1} - 3z^{-2}$$

Note the change of signs in going from $a_k^{(2)}$ to the system function.

Since $g[n]$ is $h[n]$ expanded by 2, $G(z) = H(z^2)$. We replace z by z^2 in Figure 1. We then expand each of the three sections as shown below:

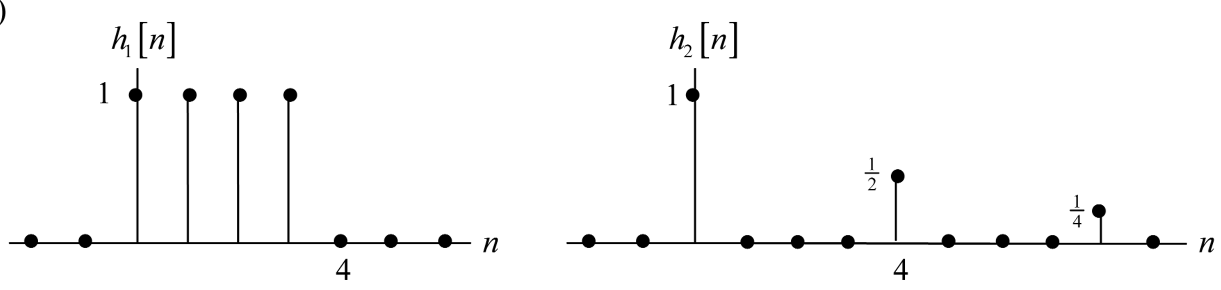


The resulting flowgraph for $G(z)$ is in the form of a 6th-order FIR lattice. We read off the six k-parameters as:

$$\begin{aligned} k_2 &= -\frac{1}{2} \\ k_4 &= 3 \\ k_6 &= 2 \\ k_p &= 0, \quad p = 1, 3, 5 \end{aligned}$$

6.34

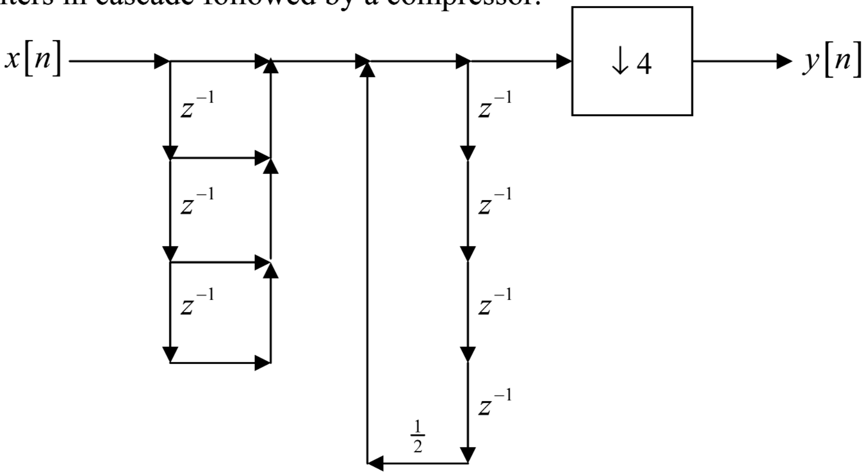
(a)



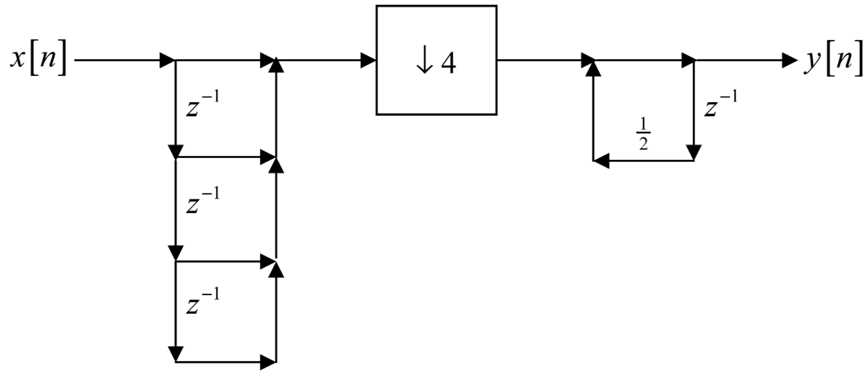
$$h_2[n] = \begin{cases} \left(\frac{1}{2}\right)^{n/4} u[n], & \text{for } n \text{ an integer multiple of 4} \\ 0, & \text{otherwise.} \end{cases}$$

The filter with impulse response $h_2[n]$ is an IIR filter.

(b) The two filters from part (a) have system functions $H_1(z) = 1 + z^{-1} + z^{-2} + z^{-3}$ and $H_2(z) = \dfrac{1}{1 - \frac{1}{2}z^{-4}}$ respectively. The flow graph below shows a direct form implementation of the two filters in cascade followed by a compressor.



We can simplify the design (although without changing the number of coefficient multipliers) by applying the identity of Figure 4.31. The result is shown below.

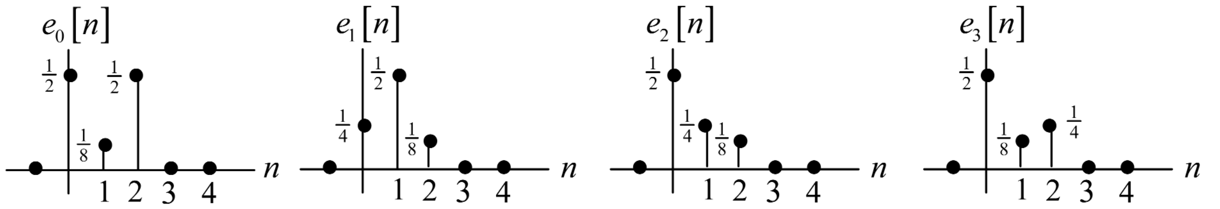


(c) Only one out of every four input samples propagates through the compressor and gets multiplied by the coefficient $\frac{1}{2}$. Thus there is $\frac{1}{4}$ multiplication per input sample.

Every output sample is derived from a multiplication of the previous sample by $\frac{1}{2}$. Thus there is one multiplication per output sample.

6.35

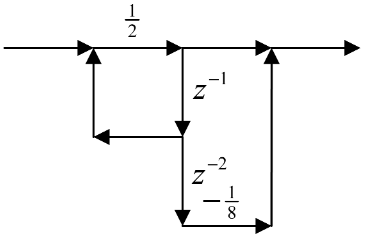
(a)



(b) First, $e_0[n]$ is a three-point system, but symmetry can be used to reduce the number of multiplies to two.

Next, $e_1[n]$ requires three multiplies per output sample.

Next $e_2[n]$ requires three multiplies per output sample. However, $e_2[n]$ can be implemented using pole-zero cancellation as shown. This requires only two multiplies per output sample.

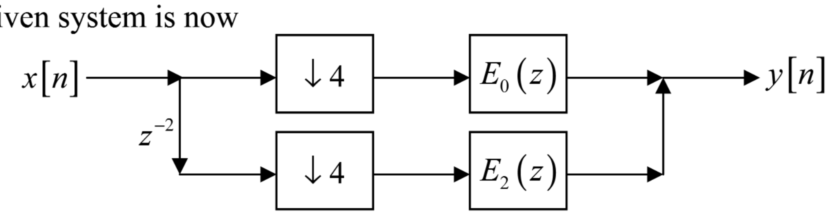


Finally, $e_3[n]$ requires three multiplies per output sample.

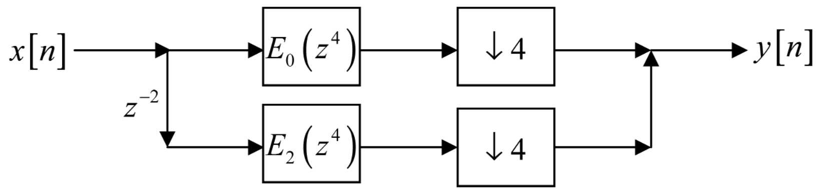
Altogether this is 10 multiplies per output sample.

The compressor reduces the required rate of multiplies relative to the input samples. We need $10/4 = 2.5$ multiplies per input sample.

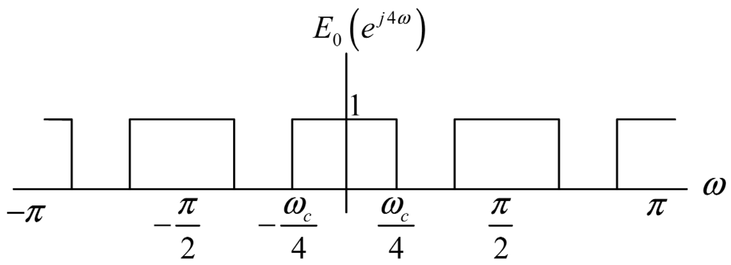
(c) The given system is now

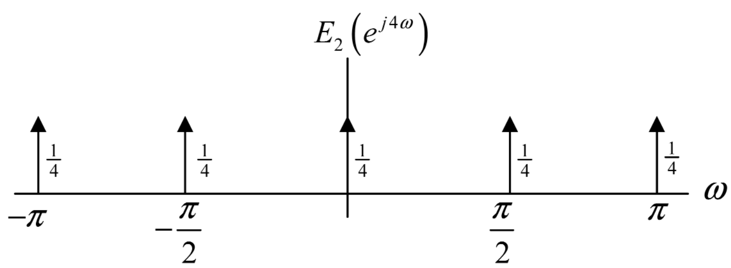


This is equivalent to



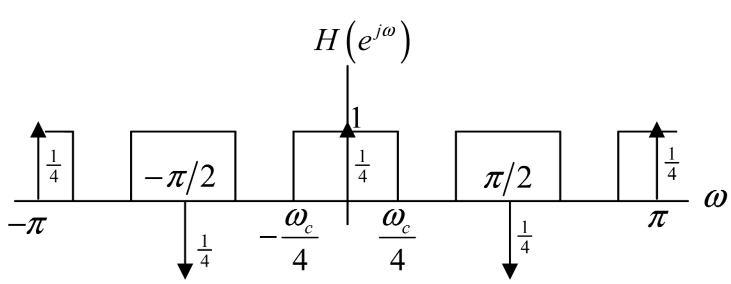
The system function is $H(z) = E_0(z^4) + z^{-2}E_2(z^4)$. This gives a frequency response of $H(e^{j\omega}) = E_0(e^{j4\omega}) + e^{-j2\omega}E_2(e^{j4\omega})$. The components of the frequency response are shown below.



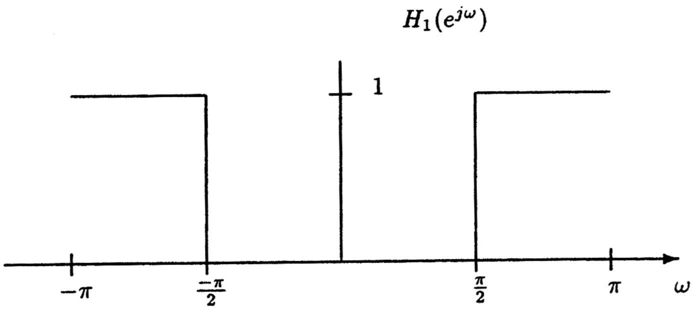


Note that the areas of the impulses scale when the frequency axis is scaled.

Including the factor of $e^{-j2\omega}$, we obtain the frequency response $H(e^{j\omega})$ shown below.

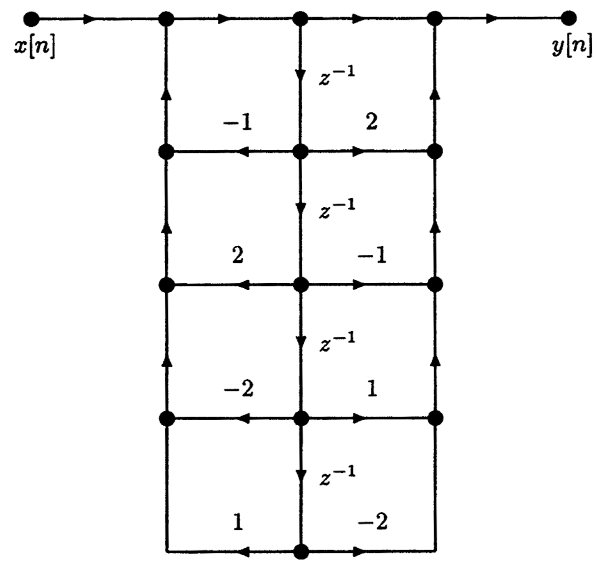


6.36. (a) $H_1(e^{j\omega}) = H(e^{j(\omega+\pi)})$.

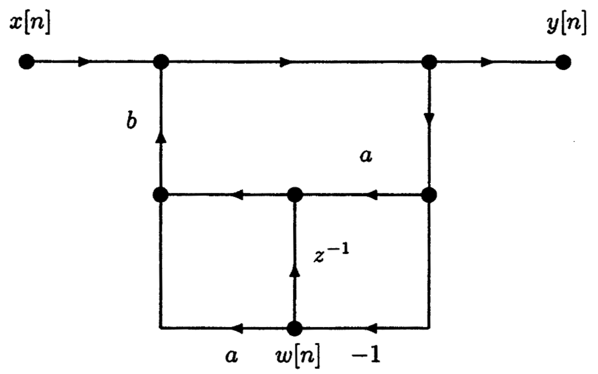


(b) For $H_1(z) = H(-z)$, replace each z^{-1} by $-z^{-1}$. Alternatively, replace each coefficient of an odd-delayed variable by its negative.

(c)



6.37.



(a)

$$\begin{aligned} y[n] &= x[n] + abw[n] + bw[n-1] + aby[n] \\ w[n] &= -y[n]. \end{aligned}$$

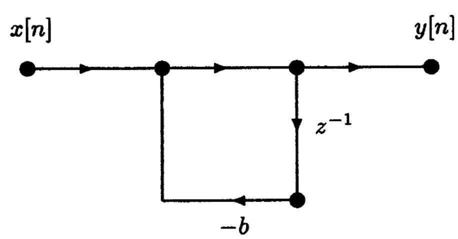
Eliminate w[n]:

$$\begin{aligned} y[n] &= x[n] - aby[n] - by[n-1] + aby[n] \\ y[n] &= x[n] - by[n-1] \end{aligned}$$

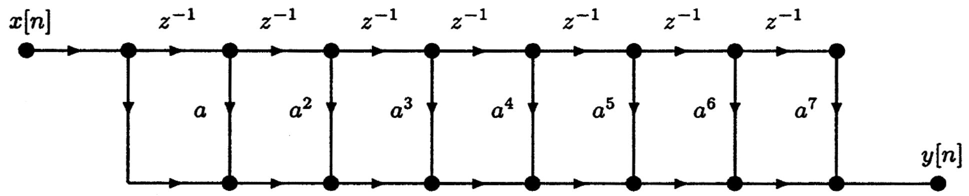
So:

$$H(z) = \frac{1}{1 + bz^{-1}}.$$

(b)



6.38. (a)



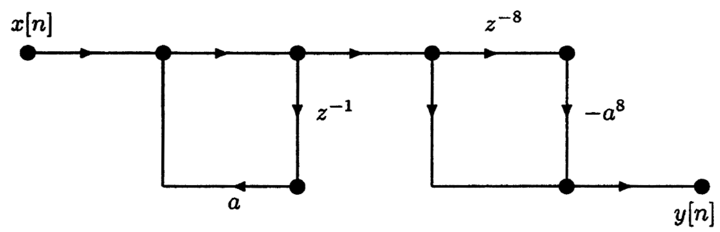
(b) From

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

it follows that

$$\sum_{n=0}^{7} a^n z^{-n} = \frac{1-a^8z^{-8}}{1-az^{-1}}.$$

(c)

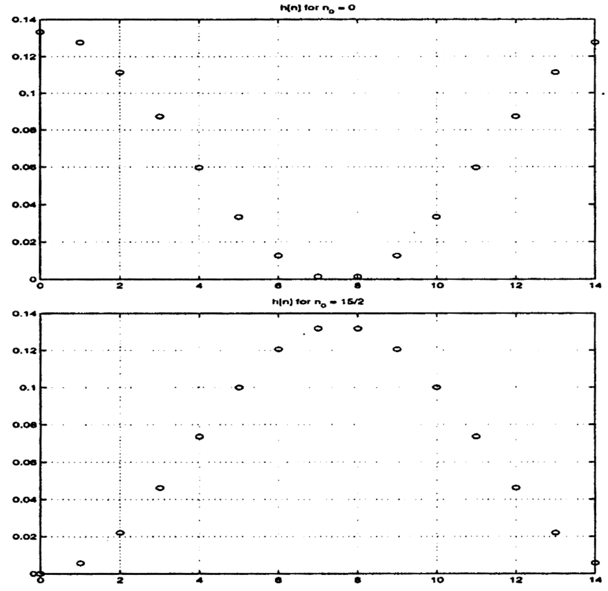


(d) (i) (c) has the most storage: 9 vs. 7.

(ii) (a) has the most arithmetic: 7 adds + 7 multiplies per sample, vs. 2 multiplies + 2 adds per sample.

6.39.

(a)



(b)

$$
\begin{aligned}
H(z) &= \frac{1}{15}\sum_{n=0}^{14}\left[1+\cos\left(\frac{2\pi}{15}(n-n_0)\right)\right]z^{-n} \\
&= \frac{1}{15}\sum_{n=0}^{14} z^{-n} + \frac{1}{15}\sum_{n=0}^{14}\frac{1}{2}\left[e^{j\frac{2\pi}{15}(n-n_0)} + e^{-j\frac{2\pi}{15}(n-n_0)}\right]z^{-n} \\
&= \frac{1}{15}\frac{1-z^{-15}}{1-z^{-1}} + \frac{1}{15}\frac{1}{2}\frac{e^{-j\frac{2\pi}{15}n_0}[1-(e^{j\frac{2\pi}{15}}z^{-1})^{15}]}{1-e^{j\frac{2\pi}{15}}z^{-1}} \\
&\quad + \frac{1}{15}\frac{1}{2}\frac{e^{j\frac{2\pi}{15}n_0}[1-(e^{-j\frac{2\pi}{15}}z^{-1})^{15}]}{1-e^{-j\frac{2\pi}{15}}z^{-1}} \\
&= \frac{1}{15}(1-z^{-15})\left[\frac{1}{1-z^{-1}} + \frac{\frac{1}{2}e^{-j\frac{2\pi}{15}n_0}}{1-e^{j\frac{2\pi}{15}}z^{-1}} + \right. \\
&\quad \left. \frac{\frac{1}{2}e^{j\frac{2\pi}{15}n_0}}{1-e^{-j\frac{2\pi}{15}}z^{-1}}\right].
\end{aligned}
$$

(c)

$$
H(e^{j\omega}) = \frac{1}{15}e^{-j7\omega}\left[\frac{\sin((15\omega)/2)}{\sin(\omega/2)} - \frac{1}{2}\frac{e^{-j\frac{\pi}{15}}\sin((15\omega)/2)}{\sin((\omega-\frac{2\pi}{15})/2)} - \frac{1}{2}\frac{e^{j\frac{\pi}{15}}\sin((15\omega)/2)}{\sin((\omega+(2\pi)/15)/2)}\right]
$$

$$
H(e^{j\omega}) = \frac{1-e^{-j15\omega}}{15}\left[\frac{1}{1-e^{-j\omega}} + \frac{\frac{1}{2}e^{-j\frac{2\pi n_0}{15}}}{1-e^{j\frac{2\pi}{15}}e^{-j\omega}} + \frac{\frac{1}{2}e^{j\frac{2\pi n_0}{15}}}{1-e^{-j\frac{2\pi}{15}}e^{-j\omega}}\right]
$$

When $n_0 = 15/2$, .

$$
H(e^{j\omega}) = \frac{1}{15}\left[\frac{e^{j\frac{\omega}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}} - \frac{\frac{1}{2}e^{j\frac{\omega-(2\pi/15)}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega-(2\pi/15)}{2}}-e^{-j\frac{\omega-(2\pi/15)}{2}}} - \right.
$$

$$\left.\frac{\frac{1}{2}e^{j\frac{\omega+(2\pi/15)}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega+(2\pi/15)}{2}}-e^{-j\frac{\omega+(2\pi/15)}{2}}}\right]$$

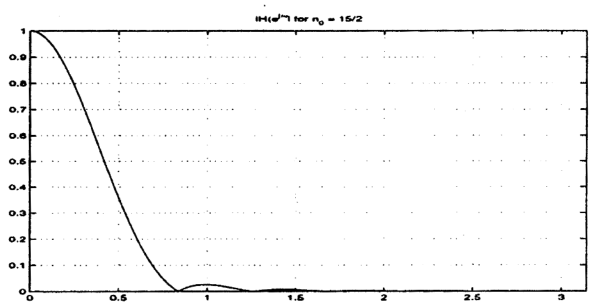
$$= \frac{1}{15}\left[\frac{e^{-j\omega 7}(e^{j\omega\frac{15}{2}}-e^{-j\omega\frac{15}{2}})}{2j\sin\frac{\omega}{2}} - \right.$$

$$\frac{\frac{1}{2}e^{-j\omega 7}e^{-j\frac{\pi}{15}}(e^{j\omega\frac{15}{2}}-e^{-j\omega\frac{15}{2}})}{2j\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} -$$

$$\left.\frac{\frac{1}{2}e^{-j\omega 7}e^{j\frac{\pi}{15}}(e^{j\frac{15}{2}\omega}-e^{-j\frac{15}{2}\omega})}{2j\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right]$$

$$= \frac{e^{-j\omega 7}}{15}\left[\frac{\sin(15\omega/2)}{\sin(\omega/2)} - \frac{\frac{1}{2}e^{-j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} - \right.$$

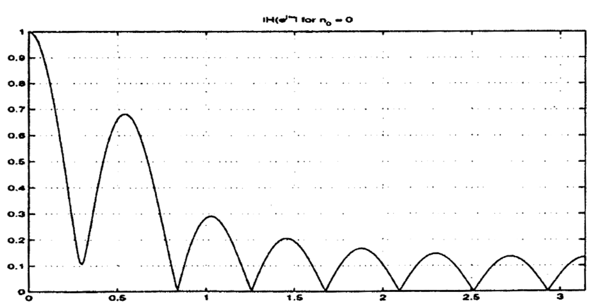
$$\left.\frac{\frac{1}{2}e^{j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right]$$



When $n_0 = 0$,

$$H(e^{j\omega}) = \frac{e^{-j\omega 7}}{15}\left[\frac{\sin(15\omega/2)}{\sin(\omega/2)} + \frac{\frac{1}{2}e^{-j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} + \right.$$

$$\left.\frac{\frac{1}{2}e^{j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right]$$



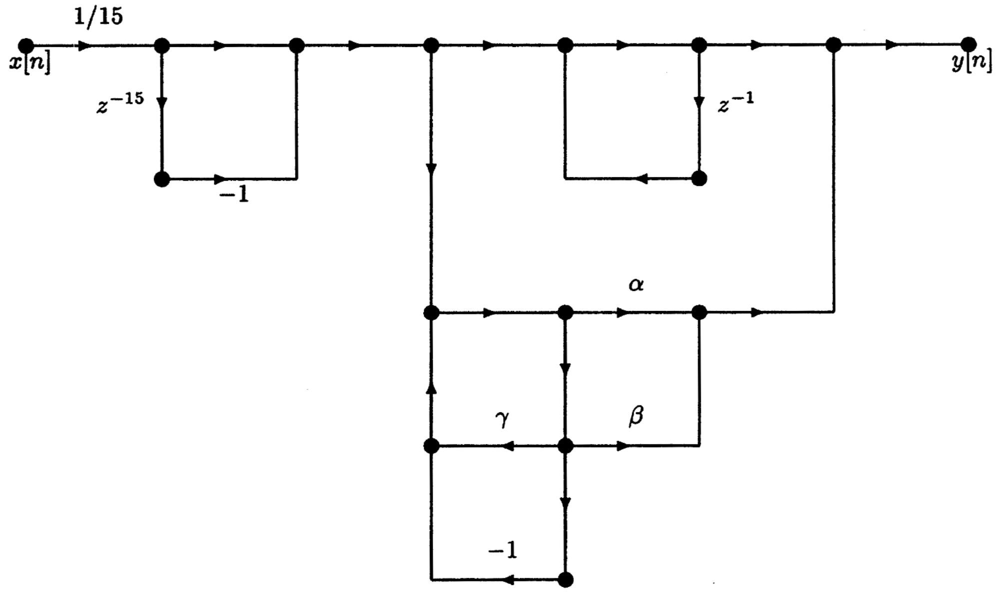
The system will have generalized linear phase if the impulse response has even symmetry (note it cannot have odd symmetry), or alternatively, if the frequency response can be expressed as:

$$H(e^{j\omega}) = e^{-jw7}A_e(e^{j\omega})$$

where $A_e(e^{j\omega})$ is a real, even, periodic function in ω. We thus conclude that the system will have generalized linear phase for $n_0 = \frac{15}{2}k$, where k is an odd integer.

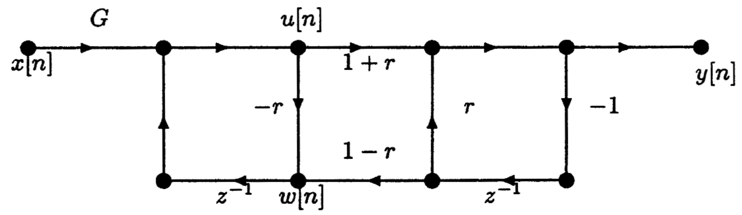
(d) Rewrite $H(z)$ as

$$H(z) = \frac{1 - z^{-15}}{15}\left[\frac{1}{1 - z^{-1}} + \frac{\cos\frac{2\pi n_0}{15} - \cos\left(\frac{2\pi}{15} + \frac{2\pi n_0}{15}\right)z^{-1}}{1 - 2\cos\frac{2\pi}{15}z^{-1} + z^{-2}}\right]$$



where $\alpha = \cos(2\pi n_0/15)$, $\beta = -\cos(2\pi(n_0+1)/15)$, and $\gamma = 2\cos(2\pi/15)$.

6.40. (a)



$$\begin{aligned} u[n] &= Gx[n] + w[n-1] \\ w[n] &= -ru[n] - (1-r)y[n-1] \\ y[n] &= (1+r)u[n] - ry[n-1]. \end{aligned}$$

(b)

$$\begin{aligned} U(z) &= GX(z) + z^{-1}W(z) \\ W(z) &= -ru(z) - (1-r)z^{-1}Y(z) \\ Y(z) &= (1+r)U(z) - rz^{-1}Y(z). \end{aligned}$$

Solve for $U(z)$ in terms of $X(z)$ and $Y(z)$:

$$U(z) = \frac{GX(z) - (1-r)z^{-2}Y(z)}{1 + rz^{-1}}$$

Then

$$Y(z) = (1+r)\left\{\frac{GX(z) - (1-r)z^{-2}Y(z)}{1+rz^{-1}}\right\} - rz^{-1}Y(z)$$

$$Y(z)(1+rz^{-1}) = G(1+r)X(z) - (1-r^2)z^{-2}Y(z) - rz^{-1}Y(z) - r^2z^{-2}Y(z)$$

$$Y(z)(1+2rz^{-1}+z^{-2}) = G(1+r)X(z)$$

$$H_1(z) = \frac{G(1+r)}{1+2rz^{-1}+z^{-2}}.$$

From the quadratic formula, the poles are at $(-r + j\sqrt{1-r^2})^{-1}$ and $(-r - j\sqrt{1-r^2})^{-1}$. The magnitude of each pole is 1. The angles are

$$-\tan^{-1}\left(\frac{\sqrt{1-r^2}}{r}\right) \quad \text{and} \quad \tan^{-1}\left(\frac{\sqrt{1-r^2}}{r}\right),$$

respectively.

(c) $U(z) = z^{-1}(GX(z) + W(z))$, $W(z) = -rU(z) - (1-r)Y(z)$, and $Y(z) = z^{-1}((1+r)U(z) - rY(z))$ lead to

$$H_2(z) = \frac{G(1+r)z^{-2}}{1+2rz^{-1}+z^{-2}} = z^{-2}H_1(z).$$

6.41. (a)

$$\begin{aligned} y_1[n] &= (1+r)x_1[n] + rx_2[n] \\ y_2[n] &= -rx_1[n] + (1-r)x_2[n]. \end{aligned}$$

(b)

$$\begin{aligned} y_1[n] &= (1+a)x_1[n] + dx_2[n] \quad (a=r=d) \\ y_2[n] &= (1+cd)x_2[n] + abx_1[n] \quad (c=d=-1). \end{aligned}$$

(c)

$$\begin{aligned} y_1[n] &= (1+e)x_1[n] + ex_2[n] \quad (e=r) \\ y_2[n] &= efx_1[n] + (1+ef)x_2[n] \quad (f=-1). \end{aligned}$$

(d) B and C preferred over A:

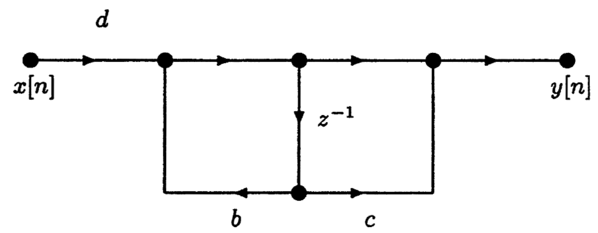
(i) coefficient quantization. If r is small, $1+r$ may not be precisely representable even in floating point. Also, network A has 4 multipliers that must be quantized, while B and C have only 1.

(ii) computational complexity. Networks B and C require fewer multiplications per output sample.

6.42.

$$H(z) = \frac{z^{-1} - 0.54}{1 - 0.54z^{-1}}.$$

(a)

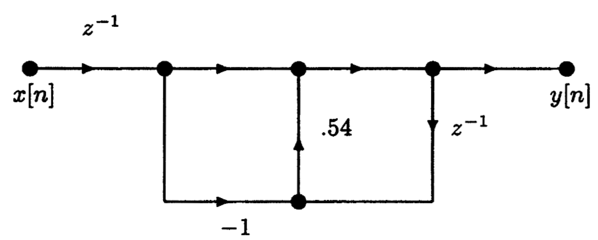


$$H(z) = \frac{cdz^{-1} + d}{1 - bz^{-1}}$$

so set $b = 0.54$, $c = -1.852$, and $d = -0.54$.

(b) With quantized coefficients $\hat{b}$, $\hat{c}$, and $\hat{d}$, $\hat{c}\hat{d} \neq 1$ and $\hat{d} \neq -\hat{b}$ in general, so the resulting system would not be allpass.

(c)

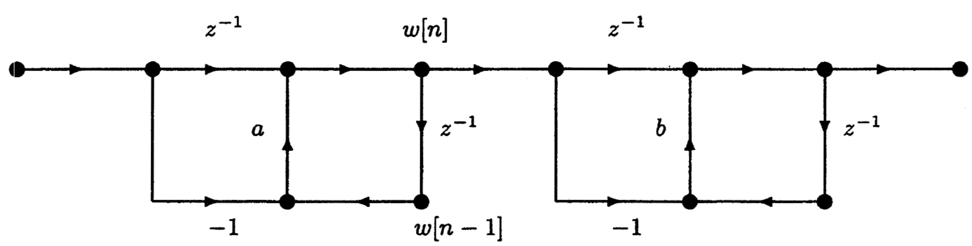


(d) Yes, since there is only one "0.54" to quantize.

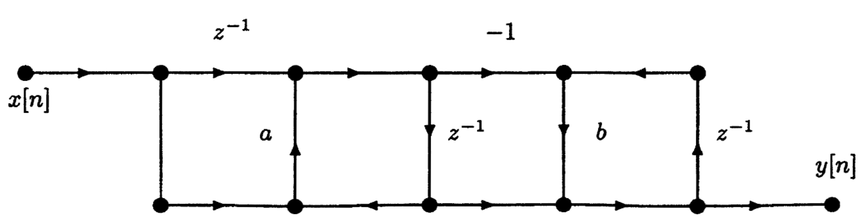
(e)

$$H(z) = \left(\frac{z^{-1} - a}{1 - az^{-1}}\right)\left(\frac{z^{-1} - b}{1 - bz^{-1}}\right)$$

Cascading two sections like (c) gives

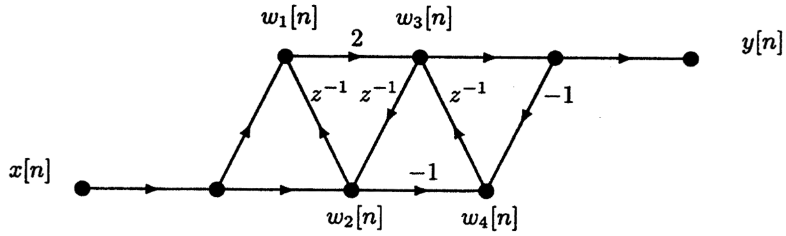


The first delay in the second section has output $w[n-1]$ so we can combine with the second delay of the first section.



(f) Yes, same reason as part (d).

6.43. (a) We have:



First, we find the system function, we have:

$$
\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= x[n] + w_3[n-1] \\
(3)\ w_3[n] &= 2w_1[n] + w_4[n-1] \\
(4)\quad y[n] &= w_3[n] \\
(5)\ w_4[n] &= -y[n] - w_2[n]
\end{aligned}
$$

Taking the Z-transform of the above equations and combining terms, we get:

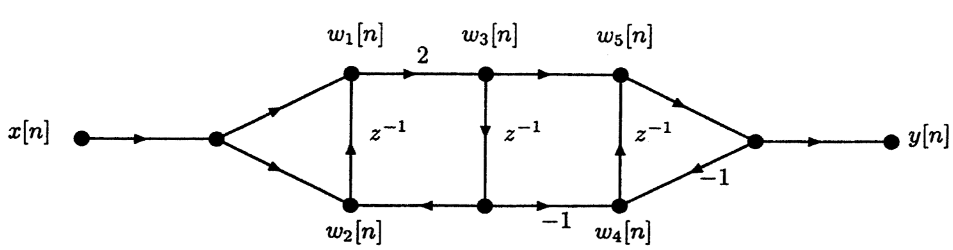
$$(1 - z^{-1})Y(z) + z^{-1}Y(z) = (2 + z^{-1})X(z).$$

The system function is thus given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + z^{-1}}{1 + z^{-1} - z^{-2}}$$

Since the system function is second order (highest order term is z^{-2}), we should be able to implement this system using only 2 delays, this can be done with a direct form II implementation. Therefore, the minimum number of delays required to implement an equivalent system is 2.

(b) Now we have:



Let's find the transfer function, we have:

$$
\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= x[n] + w_3[n-1] \\
(3)\ w_3[n] &= 2w_1[n] \\
(4)\ w_4[n] &= -w_3[n-1] - y[n] \\
(5)\ w_5[n] &= w_3[n] + w_4[n-1] \\
(6)\quad y[n] &= w_5[n]
\end{aligned}
$$

Taking the Z-transform of the above equations and combining terms, we get:

$$(1+z^{-1})Y(z) = \frac{(1-z^{-2})(2+2z^{-1})}{1-2z^{-2}}X(z).$$

The system function is thus given by:

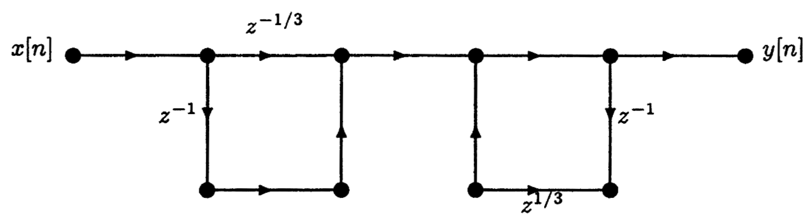
$$H(z) = \frac{Y(z)}{X(z)} = \frac{2(1+z^{-1})(1-z^{-1})}{1-2z^{-2}}.$$

Since the transfer function is not the same as the one in part a, we conclude that system B does not represent the same input-output relationship as system A. This should not be surprising since in system B we added two unidirectional wires and therefore changed the input-output relationship.

6.44.

$$H(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}.$$

(a) Direct form I:



From the graph above, it is clear that 2 delays and 2 multipliers are needed.

(b)

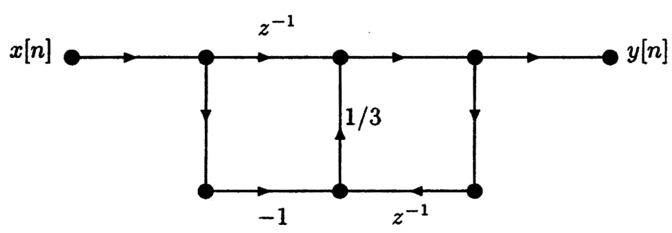
$$(1 - \frac{1}{3}z^{-1})Y(z) = (-\frac{1}{3} + z^{-1})X(z)$$

Inverse $Z-$transforming, we get:

$$y[n] - \frac{1}{3}y[n-1] = -\frac{1}{3}x[n] + x[n-1]$$

$$y[n] = \frac{1}{3}(y[n-1] - x[n]) + x[n-1]$$

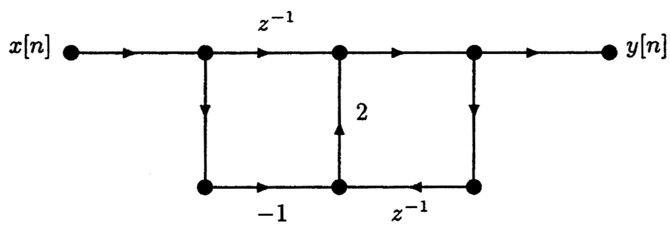
Which can be implemented with the following flow diagram:



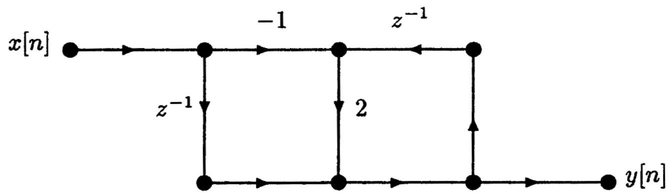
(c)

$$H(z) = \left(\frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}\right)\left(\frac{z^{-1} - 2}{1 - 2z^{-1}}\right).$$

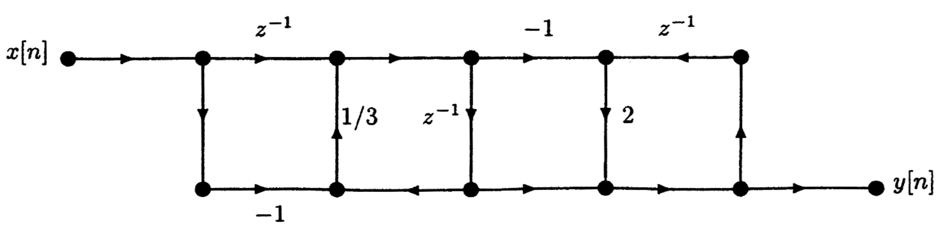
This can be implemented as the cascade of the flow graph in part (b) with the following flow graph:



However the above flow graph can be redrawn as:



Now cascading the above flow graph with the one from part (b) and grouping the delay element we get the following system with two multipliers and three delays:



6.45

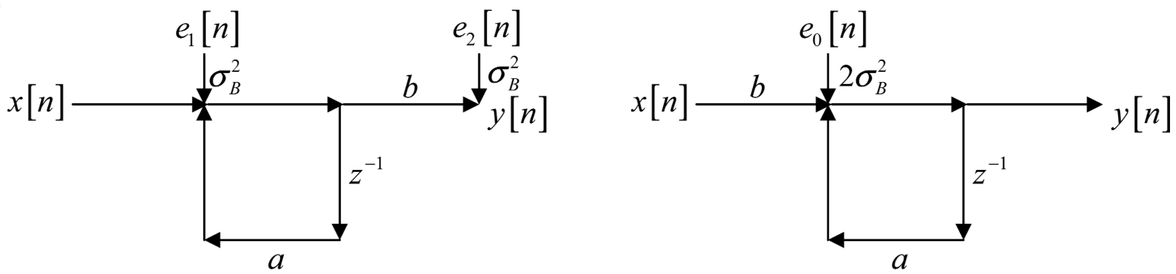
(a) From Eq. (6.120),

$$x_{\max} < \frac{1}{\sum_{m=-\infty}^{\infty} |h[m]|}.$$

For the given systems, $h[n] = ba^n u[n]$. Then we have

$$\begin{aligned} x_{\max} &< \frac{1}{\sum_{m=0}^{\infty} |ba^m|} \\ &= \frac{1}{|b|\sum_{m=0}^{\infty} |a|^m} \\ &= \frac{1-|a|}{|b|}. \end{aligned}$$

(b)



(c) For System 1 we have $H_1(z) = \dfrac{b}{1-az^{-1}}$, so $H_1(e^{j\omega}) = \dfrac{b}{1-ae^{-j\omega}}$. Then

$$\begin{aligned} \Phi_{f_1 f_1}(e^{j\omega}) &= \frac{\sigma_B^2 |b|^2}{|1-ae^{-j\omega}|^2} + \sigma_B^2 \\ &= \frac{\sigma_B^2 b^2}{1+a^2-2a\cos\omega} + \sigma_B^2. \end{aligned}$$

For System 2 we have $H_0(z) = \dfrac{1}{1-az^{-1}}$, so $H_0(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}}$. Then

$$\begin{aligned} \Phi_{f_2 f_2}(e^{j\omega}) &= \frac{2\sigma_B^2}{|1-ae^{-j\omega}|^2} \\ &= \frac{2\sigma_B^2}{1+a^2-2a\cos\omega}. \end{aligned}$$

(d) For System 1,

$$\begin{aligned}\sigma_{f_1}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{f_1 f_1}\left(e^{j\omega}\right)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sigma_B^2|b|^2}{\left|1-ae^{-j\omega}\right|^2}d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma_B^2 d\omega \\ &= \frac{\sigma_B^2 b^2}{1-a^2}+\sigma_B^2,\end{aligned}$$

where the method of Example 6.11 was used to evaluate the integral.

For System 2,

$$\begin{aligned}\sigma_{f_2}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{f_2 f_2}\left(e^{j\omega}\right)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2\sigma_B^2}{\left|1-ae^{-j\omega}\right|^2}d\omega \\ &= \frac{2\sigma_B^2}{1-a^2}.\end{aligned}$$

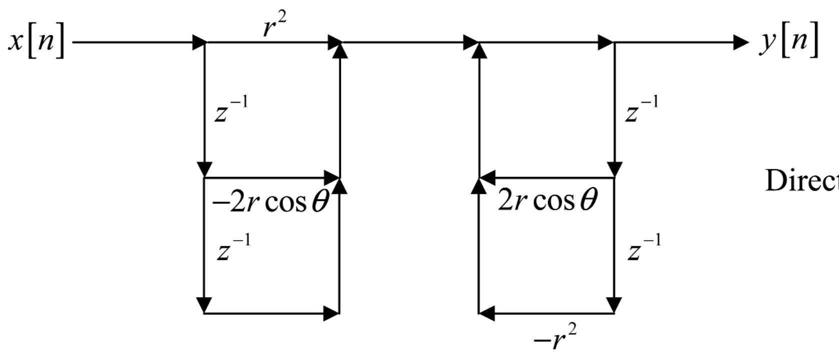
6.46

(a) For the allpass system

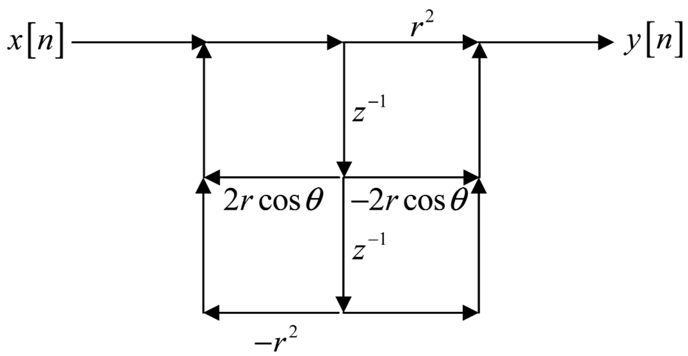
$$H(z)=\frac{\left(z^{-1}-a^*\right)\left(z^{-1}-a\right)}{\left(1-az^{-1}\right)\left(1-a^*z^{-1}\right)}$$

$$=\frac{z^{-2}-\left(a+a^*\right)z^{-1}+|a|^2}{1-\left(a+a^*\right)z^{-1}+|a|^2 z^{-2}}$$

$$=\frac{z^{-2}-2r\cos\theta z^{-1}+r^2}{1-2r\cos\theta z^{-1}+r^2z^{-2}}.$$

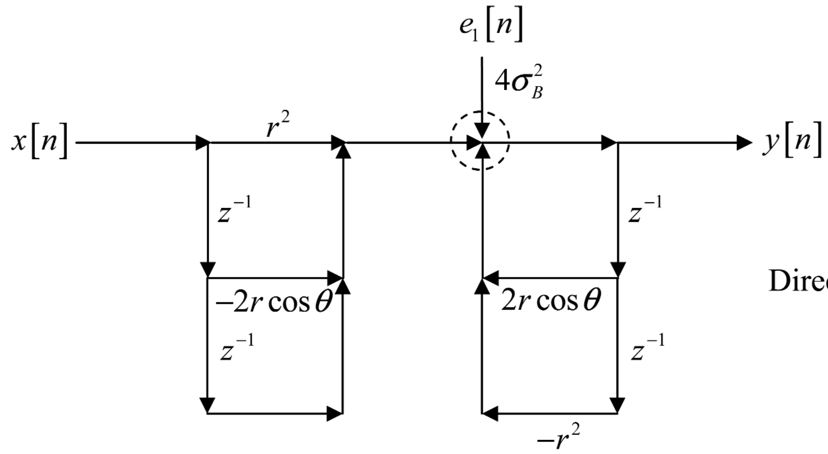


Direct Form I

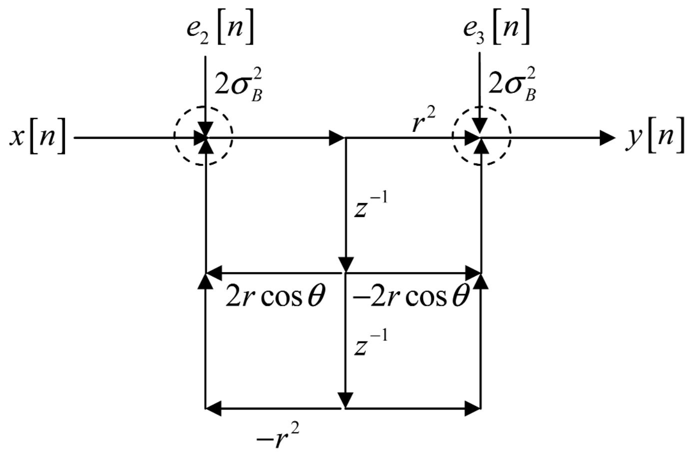


Direct Form II

(b), (c)



Direct Form I



Direct Form II

(d) In the direct form II realization, the noise power spectrum at the output is given by

$$\Phi_{f_2 f_2}\left(e^{j\omega}\right) = 2\sigma_B^2 \left|H\left(e^{j\omega}\right)\right|^2 + 2\sigma_B^2.$$

Since the system is allpass, $\left|H\left(e^{j\omega}\right)\right| = 1$. Consequently, the parameters of the system have no effect on the output noise power.

In the direct form I realization, the noise power spectrum at the output is given by

$$\Phi_{f_1 f_1}\left(e^{j\omega}\right) = \frac{4\sigma_B^2}{\left|\left(1-ae^{-j\omega}\right)\left(1-a^* e^{-j\omega}\right)\right|^2}.$$

This spectrum will show a peak near $\omega = \theta$ caused by the system poles. This spectral peak will become more prominent as $r \to 1$, enhancing the output noise power.

(e) For the direct form II realization,

$$\begin{aligned}
\sigma_{f_2}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{f_2 f_2}\left(e^{j\omega}\right) d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sigma_B^2 d\omega \\
&= 4\sigma_B^2.
\end{aligned}$$

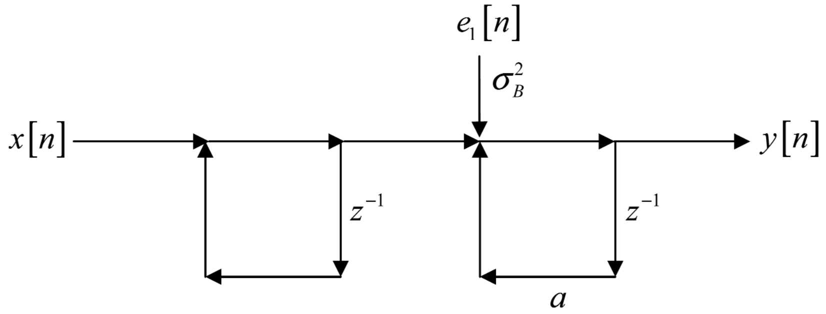
For the direct form I realization,

$$\begin{aligned}
\sigma_{f_1}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{f_1 f_1}\left(e^{j\omega}\right) d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{4\sigma_B^2}{\left|\left(1-ae^{-j\omega}\right)\left(1-a^* e^{-j\omega}\right)\right|^2} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{4\sigma_B^2}{\left|1-2r\cos\theta e^{-j\omega}+r^2 e^{-j2\omega}\right|^2} d\omega \\
&= 4\sigma_B^2\left(\frac{1+r^2}{1-r^2}\right)\frac{1}{1-2r\cos(2\theta)+r^4},
\end{aligned}$$

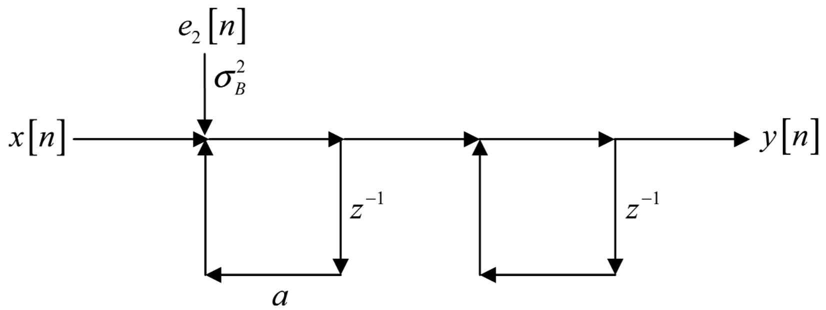
where the technique of Example 6.12 was used to evaluate the integral.

6.47

(a) Flow graph #1:



Flow graph #2:



(b) The power density spectrum of the output noise for flow graph #1 is

$$\Phi_{f_1 f_1}\left(e^{j\omega}\right) = \sigma_B^2 \left|\frac{1}{1-ae^{j\omega}}\right|^2 = \frac{\sigma_B^2}{\left|1-ae^{j\omega}\right|^2}.$$

The total output noise power is

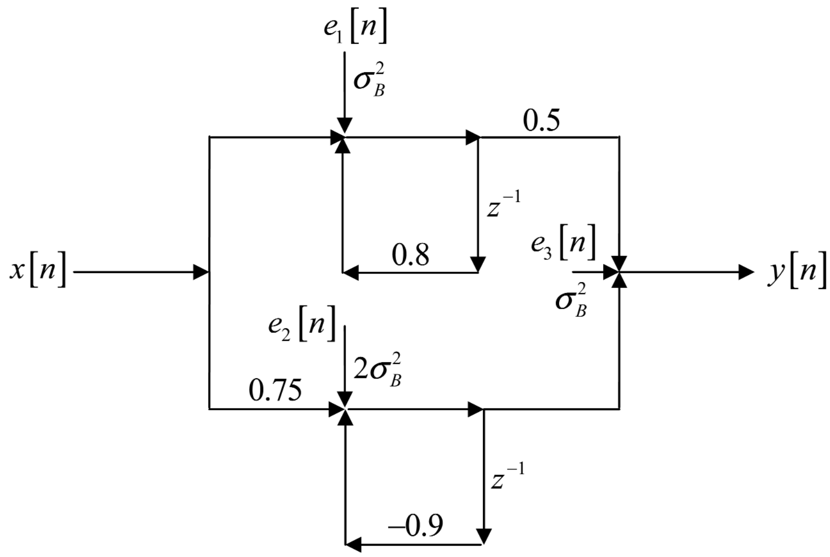
$$\begin{aligned}\sigma_{f_1}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{f_1 f_1}\left(e^{j\omega}\right) d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sigma_B^2}{\left|1-ae^{j\omega}\right|^2} d\omega \\ &= \frac{\sigma_B^2}{1-a^2},\end{aligned}$$

where the method of Example 6.11 was used to evaluate the integral.

(c) The noise in flow graph #2 is filtered through a cascade of two stages. The second stage has a pole on the unit circle at $\omega = 0$. This pole will cause a peak in the noise power density spectrum. Consequently, flow graph #2 would be expected to produce the largest total noise power at the output. (In fact, the noise power at the output of flow graph #2 will be infinite.)

6.48

(a)



(b) Let $H_1(z)$ be the system function of the upper branch, i.e. $H_1(z)=\dfrac{0.5}{1-0.8z^{-1}}$. Let $H_2(z)$ be the system function of the lower branch to the right of the noise source, i.e. $H_2(z)=\dfrac{1}{1+0.9z^{-1}}$. Then the power density spectrum of the output noise is given by

$$\begin{aligned}\Phi_{ff}\left(e^{j\omega}\right)&=\sigma_B^2\left|H_1\left(e^{j\omega}\right)\right|^2+2\sigma_B^2\left|H_2\left(e^{j\omega}\right)\right|^2+\sigma_B^2\\&=\sigma_B^2\frac{(0.5)^2}{\left|1-0.8e^{-j\omega}\right|^2}+2\sigma_B^2\frac{1}{\left|1+0.9e^{-j\omega}\right|^2}+\sigma_B^2\\&=\frac{0.25\sigma_B^2}{1.64-1.6\cos\omega}+\frac{2\sigma_B^2}{1.81+1.8\cos\omega}+\sigma_B^2.\end{aligned}$$

(c) The total noise component of the output is given by

$$\begin{aligned}\sigma_f^2&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{ff}\left(e^{j\omega}\right)d\omega\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{0.25\sigma_B^2}{\left|1-0.8e^{-j\omega}\right|^2}d\omega+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2\sigma_B^2}{\left|1+0.9e^{-j\omega}\right|^2}d\omega+\frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma_B^2d\omega\\&=\frac{0.25\sigma_B^2}{1-(0.8)^2}+\frac{2\sigma_B^2}{1-(-0.9)^2}+\sigma_B^2\\&=12.2\sigma_B^2,\end{aligned}$$

where the method of Example 6.11 was used to evaluate the integrals.

6.49

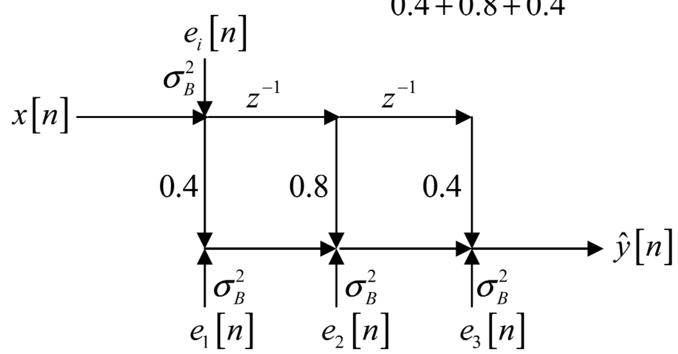
(a) From Eq. (6.120),

$$x_{\max} < \frac{1}{\sum_{m=-\infty}^{\infty} \left|h[m]\right|}.$$

Substituting the coefficients of the given impulse response gives

$$x_{\max} < \frac{1}{0.4+0.8+0.4} = 0.625.$$

(b)



Since quantization is to $(B+1)=16$ bits, we have $\sigma_B^2 = \dfrac{2^{-2\times 15}}{12} = 77.6\times 10^{-12}$.

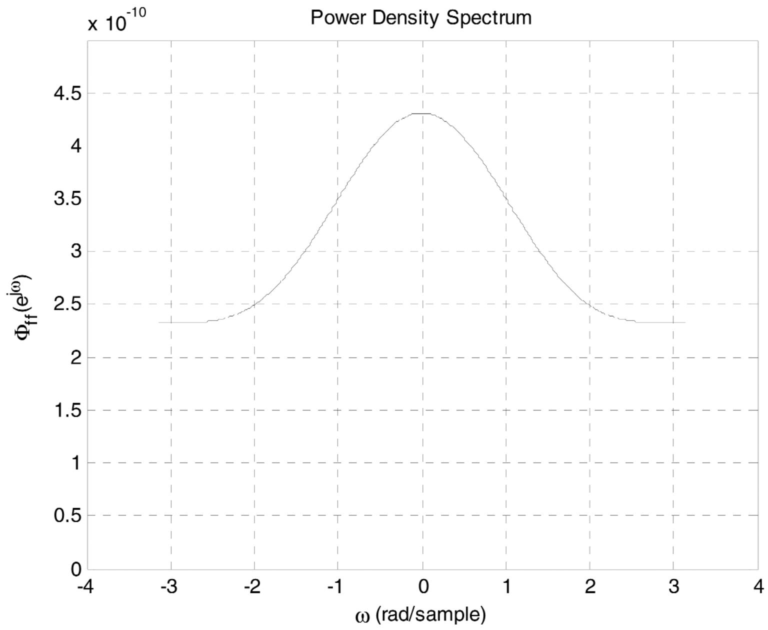
(c) Let $H(e^{j\omega})$ be the frequency response of the digital filter. The total noise power at the output is given by

$$\begin{aligned}
\sigma_f^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{ff}\left(e^{j\omega}\right) d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma_B^2 \left|H\left(e^{-j\omega}\right)\right|^2 d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} 3\sigma_B^2 d\omega \\
&= \sigma_B^2 \sum_{n=0}^{2} h^2[n] + 3\sigma_B^2 \\
&= \left[(0.4)^2 + (0.8)^2 + (0.4)^2 + 3\right]\sigma_B^2 \\
&= 307\times 10^{-12}.
\end{aligned}$$

(d) From the given impulse response, $H(z) = 0.4 + 0.8z^{-1} + 0.4z^{-2}$. Then

$$\begin{aligned}
\Phi_{ff}\left(e^{j\omega}\right) &= \sigma_B^2 \left|H\left(e^{j\omega}\right)\right|^2 + 3\sigma_B^2 \\
&= \sigma_B^2 \left|0.4 + 0.8e^{-j\omega} + 0.4e^{-j2\omega}\right|^2 + 3\sigma_B^2 \\
&= 0.64\left(1+\cos\omega\right)^2 \sigma_B^2 + 3\sigma_B^2 \\
&= \left(3.96 + 1.28\cos\omega + 0.32\cos 2\omega\right)\sigma_B^2 \\
&= 307\times 10^{-12} + 99.3\times 10^{-12}\cos\omega + 24.8\times 10^{-12}\cos 2\omega.
\end{aligned}$$

This power density spectrum is plotted below.



7.1. Using the partial fraction technique, we see

$$H_c(s) = \frac{s+a}{(s+a)^2+b^2} = \frac{0.5}{s+a+jb} + \frac{0.5}{s+a-jb}$$

Now we can use the Laplace transform pair

$$e^{-\alpha t}u(t) \longleftrightarrow \frac{1}{s+\alpha}$$

to get

$$h_c(t) = \frac{1}{2}\left(e^{-(a+jb)t} + e^{-(a-jb)t}\right)u(t).$$

(a) Therefore,

$$\begin{aligned} h_1[n] &= h_c(nT) = \frac{1}{2}\left[e^{-(a+jb)nT} + e^{-(a-jb)nT}\right]u[n] \\ H_1(z) &= \frac{0.5}{1-e^{-(a+jb)T}z^{-1}} + \frac{0.5}{1-e^{-(a-jb)T}z^{-1}}, \quad |z| > e^{-aT} \end{aligned}$$

(b) Since

$$s_c(t) = \int_{-\infty}^{t} h_c(\tau)d\tau \longleftrightarrow \frac{H_c(s)}{s} = S_c(s)$$

we get

$$S_c(s) = \frac{s+a}{s(s+a+jb)(s+a-jb)} = \frac{A_1}{s} + \frac{A_2}{s+a+jb} + \frac{A_2^*}{s+a-jb}$$

where

$$A_1 = \frac{a}{a^2+b^2}, \qquad A_2 = -\frac{0.5}{a+jb}$$

Though the system $h_2[n]$ is related by step invariance to $h_c(t)$, the signal $s_2[n]$ is related to $s_c(t)$ by impulse invariance. Therefore, we know the poles of the partial fraction expansion of $S_c(s)$ above must transform as $z_k = e^{s_k T}$, and we can find

$$S_2(z) = \frac{A_1}{1-z^{-1}} + \frac{A_2}{1-e^{-(a+jb)T}z^{-1}} + \frac{A_2^*}{1-e^{-(a-jb)T}z^{-1}}$$

Now, since the relationship between the step response and the impulse response is

$$s_2[n] = \sum_{k=-\infty}^{n} h_2[k] = \sum_{k=-\infty}^{\infty} h_2[k]u[n-k] = h_2[n] * u[n]$$

$$S_2(z) = \frac{H_2(z)}{1-z^{-1}}$$

We can finally solve for $H_2(z)$

$$\begin{aligned} H_2(z) &= S_2(z)(1-z^{-1}) \\ &= A_1 + A_2\frac{1-z^{-1}}{1-e^{-(a+jb)T}z^{-1}} + A_2^*\frac{1-z^{-1}}{1-e^{-(a-jb)T}z^{-1}}, \quad |z| > e^{-aT} \end{aligned}$$

where A_1 and A_2 are as given above.

(c)

$$\begin{aligned}
s_1[n] &= \sum_{k=-\infty}^{n} h_1[k] = \frac{1}{2}\sum_{k=0}^{n}\left(e^{-(a+jb)kT} + e^{-(a-jb)kT}\right) \\
&= \frac{1}{2}\left[\frac{1-e^{-(a+jb)(n+1)T}}{1-e^{-(a+jb)T}} + \frac{1-e^{-(a-jb)(n+1)T}}{1-e^{-(a-jb)T}}\right]u[n] \\
&= \left[B_1 + B_2 e^{-(a+jb)Tn} + B_2^* e^{-(a-jb)Tn}\right]u[n]
\end{aligned}$$

where

$$B_1 = \frac{1-e^{-aT}\cos bT}{1-2e^{-aT}\cos bT + e^{-2aT}}, \qquad B_2 = -\frac{e^{-(a+jb)T}}{1-e^{-(a+jb)T}}$$

¿From this we can see that

$$\begin{aligned}
S_1(z) &= \frac{B_1}{1-z^{-1}} + \frac{B_2}{1-e^{-(a+jb)T}z^{-1}} + \frac{B_2^*}{1-e^{-(a-jb)T}z^{-1}} \\
&\neq S_2(z)
\end{aligned}$$

since the partial fraction constants are different. Therefore, $s_1[n] \neq s_2[n]$, the two step responses are not equal.

Taking the inverse z-transform of $H_2(z)$

$$\begin{aligned}
h_2[n] &= A_1\delta[n] + A_2\left[e^{-(a+jb)Tn}u[n] - e^{-(a+jb)T(n-1)}u[n-1]\right] \\
&\quad + A_2^*\left[e^{-(a-jb)Tn}u[n] - e^{-(a-jb)T(n-1)}u[n-1]\right]
\end{aligned}$$

where A_1 and A_2 are as defined earlier. By comparing $h_1[n]$ and $h_2[n]$ one sees that $h_1[n] \neq h_2[n]$.

The overall idea this problem illustrates is that a filter designed with impulse invariance is different from a filter designed with step invariance.

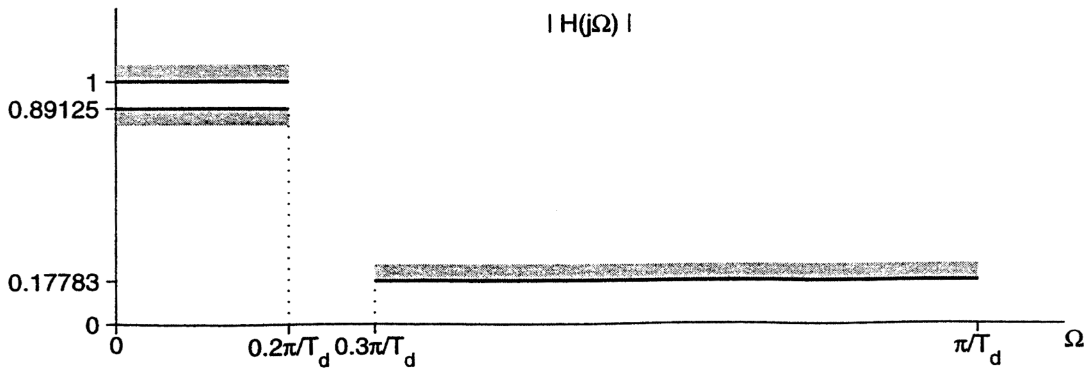
7.2. Recall that $\Omega = \omega/T_d$.

(a) Then

$$0.89125 \le |H(j\Omega)| \le 1, \quad 0 \le |\Omega| \le 0.2\pi/T_d$$

$$|H(j\Omega)| \le 0.17783, \qquad 0.3\pi/T_d \le |\Omega| \le \pi/T_d$$

The plot of the tolerance scheme is



(b) As in the book's example, since the Butterworth frequency response is monotonic, we can solve

$$|H_c(j0.2\pi/T_d)|^2 = \frac{1}{1 + \left(\dfrac{0.2\pi}{\Omega_c T_d}\right)^{2N}} = (0.89125)^2$$

$$|H_c(j0.3\pi/T_d)|^2 = \frac{1}{1 + \left(\dfrac{0.3\pi}{\Omega_c T_d}\right)^{2N}} = (0.17783)^2$$

to get $\Omega_c T_d = 0.70474$ and $N = 5.8858$. Rounding up to $N = 6$ yields $\Omega_c T_d = 0.7032$ to meet the specifications.

(c) We see that the poles of the magnitude-squared function are again evenly distributed around a circle of radius 0.7032. Therefore, $H_c(s)$ is formed from the left half-plane poles of the magnitude-squared function, and the result is the same for any value of T_d. Correspondingly, $H(z)$ does not depend on T_d.

7.3. We are given the digital filter constraints

$$\begin{aligned} 1-\delta_1 \le |H(e^{j\omega})| \le 1+\delta_1, &\quad 0 \le |\omega| \le \omega_p \\ |H(e^{j\omega})| \le \delta_2, &\quad \omega_s \le |\omega| \le \pi \end{aligned}$$

and the analog filter constraints

$$\begin{aligned} 1-\hat{\delta}_1 \le |H_c(j\Omega)| \le 1, &\quad 0 \le |\Omega| \le \Omega_p \\ |H_c(j\Omega)| \le \hat{\delta}_2, &\quad \Omega_s \le |\Omega| \end{aligned}$$

(a) If we divide the digital frequency specifications by $(1+\delta_1)$ we get

$$\begin{aligned} 1-\hat{\delta}_1 &= \frac{1-\delta_1}{1+\delta_1} \\ \hat{\delta}_1 &= \frac{2\delta_1}{1+\delta_1} \\ \hat{\delta}_2 &= \frac{\delta_2}{1+\delta_1} \end{aligned}$$

(b) Solving the equations in Part (a) for δ_1 and δ_2, we find

$$\begin{aligned} \delta_1 &= \frac{\hat{\delta}_1}{2-\hat{\delta}_1} \\ \delta_2 &= \frac{2\hat{\delta}_2}{2-\hat{\delta}_1} \end{aligned}$$

In the example, we were given

$$\begin{aligned} \hat{\delta}_1 &= 1-0.89125 = 0.10875 \\ \hat{\delta}_2 &= 0.17783 \end{aligned}$$

Plugging in these values into the equations for δ_1 and δ_2, we find

$$\begin{aligned} \delta_1 &= 0.0575 \\ \delta_2 &= 0.1881 \end{aligned}$$

The filter $H'(z)$ satisfies the discrete-time filter specifications where $H'(z) = (1+\delta_1)H(z)$ and $H(z)$ is the filter designed in the example. Thus,

$$\begin{aligned} H'(z) &= 1.0575\left[\frac{0.2871-0.4466z^{-1}}{1-1.2971z^{-1}+0.6949z^{-2}} + \frac{-2.1428+1.1455z^{-1}}{1-1.0691z^{-1}+0.3699z^{-2}}\right. \\ &\quad \left. + \frac{1.8557-0.6303z^{-1}}{1-0.9972z^{-1}+0.2570z^{-2}}\right] \\ &= \frac{0.3036-0.4723z^{-1}}{1-1.2971z^{-1}+0.6949z^{-2}} + \frac{-2.2660+1.2114z^{-1}}{1-1.0691z^{-1}+0.3699z^{-2}} \\ &\quad + \frac{1.9624-0.6665z^{-1}}{1-0.9972z^{-1}+0.2570z^{-2}} \end{aligned}$$

(c) Following the same procedure used in part (b) we find

$$\begin{aligned} H'(z) &= 1.0575\left[\frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})}\right. \\ &\quad \left. \times \frac{1}{1-0.9044z^{-1}+0.2155z^{-2}}\right] \\ &= \frac{0.0007802(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})} \\ &\quad \times \frac{1}{1-0.9044z^{-1}+0.2155z^{-2}} \end{aligned}$$

(c) Following the same procedure used in part (b) we find

$$\begin{aligned}
H'(z) &= 1.0575\left[\frac{0.0007378(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})}\right. \\
&\quad \left.\times\frac{1}{1-0.9044z^{-1}+0.2155z^{-2}}\right] \\
&= \frac{0.0007802(1+z^{-1})^6}{(1-1.2686z^{-1}+0.7051z^{-2})(1-1.0106z^{-1}+0.3583z^{-2})} \\
&\quad \times\frac{1}{1-0.9044z^{-1}+0.2155z^{-2}}
\end{aligned}$$

7.4. (a) In the impulse invariance design, the poles transform as $z_k = e^{s_k T_d}$ and we have the relationship

$$\frac{1}{s+a} \longleftrightarrow \frac{T_d}{1-e^{-aT_d}z^{-1}}$$

Therefore,

$$\begin{aligned} H_c(s) &= \frac{2/T_d}{s+0.1} - \frac{1/T_d}{s+0.2} \\ &= \frac{1}{s+0.1} - \frac{0.5}{s+0.2} \end{aligned}$$

The above solution is not unique due to the periodicity of $z = e^{j\omega}$. A more general answer is

$$H_c(s) = \frac{2/T_d}{s+\left(0.1+j\frac{2\pi k}{T_d}\right)} - \frac{1/T_d}{s+\left(0.2+j\frac{2\pi l}{T_d}\right)}$$

where k and l are integers.

(b) Using the inverse relationship for the bilinear transform,

$$z = \frac{1+(T_d/2)s}{1-(T_d/2)s}$$

we get

$$\begin{aligned} H_c(s) &= \frac{2}{1-e^{-0.2}\left(\frac{1-s}{1+s}\right)} - \frac{1}{1-e^{-0.4}\left(\frac{1-s}{1+s}\right)} \\ &= \frac{2(s+1)}{s(1+e^{-0.2})+(1-e^{-0.2})} - \frac{(s+1)}{s(1+e^{-0.4})+(1-e^{-0.4})} \\ &= \left(\frac{2}{1+e^{-0.2}}\right)\left(\frac{s+1}{s+\frac{1-e^{-0.2}}{1+e^{-0.2}}}\right) - \left(\frac{1}{1+e^{-0.4}}\right)\left(\frac{s+1}{s+\frac{1-e^{-0.4}}{1+e^{-0.4}}}\right) \end{aligned}$$

Since the bilinear transform does not introduce any ambiguity, the representation is unique.

7.5. (a) We must use the minimum specifications!

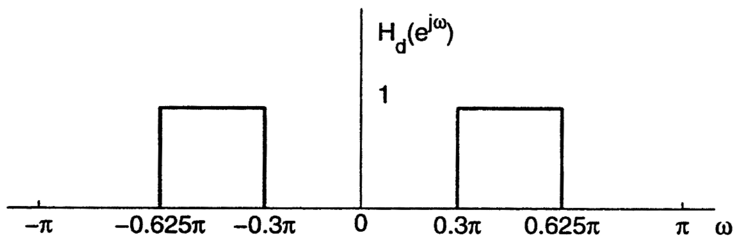
$$\begin{aligned} \delta &= 0.01 \\ \Delta\omega &= 0.05\pi \\ \mathrm{A} &= -20\log_{10}\delta = 40 \end{aligned}$$

$$M+1 = \frac{\mathrm{A}-8}{2.285\Delta\omega} + 1 = 90.2 \rightarrow 91$$

$$\beta = 0.5842(\mathrm{A}-21)^{0.4} + 0.07886(\mathrm{A}-21) = 3.395$$

(b) Since it is a linear phase filter with order 90, it has a delay of $90/2 = 45$ samples.

(c)



$$h_d[n] = \frac{\sin(.625\pi(n-45)) - \sin(.3\pi(n-45))}{\pi(n-45)}$$

7.6. (a) The Kaiser formulas say that a discontinuity of height 1 produces a peak error of δ. If a filter has a discontinuity of a different height the peak error should be scaled appropriately. This filter can be thought of as the sum of two filters. This first is a lowpass filter with a discontinuity of 1 and a peak error of δ. The second is a highpass filter with a discontinuity of 2 and a peak error of 2δ. In the region $0.3\pi \leq |\omega| \leq 0.475\pi$, the two peak errors add but must be less or equal to than 0.06.

$$\begin{aligned} \delta + 2\delta &\leq 0.06 \\ \delta_{\text{max}} &= 0.02 \end{aligned}$$

$$A = -20\log(0.02) = 33.9794$$

$$\beta = 0.5842(33.9794 - 21)^{0.4} + 0.07886(33.9794 - 21) = 2.65$$

(b) The transition width can be

$$\begin{aligned} \Delta\omega &= 0.3\pi - 0.2\pi \\ &= 0.1\pi \text{ rad} \end{aligned} \quad \text{or} \quad \begin{aligned} \Delta\omega &= 0.525\pi - 0.475\pi \\ &= 0.05\pi \text{ rad} \end{aligned}$$

We must choose the smallest transition width so $\Delta\omega_{\text{max}} = 0.05\pi$ rad. The corresponding value of M is

$$M = \frac{33.9794 - 8}{2.285(0.05\pi)} = 72.38 \rightarrow 73$$

7.7. Using the relation $\omega = \Omega T$, the passband cutoff frequency, ω_p, and the stopband cutoff frequency, ω_s, are found to be

$$\begin{aligned}\omega_p &= 2\pi(1000)10^{-4}\\ &= 0.2\pi \text{ rad}\\ \omega_s &= 2\pi(1100)10^{-4}\\ &= 0.22\pi \text{ rad}\end{aligned}$$

Therefore, the specifications for the discrete-time frequency response $H_d(e^{jw})$ are

$$\begin{aligned}0.99 \le \left|H_d(e^{jw})\right| &\le 1.01, \qquad 0 \le |\omega| \le 0.20\pi\\ \left|H_d(e^{jw})\right| &\le 0.01, \qquad 0.22\pi \le |\omega| \le \pi\end{aligned}$$

7.8. Optimal Type I filters must have either $L+2$ or $L+3$ alternations. The filter is 9 samples long so its order is 8 and $L = M/2 = 4$. Thus, to be optimal, the filter must have either 6 or 7 alternations.

Filter 1: 6 alternations
Meets optimal conditions

Filter 2: 7 alternations
Meets optimal conditions

7.9. Using the relation $\omega = \Omega T$, the cutoff frequency ω_c for the resulting discrete-time filter is

$$\begin{aligned}\omega_c &= \Omega_c T \\ &= [2\pi(1000)][0.0002] \\ &= 0.4\pi \text{ rad}\end{aligned}$$

7.10. Using the bilinear transform frequency mapping equation,

$$\begin{aligned}
\omega_c &= 2\tan^{-1}\left(\frac{\Omega_c T}{2}\right) \\
&= 2\tan^{-1}\left(\frac{2\pi(2000)(0.4\times 10^{-3})}{2}\right) \\
&= 0.7589\pi \text{ rad}
\end{aligned}$$

7.11. Using the relation $\omega = \Omega T$,

$$\begin{aligned}\Omega_c &= \frac{\omega_c}{T} \\ &= \frac{\pi/4}{0.0001} \\ &= 2500\pi \\ &= 2\pi(1250)\ \frac{\text{rad}}{\text{s}}\end{aligned}$$

7.12. Using the bilinear transform frequency mapping equation,

$$\begin{aligned}
\Omega_c &= \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right) \\
&= \frac{2}{0.001}\tan\left(\frac{\pi/2}{2}\right) \\
&= 2000\ \frac{\text{rad}}{\text{s}} \\
&= 2\pi(318.3)\ \frac{\text{rad}}{\text{s}}
\end{aligned}$$

7.13. Using the relation $\omega = \Omega T$,

$$\begin{aligned} T &= \frac{\omega_c}{\Omega_c} \\ &= \frac{2\pi/5}{2\pi(4000)} \\ &= 50\ \mu\text{s} \end{aligned}$$

This value of T is unique. Although one can find other values of T that will alias the continuous-time frequency $\Omega_c = 2\pi(4000)$ rad/s to the discrete-time frequency $\omega_c = 2\pi/5$ rad, the resulting aliased filter will not be the ideal lowpass filter.

7.14. Using the bilinear transform frequency mapping equation,

$$\begin{aligned}\Omega_c &= \frac{2}{T}\tan\left(\frac{\omega_c + 2\pi k}{2}\right), \quad \text{k an integer} \\ &= \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)\end{aligned}$$

$$T = \frac{2}{2\pi(300)}\tan\left(\frac{3\pi/5}{2}\right) = 1.46 \text{ ms}$$

The only ambiguity in the above is the periodicity in ω. However, the periodicity of the tangent function "cancels" the ambiguity and so T is unique.

7.15. This filter requires a maximal passband error of $\delta_p = 0.05$, and a maximal stopband error of $\delta_s = 0.1$. Converting these values to dB gives

$$\delta p = -26 \text{ dB}$$
$$\delta s = -20 \text{ dB}$$

This requires a window with a peak approximation error less than -26 dB. Looking in Table 7.1, the Hanning, Hamming, and Blackman windows meet this criterion.

Next, the minimum length L required for each of these filters can be found using the "approximate width of mainlobe" column in the table since the mainlobe width is about equal to the transition width. Note that the actual length of the filter is $L = M + 1$.

Hanning:

$$\begin{aligned} 0.1\pi &= \frac{8\pi}{M} \\ M &= 80 \end{aligned}$$

Hamming:

$$\begin{aligned} 0.1\pi &= \frac{8\pi}{M} \\ M &= 80 \end{aligned}$$

Blackman:

$$\begin{aligned} 0.1\pi &= \frac{12\pi}{M} \\ M &= 120 \end{aligned}$$

7.16. Since filters designed by the window method inherently have $\delta_1 = \delta_2$ we must use the smaller value for δ.

$$\begin{aligned} \delta &= 0.02 \\ A &= -20\log_{10}(0.02) = 33.9794 \end{aligned}$$

$$\beta = 0.5842(33.9794 - 21)^{0.4} + 0.07886(33.9794 - 21) = 2.65$$

$$M = \frac{A-8}{2.285\Delta\omega} = \frac{33.9794 - 8}{2.285(0.65\pi - 0.63\pi)} = 180.95 \rightarrow 181$$

7.17. Using the relation $\omega = \Omega T$, the specifications which should be used to design the prototype continuous-time filter are

$$\begin{aligned} -0.02 < H(j\Omega) &< 0.02, & 0 \le |\Omega| &\le 2\pi(20) \\ 0.95 < H(j\Omega) &< 1.05, & 2\pi(30) \le |\Omega| &\le 2\pi(70) \\ -0.001 < H(j\Omega) &< 0.001, & 2\pi(75) \le |\Omega| &\le 2\pi(100) \end{aligned}$$

Note: Typically, a continuous-time filter's passband tolerance is between 1 and $1 - \delta_1$ since historically most continuous-time filter approximation methods were developed for passive systems which have a gain less than one. If necessary, specifications using this convention can be obtained from the above specifications by scaling the magnitude response by $\frac{1}{1.05}$.

7.18. Using the bilinear transform frequency mapping equation,

$$\begin{aligned}\Omega_s &= \frac{2}{T}\tan\left(\frac{\omega_s}{2}\right) = \frac{2}{2\times 10^{-3}}\tan\left(\frac{0.2\pi}{2}\right) = 2\pi(51.7126)\ \frac{\text{rad}}{\text{s}}\\ \Omega_p &= \frac{2}{T}\tan\left(\frac{\omega_p}{2}\right) = \frac{2}{2\times 10^{-3}}\tan\left(\frac{0.3\pi}{2}\right) = 2\pi(81.0935)\ \frac{\text{rad}}{\text{s}}\end{aligned}$$

Thus, the specifications which should be used to design the prototype continuous-time filter are

$$\begin{aligned}|H_c(j\Omega)| &< 0.04, \qquad |\Omega| \le 2\pi(51.7126)\\ 0.995 < |H_c(j\Omega)| &< 1.005, \qquad |\Omega| \ge 2\pi(81.0935)\end{aligned}$$

Note: Typically, a continuous-time filter's passband tolerance is between 1 and $1-\delta_1$ since historically most continuous-time filter approximation methods were developed for passive systems which have a gain less than one. If necessary, specifications using this convention can be obtained from the above specifications by scaling the magnitude response by $\frac{1}{1.005}$.

7.19. Using the relation $\omega = \Omega T$,

$$\begin{aligned} T &= \frac{\omega}{\Omega} \\ &= \frac{\pi/4}{2\pi(300)} \\ &= 417\ \mu\text{s} \end{aligned}$$

This choice of T is unique. It is possible to find other values of T that alias one of the given continuous-time band edges to its corresponding discrete-time band edge. However, this is the only value of T that maps both band edges correctly.

7.20. *True*. The bilinear transform is a frequency mapping. The value of $H(s)$ for a particular value of s gets mapped to $H(e^{j\omega})$ at a particular value of ω according to the mapping

$$s = \frac{2}{T_d}\left(\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}\right).$$

The continuous frequency axis gets warped onto the discrete-time frequency axis, but the magnitude values do not change. If $H(s)$ is constant for all s, then $H(e^{jw})$ must also be constant.

7.21. (a) Using the bilinear transform frequency mapping equation,

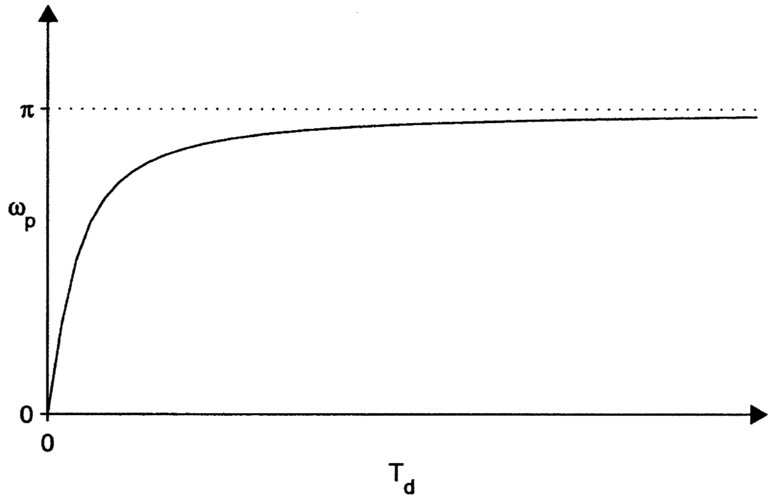
$$\Omega_p = \frac{2}{T_d}\tan\left(\frac{\omega_p}{2}\right)$$

we have

$$\begin{aligned} T_d &= \frac{2}{\Omega_p}\tan\left(\frac{\pi}{4}\right) \\ &= \frac{2}{\Omega_p} \end{aligned}$$

(b)

$$\omega_p = 2\tan^{-1}\left(\frac{\Omega_p T_d}{2}\right)$$

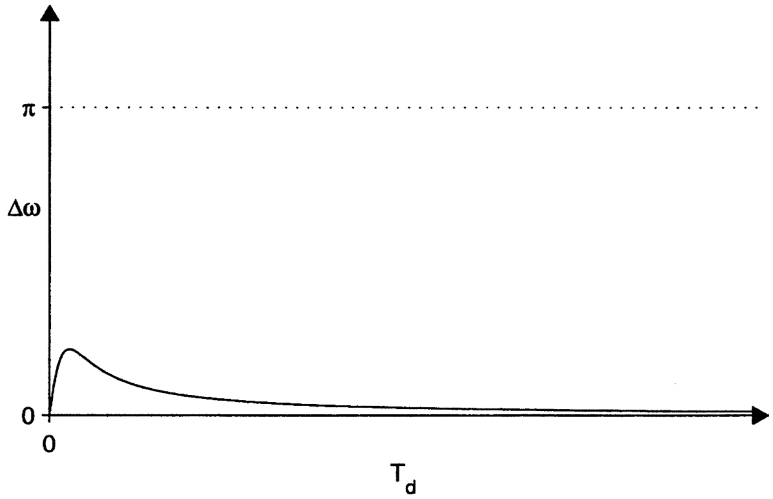


(c)

$$\omega_s = 2\tan^{-1}\left(\frac{\Omega_s T_d}{2}\right)$$

$$\omega_p = 2\tan^{-1}\left(\frac{\Omega_p T_d}{2}\right)$$

$$\Delta\omega = \omega_s - \omega_p = 2\left[\tan^{-1}\left(\frac{\Omega_s T_d}{2}\right) - \tan^{-1}\left(\frac{\Omega_p T_d}{2}\right)\right]$$



7.22

A. Strictly speaking, the input $x_c(t)$ must be bandlimited to 5000 Hz to ensure that there is no aliasing when sampled at 10000 samples/sec. As a practical matter, it may be adequate to bandlimit the input to 7000 Hz. Frequency components between 5000 and 7000 Hz will alias to the range $\Omega = 2\pi 3000$ to $2\pi 5000$ rad/s, or $\omega = 0.6\pi$ to π, using $\omega = \Omega T$. Thus the aliased components will fall in the stopband of the discrete-time lowpass filter.

B. For the continuous-time system, the passband edge is $\Omega_p = \omega_p / T = 0.4\pi \times 10000 = 2\pi 2000$ rad/s. The stopband edge is $\Omega_s = \omega_s / T = 0.6\pi \times 10000 = 2\pi 3000$ rad/s. Within the passband the specifications are

$$(1-\delta_1) \le |H_{eff}(j\Omega)| \le (1+\delta_1), \quad |\Omega| \le \Omega_p$$

$$0.99 \le |H_{eff}(j\Omega)| \le 1.02, \quad |\Omega| \le 2\pi 2000.$$

Within the stopband the specifications are

$$|H_{eff}(j\Omega)| \le \delta_2, \quad \Omega_s \le \Omega \le 2\pi 5000$$

$$|H_{eff}(j\Omega)| \le 0.001, \quad 2\pi 3000 \le \Omega \le 2\pi 5000.$$

C. The given filter is a linear phase filter whose impulse response has a length of 28 samples. The group delay of the filter is $\alpha = 27/2 = 13.5$ samples. Since samples are spaced 10^{-4} seconds apart, the delay in seconds is $13.5 \times 10^{-4} = 1.35$ ms.

7.23. (a) Applying the bilinear transform yields

$$\begin{aligned} H(z) &= H_c(s)\,\Big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{T_d}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right), \qquad |z|>1 \end{aligned}$$

which has the impulse response

$$h[n] = \frac{T_d}{2}\left(u[n] + u[n-1]\right)$$

(b) The difference equation is

$$y[n] = \frac{T_d}{2}\left(x[n] + x[n-1]\right) + y[n-1]$$

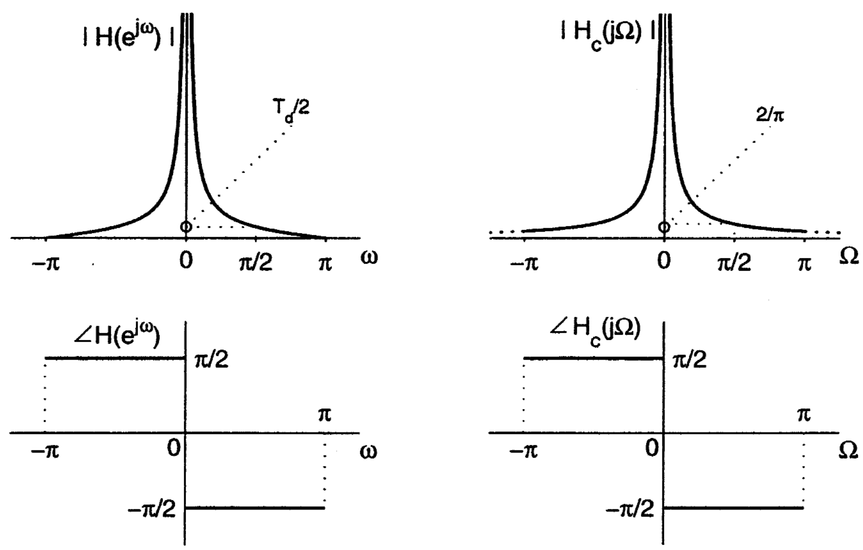
This system is not implementable since it has a pole on the unit circle and is therefore not stable.

(c) Since this system is not stable, it does not strictly have a frequency response. However, if we ignore this mathematical subtlety we get

$$\begin{aligned} H(e^{j\omega}) &= \frac{T_d}{2}\left(\frac{1+e^{-j\omega}}{1-e^{-j\omega}}\right) \\ &= \frac{T_d}{2}\left(\frac{e^{j\omega/2}+e^{-j\omega/2}}{e^{j\omega/2}-e^{-j\omega/2}}\right) \\ &= \frac{T_d}{2j}\cot(\omega/2) \end{aligned}$$

and since the Laplace transform evaluated along the $j\Omega$ axis is the continous-time Fourier transform we also have

$$H_c(j\Omega) = \frac{1}{j\Omega}$$



In general, we see that we will not be able to approximate the high frequencies, but we can approximate the lower frequencies if we choose $T_d = 4/\pi$.

(d) Applying the bilinear transform yields

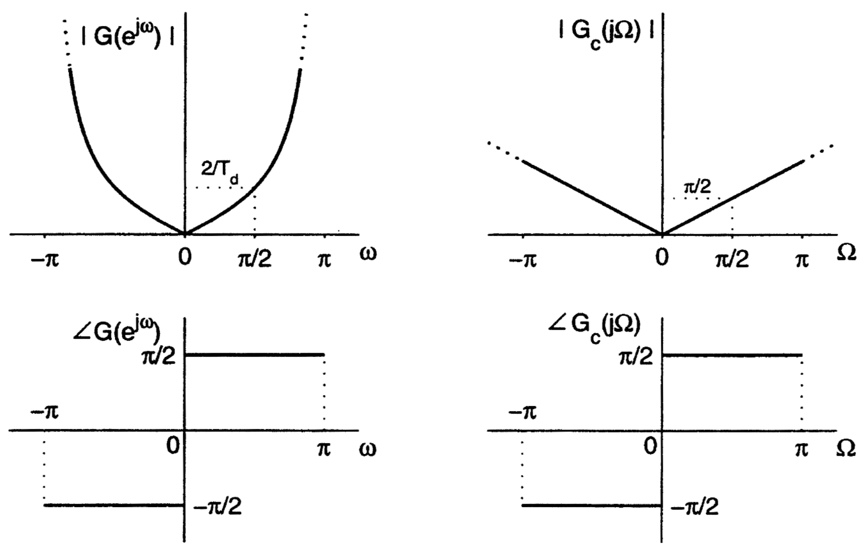
$$\begin{aligned} G(z) &= H_c(s)\big|_{s=\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{2}{T_d}\left[\frac{1-z^{-1}}{1+z^{-1}}\right], \qquad |z|>1 \end{aligned}$$

which has the impulse response

$$\begin{aligned} g[n] &= \frac{2}{T_d}\left[(-1)^n u[n] - (-1)^{n-1}u[n-1]\right] \\ &= \frac{2}{T_d}\left[2(-1)^n u[n] - \delta[n]\right] \end{aligned}$$

(e) This system does not strictly have a frequency response either, due to the pole on the unit circle. However, ignoring this fact again we get

$$\begin{aligned} G(e^{j\omega}) &= \frac{2}{T_d}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right] \\ &= \frac{2}{T_d}\left(\frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}}\right) \\ &= \frac{2j}{T_d}\tan(\omega/2) \\ G(j\Omega) &= j\Omega \end{aligned}$$

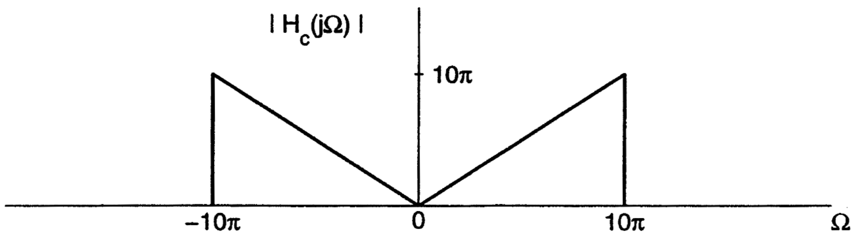


Again, we see that we will not be able to approximate the high frequencies, but we can approximate the lower frequencies if we choose $T_d = 4/\pi$.

(f) If the same value of T_d is used for each bilinear transform, then the two systems are inverses of each other, since then

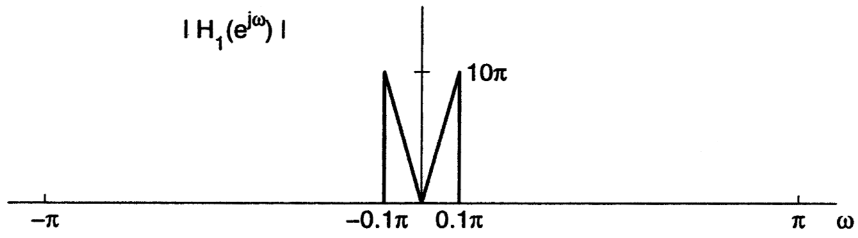
$$H(e^{j\omega})G(e^{j\omega}) = 1$$

7.24. We start with $|H_c(j\Omega)|$,



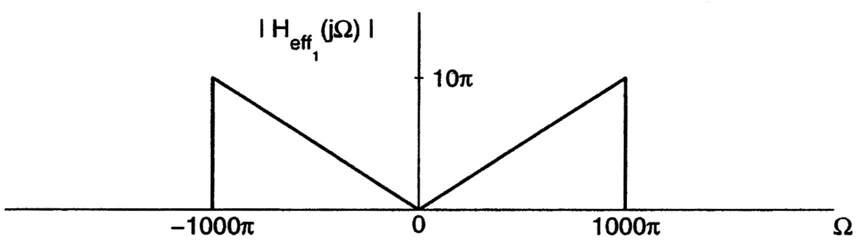
(a) By impulse invariance we scale the frequency axis by T_d to get

$$|H_1(e^{j\omega})| = \left| \sum_{k=-\infty}^{\infty} H_c\left(j\frac{\omega}{T_d} + j\frac{2\pi k}{T_d}\right) \right|$$



Then, to get the overall system response we scale the frequency axis by T and bandlimit the result according to the equation

$$|H_{\text{eff}_1}(j\Omega)| = \begin{cases} |H_1(e^{j\omega T})|, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$

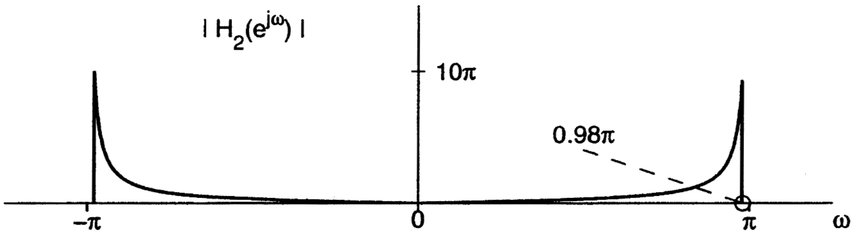


(b) Using the frequency mapping relationships of the bilinear transform,

$$\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right),$$
$$\omega = 2\tan^{-1}\left(\frac{\Omega T_d}{2}\right),$$

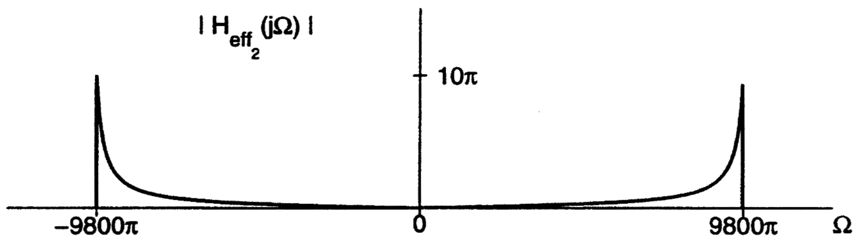
we get

$$|H_2(e^{j\omega})| = \begin{cases} |\tan\left(\frac{\omega}{2}\right)|, & |\omega| < 2\tan^{-1}(10\pi) = 0.98\pi \\ 0, & \text{otherwise} \end{cases}$$



Then, to get the overall system response we scale the frequency axis by T and bandlimit the result according to the equation

$$|H_{\text{eff}_2}(j\Omega)| = \begin{cases} |H_2(e^{j\omega T})|, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$



7.25. (a) By using Parseval's theorem,

$$\begin{aligned} \epsilon^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |E(e^{j\omega})|^2 d\omega \\ &= \sum_{n=-\infty}^{\infty} |e[n]|^2 \end{aligned}$$

where

$$e[n] = \begin{cases} h_d[n], & n < 0, \\ h_d[n] - h[n], & 0 \le n \le M, \\ h_d[n], & n > M \end{cases}$$

(b) Since we only have control over $e[n]$ for $0 \le m \le M$, we get that ϵ^2 is minimized if $h[n] = h_d[n]$ for $0 \le n \le M$.

(c)

$$w[n] = \begin{cases} 1, & 0 \le n \le M, \\ 0, & \text{otherwise.} \end{cases}$$

which is a rectangular window.

7.26. (a) **Answer:** Only the bilinear transform design will guarantee that a minimum phase discrete-time filter is created from a minimum phase continuous-time filter. For the following explanations remember that a discrete-time minimum phase system has all its poles and zeros inside the unit circle.

Impulse Invariance: Impulse invariance maps left-half s-plane poles to the interior of the z-plane unit circle. However, left-half s-plane zeros will *not necessarily* be mapped inside the z-plane unit circle. Consider:

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} = \frac{\sum_{k=1}^{N} A_k \prod_{\substack{j=1\\ j\neq k}}^{N} (s - s_j)}{\prod_{\ell=1}^{N} (s - s_\ell)}$$

$$H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}} = \frac{\sum_{k=1}^{N} T_d A_k \prod_{\substack{j=1\\ j\neq k}}^{N} (1 - e^{s_j T_d} z^{-1})}{\prod_{\ell=1}^{N} (1 - e^{s_\ell T_d} z^{-1})}$$

If we define $\text{Poly}_k(z) = \prod_{\substack{j=1\\ j\neq k}}^{N} (1 - e^{s_j T_d} z^{-1})$, we can note that all the roots of $\text{Poly}_k(z)$ are inside the unit circle. Since the numerator of $H(z)$ is a sum of $A_k \text{Poly}_k(z)$ terms, we see that there are *no guarantees* that the roots of the numerator polynomial are inside the unit circle. In other words, the sum of minimum phase filters is not necessarily minimum phase. By considering the specific example of

$$H_c(s) = \frac{s + 10}{(s + 1)(s + 2)},$$

and using $T = 1$, we can show that a minimum phase filter is transformed into a non-minimum phase discrete time filter.

Bilinear Transform: The bilinear transform maps a pole or zero at $s = s_0$ to a pole or zero (respectively) at $z_0 = \frac{1+\frac{T}{2}s_0}{1-\frac{T}{2}s_0}$. Thus,

$$|z_0| = \left| \frac{1 + \frac{T}{2}s_0}{1 - \frac{T}{2}s_0} \right|$$

Since $H_c(s)$ is minimum phase, all the poles of $H_c(s)$ are located in the left half of the s-plane. Therefore, a pole $s_0 = \sigma + j\Omega$ must have $\sigma < 0$. Using the relation for s_0, we get

$$\begin{aligned} |z_0| &= \sqrt{\frac{(1 + \frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}{(1 - \frac{T}{2}\sigma)^2 + (\frac{T}{2}\Omega)^2}} \\ &< 1 \end{aligned}$$

Thus, all poles and zeros will be inside the z-plane unit circle and the discrete-time filter will be minimum phase as well.

(b) **Answer:** Only the bilinear transform design will result in an allpass filter.

Impulse Invariance: In the impulse invariance design we have

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)$$

The aliasing terms can destroy the allpass nature of the continuous-time filter.

Bilinear Transform: The bilinear transform only warps the frequency axis. The magnitude response is not affected. Therefore, an allpass filter will map to an allpass filter.

(c) **Answer:** Only the bilinear transform will guarantee

$$H(e^{j\omega})|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

Impulse Invariance: Since impulse invariance may result in aliasing, we see that

$$H(e^{j0}) = H_c(j0)$$

if and only if

$$H(e^{j0}) = \sum_{k=-\infty}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = H_c(j0)$$

or equivalently

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = 0$$

which is generally not the case.

Bilinear Transform: Since, under the bilinear transformation, $\Omega = 0$ maps to $\omega = 0$,

$$H(e^{j0}) = H_c(j0)$$

for all $H_c(s)$.

(d) **Answer:** Only the bilinear transform design is guaranteed to create a bandstop filter from a bandstop filter.
If $H_c(s)$ is a bandstop filter, the bilinear transform will preserve this because it just warps the frequency axis; however aliasing (in the impulse invariance technique) can fill in the stop band.

(e) **Answer:** The property holds under the bilinear transform, but not under impulse invariance.

Impulse Invariance: Impulse invariance may result in aliasing. Since the order of aliasing and multiplication are not interchangeable, the desired identity does not hold. Consider $H_{a_1}(s) = H_{a_2}(s) = e^{-sT/2}$.

Bilinear Transform: By the bilinear transform,

$$\begin{aligned} H(z) &= H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &\equiv H_{c_1}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) H_{c_2}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_1(z)H_2(z) \end{aligned}$$

(f) **Answer:** The property holds for both impulse invariance and the bilinear transform.

Impulse Invariance:

$$\begin{aligned} H(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) \\ &= \sum_{k=-\infty}^{\infty} H_{c1}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) + \sum_{k=-\infty}^{\infty} H_{c2}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi}{T_d}k\right)\right) \\ &= H_1(e^{j\omega}) + H_2(e^{j\omega}) \end{aligned}$$

Bilinear Transform:

$$\begin{aligned} H(z) &= H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_{c_1}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) + H_{c_2}\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right) \\ &= H_1(z) + H_2(z) \end{aligned}$$

(g) **Answer:** Only the bilinear transform will result in the desired relationship.
Impulse Invariance: By impulse invariance,

$$\begin{aligned} H_1\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} H_{c_1}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right) \\ H_2\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} H_{c_2}\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right) \end{aligned}$$

We can clearly see that due to the aliasing, the phase relationship is not guaranteed to be maintained.

Bilinear Transform: By the bilinear transform,

$$\begin{aligned} H_1(e^{j\omega}) &= H_{c_1}\left(j\frac{2}{T_d}\tan(\omega/2)\right) \\ H_2(e^{j\omega}) &= H_{c_2}\left(j\frac{2}{T_d}\tan(\omega/2)\right) \end{aligned}$$

therefore,

$$\frac{H_1(e^{j\omega})}{H_2(e^{j\omega})} = \frac{H_{c_1}\left(j\frac{2}{T_d}\tan(\omega/2)\right)}{H_{c_2}\left(j\frac{2}{T_d}\tan(\omega/2)\right)} = \begin{cases} e^{-j\pi/2}, & 0 < \omega < \pi \\ e^{j\pi/2}, & -\pi < \omega < 0 \end{cases}$$

7.27. (a) Since

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)$$

and we desire

$$H(e^{j\omega})\,|_{\omega=0} = H_c(j\Omega)\,|_{\Omega=0},$$

we see that

$$H(e^{j\omega})|_{\omega=0} = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} + \frac{2\pi k}{T_d}\right)\right)|_{\omega=0} = H_c(j\Omega)|_{\Omega=0}$$

requires

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} H_c\left(j\frac{2\pi k}{T_d}\right) = 0.$$

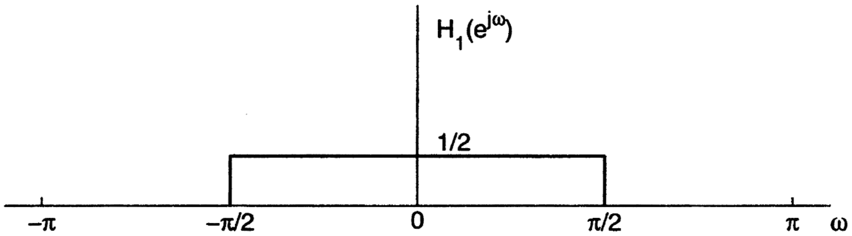
(b) Since the bilinear transform maps $\Omega = 0$ to $\omega = 0$, the condition will hold for any choice of $H_c(j\Omega)$.

7.28.

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| \le \pi \end{cases}$$

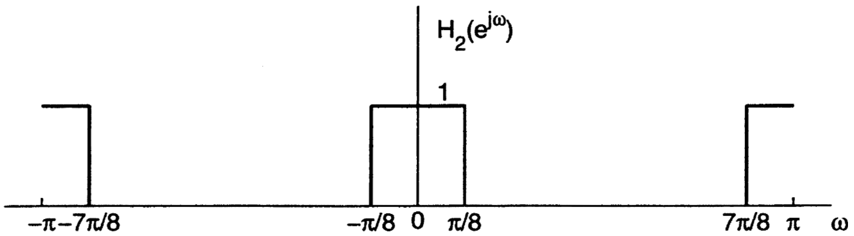
(a)

$$\begin{aligned} h_1[n] &= h[2n] \\ H_1(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[2n]e^{j\omega n} \\ &= \sum_{n \text{ even}} h[n]e^{\frac{j\omega n}{2}} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[h[n] + (-1)^n h[n]\right] e^{j\frac{\omega n}{2}} \\ &= \frac{1}{2}H(e^{j\frac{\omega}{2}}) + \frac{1}{2}H\left(e^{j\frac{\omega+2\pi}{2}}\right) \end{aligned}$$



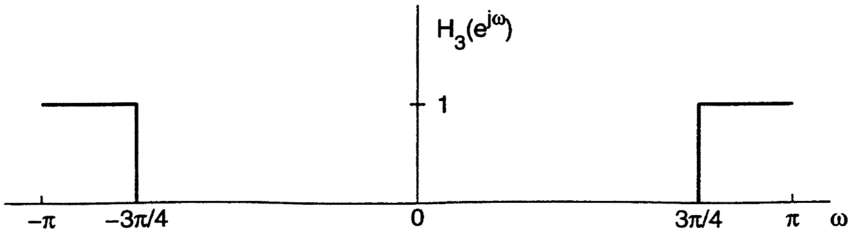
(b)

$$\begin{aligned} H_2(e^{j\omega}) &= \sum_{n \text{ even}} h[n/2]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega 2n} \\ &= H\left(e^{j2\omega}\right) \end{aligned}$$



(c)

$$H_3(e^{j\omega}) = H\left(e^{j(\omega+\pi)}\right)$$



7.29. (a) We have

$$\begin{aligned} s &= \frac{1-z^{-1}}{1+z^{-1}} \\ j\Omega &= \frac{1-e^{-j\omega}}{1+e^{-j\omega}} \\ &= \frac{e^{j\omega/2}-e^{-j\omega/2}}{e^{j\omega/2}+e^{-j\omega/2}} \\ \Omega &= \tan\left(\frac{\omega}{2}\right) \end{aligned}$$

$$\Omega_p = \tan\left(\frac{\omega_{p_1}}{2}\right) \longleftrightarrow \omega_{p_1} = 2\tan^{-1}(\Omega_p)$$

(b)

$$\begin{aligned} s &= \frac{1+z^{-1}}{1-z^{-1}} \\ j\Omega &= \frac{1+e^{-j\omega}}{1-e^{-j\omega}} \\ &= \frac{e^{j\omega/2}+e^{-j\omega/2}}{e^{j\omega/2}-e^{-j\omega/2}} \\ \Omega &= -\cot\left(\frac{\omega}{2}\right) \\ &= \tan\left(\frac{\omega-\pi}{2}\right) \end{aligned}$$

$$\Omega_p = \tan\left(\frac{\omega_{p_2}-\pi}{2}\right) \longleftrightarrow \omega_{p_2} = \pi + 2\tan^{-1}(\Omega_p)$$

(c)

$$\tan\left(\frac{\omega_{p_2}-\pi}{2}\right) = \tan\left(\frac{\omega_{p_1}}{2}\right)$$

$$\Rightarrow \omega_{p_2} = \omega_{p_1} + \pi$$

(d)

$$H_2(z) = H_1(z)|_{z=-z}$$

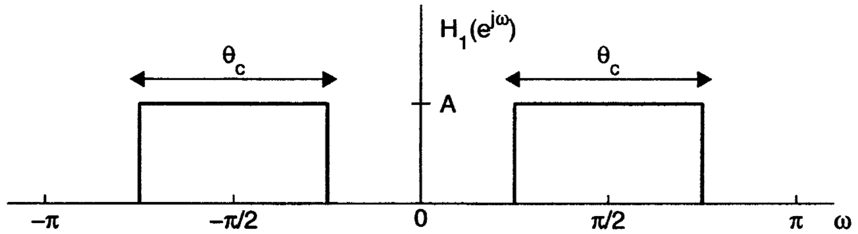
The even powers of z do not get changed by this transformation, while the coefficients of the odd powers of z change sign.
Thus, replace $A, C, 2$ with $-A, -C, -2$.

7.30. (a) Substituting $Z = e^{j\theta}$ and $z = e^{j\omega}$ we get,

$$\begin{aligned} e^{j\theta} &= -e^{j2\omega} \\ &= e^{j(2\omega+\pi)} \end{aligned}$$

$$\theta = 2\omega + \pi \quad \longleftrightarrow \quad \omega = \frac{\theta - \pi}{2}$$

(b)



(c)

$$\begin{aligned} h[n] &\longleftrightarrow H(e^{j\theta}) \\ h_1[n] &\longleftrightarrow H\left(e^{j(2\omega+\pi)}\right) \end{aligned}$$

In the frequency domain, we first shift by π and then we upsample by 2. In the time domain, we can write that as

$$h_1[n] = \begin{cases} (-1)^{n/2}h[n/2], & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

(d) In general, a filter

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{M-1}z^{M-1} + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_{N-1}z^{N-1} + a_N z^{-N}}$$

will transform under $H_1(z) = H(-z^2)$ to

$$H_1(z) = \frac{b_0 - b_1 z^{-2} + b_2 z^{-4} + \cdots - b_{M-1}z^{2M-2} + b_M z^{-2M}}{a_0 - a_1 z^{-2} + a_2 z^{-4} + \cdots - a_{M-1}z^{2N-2} + a_N z^{-2N}}$$

where we are assuming here that M and N are even. All the delay terms increase by a factor of two, and the sign of the coefficient in front of any odd delay term is negated.
The given difference equations therefore become

$$\begin{aligned} g[n] &= x[n] + a_1 g[n-2] - b_1 f[n-4] \\ f[n] &= -a_2 g[n-2] - b_2 f[n-2] \\ y[n] &= c_1 f[n] + c_2 g[n-2] \end{aligned}$$

To avoid any possible confusion please note that the b_k and a_k in these difference equations are not the same b_k and a_k shown above for the general case.

7.31. We are given

$$H(z) = H_c(s)\left.\right|_{s=\beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right]}$$

where α is a nonzero integer and β is a real number.

(a) *It is true for* $\beta > 0$.
Proof:

$$\begin{aligned}
s &= \beta\left[\frac{1-z^{-\alpha}}{1+z^{-\alpha}}\right] \\
s + sz^{-\alpha} &= \beta - \beta z^{-\alpha} \\
s - \beta &= -\beta z^{-\alpha} - sz^{-\alpha} \\
\beta - s &= z^{-\alpha}(\beta + s) \\
z^{-\alpha} &= \frac{\beta - s}{\beta + s} \\
z^{\alpha} &= \frac{\beta + s}{\beta - s}
\end{aligned}$$

The poles s_k of a stable, causal, continuous-time filter satisfy the condition $\mathcal{R}e\{s\} < 0$. We want these poles to map to the points z_k in the z-plane such that $|z_k| < 1$. With $\alpha > 0$ it is also true that if $|z_k| < 1$ then $|z_k^{\alpha}| < 1$. Letting $s_k = \sigma + j\omega$ we see that

$$\begin{aligned}
|z_k| &< 1 \\
|z_k^{\alpha}| &< 1 \\
|\beta + \sigma + j\Omega| &< |\beta - \sigma - j\Omega| \\
(\beta + \sigma)^2 + \Omega^2 &< (\beta - \sigma)^2 + \Omega^2 \\
2\sigma\beta &< -2\sigma\beta
\end{aligned}$$

But since the continuous-time filter is stable we have $\mathcal{R}e\{s_k\} < 0$ or $\sigma < 0$. That leads to

$$-\beta < \beta$$

This can only be true if $\beta > 0$.

(b) *It is true for* $\beta < 0$. The proof is similar to the last proof except now we have $|z^{\alpha}| > 1$.

(c) We have

$$\begin{aligned}
z^2 &= \left.\frac{1+s}{1-s}\right|_{s=j\Omega} \\
|z^2| &= 1 \\
|z| &= 1
\end{aligned}$$

Hence, the $j\Omega$ axis of the s-plane is mapped to the unit circle of z-plane.

(d) First, find the mapping between Ω and ω.

$$\begin{aligned}
j\Omega &= \frac{1 - e^{-j2\omega}}{1 + e^{-j2\omega}} \\
&= \frac{e^{j\omega} - e^{-j\omega}}{e^{j\omega} + e^{-j\omega}} \\
\Omega &= \tan(\omega) \\
\omega &= \tan^{-1}(\Omega)
\end{aligned}$$

Therefore,

$$1 - \delta_1 \le |H(e^{j\omega})| \le 1 + \delta_1, \qquad \left\{|\omega| \le \frac{\pi}{4}\right\} \cup \left\{\frac{3\pi}{4} < |\omega| < \pi\right\}$$

Note that the highpass region $3\pi/4 \le |w| \le \pi$ is included because $\tan(\omega)$ is periodic with period π.

7.32. (a)

$$s = \frac{1+z^{-1}}{1-z^{-1}} \longleftrightarrow z = \frac{s+1}{s-1}$$

Now, we evaluate the above expressions along the $j\Omega$ axis of the s-plane

$$\begin{aligned} z &= \frac{j\Omega+1}{j\Omega-1} \\ |z| &= 1 \end{aligned}$$

(b) We want to show $|z| < 1$ if $\mathcal{R}e\{s\} < 0$.

$$\begin{aligned} z &= \frac{\sigma + j\Omega + 1}{\sigma + j\Omega - 1} \\ |z| &= \frac{\sqrt{(\sigma+1)^2 + \Omega^2}}{\sqrt{(\sigma-1)^2 + \Omega^2}} \end{aligned}$$

Therefore, if $|z| < 1$

$$\begin{aligned} (\sigma+1)^2 + \Omega^2 &< (\sigma-1)^2 + \Omega^2 \\ \sigma &< -\sigma \end{aligned}$$

it must also be true that $\sigma < 0$. We have just shown that the left-half s-plane maps to the interior of the z-plane unit circle. Thus, any pole of $H_c(s)$ inside the left-half s-plane will get mapped to a pole inside the z-plane unit circle.

(c) We have the relationship

$$\begin{aligned} j\Omega &= \frac{1+e^{-j\omega}}{1-e^{-j\omega}} \\ &= \frac{e^{j\omega/2} + e^{-j\omega/2}}{e^{j\omega/2} - e^{-j\omega/2}} \\ \Omega &= -\cot(\omega/2) \end{aligned}$$

$$\begin{aligned} |\Omega_s| &= |\cot(\pi/6)| = \sqrt{3} \\ |\Omega_{p_1}| &= |\cot(\pi/2)| = 0 \\ |\Omega_{p_2}| &= |\cot(\pi/4)| = 1 \end{aligned}$$

Therefore, the constraints are

$$\begin{aligned} 0.95 \le |H_c(j\Omega)| \le 1.05, &\quad 0 \le |\Omega| \le 1 \\ |H_c(j\Omega)| \le 0.01, &\quad \sqrt{3} \le |\Omega| \end{aligned}$$

7.33

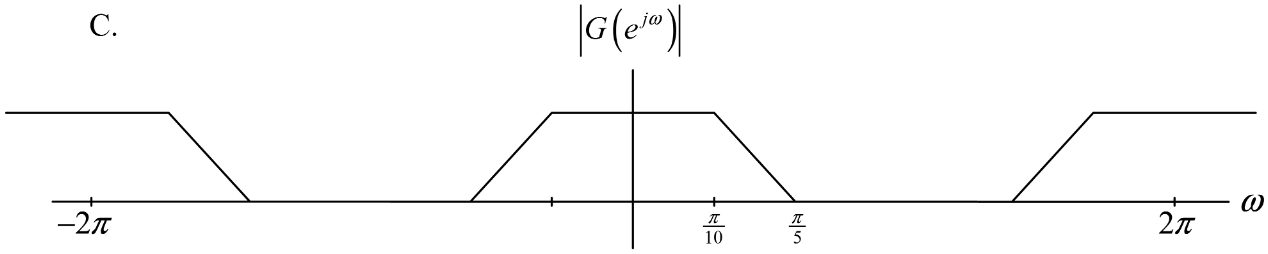
A. Given $\delta_s\big|_{dB} = -60$ dB, we have $A = 60$. Then $\beta = \frac{A-8}{8} = \frac{60-8}{8} = 6.5$ and

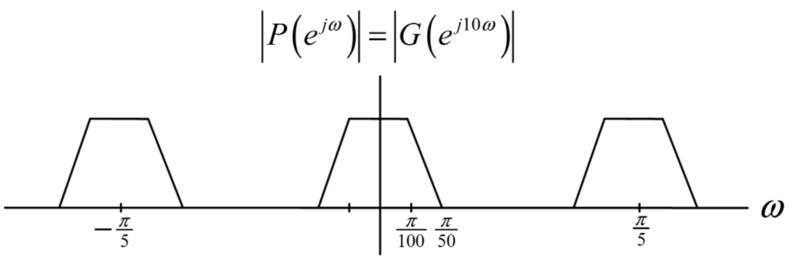
$$M = \frac{(60-8)\pi}{7\left(\frac{\pi}{50} - \frac{\pi}{100}\right)} = 743.$$

B. For filter $g[n]$ we require $\omega_p' = \frac{\pi}{10}$, $\omega_s' = \frac{\pi}{5}$, and $\delta_s'\big|_{dB} = \delta_s\big|_{dB} = -60$ dB. Then we have

$A' = 60$, $\beta' = 6.5$, and $M' = \frac{(60-8)\pi}{7\left(\frac{\pi}{5} - \frac{\pi}{10}\right)} = 74.3 \cong 74$.

C.





D. The filter $q[n]$ will be cascaded with $p[n]$. To ensure that the original specifications are met, we require that $q[n]$ satisfy $\omega_p'' = \frac{\pi}{100}$, $\omega_s'' = \frac{\pi}{5} - \frac{\pi}{50} = 0.18\pi$, and $\delta_s''\big|_{dB} = -60$ dB.

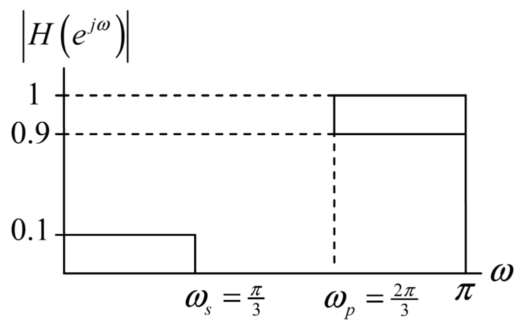
E. For the filter $q[n]$ we have $A'' = 60$, $\beta'' = 6.5$, and $M'' = \dfrac{(60-8)\pi}{7(0.18\pi - 0.01\pi)} = 43.7 \cong 44$.

Now if $g[n]$ has 75 samples, then $p[n]$ will have 741 samples. The convolution, $h'[n] = q[n] * p[n]$, will therefore have 785 samples.

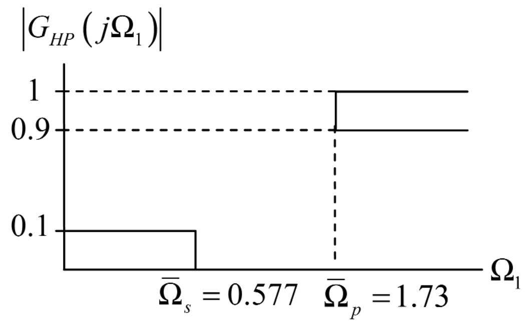
F. Convolving the input with $q[n]$ requires 45 multiplications per output sample. Then convolving with $p[n]$ requires 75 multiplications per output sample (not counting multiplication by zero). The total for this approach is 120 multiplications per output sample. For the original filter $h[n]$, 744 multiplications per output sample were required.

7.34

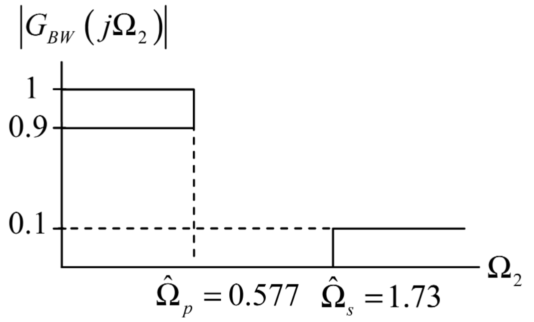
A. We find the passband and stopband edges of the discrete-time filter by using the transformation $\omega = \Omega T$, where $\frac{1}{T} = 24,000$. The specifications for the discrete-time filter are shown below.



B. The mapping between ω and Ω_1 is $\Omega_1 = \tan(\omega/2)$. Then $\bar{\Omega}_p = \tan(2\pi/6) = 1.73$ and $\bar{\Omega}_s = \tan(\pi/6) = 0.577$. The specifications for $|G_{HP}(j\Omega_1)|$ are shown below.



C. The mapping between Ω_2 and Ω_1 is $\Omega_2 = 1/\Omega_1$. Then $\hat{\Omega}_p = 1/1.73 = 0.577$ and $\hat{\Omega}_s = 1/0.577 = 1.73$. The specifications for $\left|G_{BW}\left(j\Omega_2\right)\right|$ are shown below.



One way of writing the frequency response of a Butterworth filter is

$$\left|G_{BW}\left(j\Omega_2\right)\right|^2 = \frac{1}{1+\varepsilon^2\left(\dfrac{\Omega_2}{\hat{\Omega}_p}\right)^{2N}}.$$

At $\Omega_2 = \hat{\Omega}_p$ we have

$$\frac{1}{1+\varepsilon^2} = 0.9^2 = 0.81$$

so $\varepsilon^2 = 0.235$.

At $\Omega_2 = \hat{\Omega}_s$ we have

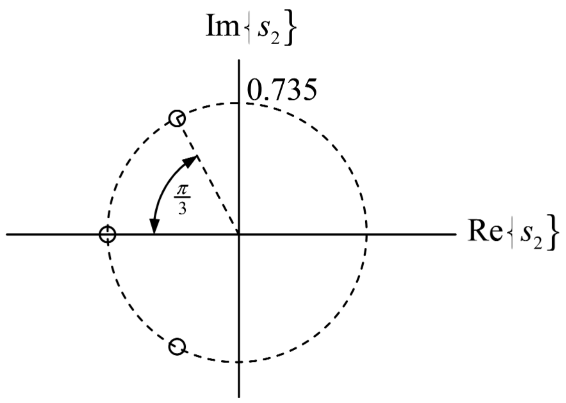
$$\frac{1}{1+0.235\left(\dfrac{1.73}{0.577}\right)^{2N}} = 0.1^2 = 0.01.$$

Solving gives $N = 2.75$. Since the filter order must be an integer we round up to $N = 3$.
We now have

$$\begin{aligned}\left|G_{BW}\left(j\Omega_2\right)\right|^2 &= \frac{1}{1+0.235\left(\dfrac{\Omega_2}{0.577}\right)^6} \\ &= \frac{1}{1+\left(0.785\dfrac{\Omega_2}{0.577}\right)^6} \\ &= \frac{1}{1+\left(\dfrac{\Omega_2}{0.735}\right)^6}.\end{aligned}$$

We therefore have $\Omega_c = 0.735$ and $N = 3$.

D. An analog Butterworth filter has poles on a circle in the s-plane. We have



$$G_{BW}(s_2) = \frac{(0.735)^3}{(s+0.735)\left(s-0.735e^{j\frac{2\pi}{3}}\right)\left(s-0.735e^{-j\frac{2\pi}{3}}\right)}$$

$$= \frac{0.397}{s^3+1.47s^2+1.08s+0.397}.$$

7.35

A. From the figure, $H\left(e^{j\omega}\right)$ exhibits eight alternations of the error on the interval $0 \le \omega \le \pi$ as an approximation to an ideal lowpass filter with the given parameters. Because a lowpass filter designed with the Parks-McClellan algorithm has either $L+2$ or $L+3$ alternations and because we are told that there is another filter out there that meets the specs with $N_2 > N_1$, we should consider the $L+3$ case to find the smaller value of N.

With $L+3=8$ alternations, $L=5$. Then, since $(N_1-1)/2=5$, we have $N_1=11$ as the only possible value.

B. Since there are 8 alternations, L can be no greater than 6. Therefore $(N_2-1)/2 \le 6$, which implies $N_2 \le 13$. Since the only other possible value of N for a lowpass filter was found in A, we have $N_2=13$ as the only possible value.

C. Yes. Since both filters have identical frequency responses, they must have identical impulse responses.

D. While the alternation theorem states that *for a given* r there is a unique r th order polynomial that satisfies it, the theorem makes no claim about how this polynomial may or may not relate to a polynomial satisfying the alternation theorem for a different value of r.

It turns out that in this case, the single 5$^{\text{th}}$ order polynomial satisfying the alternation theorem for $r_1 = L_1 = 5$ is identical to the single 6$^{\text{th}}$ order polynomial satisfying the alternation theorem for $r_2 = L_2 = 6$.

7.36

(a) The Parks-McClellan algorithm minimizes

$$\max_{\omega:|\omega|\in[0,\omega_1]\cup[\omega_2,\omega_3]\cup[\omega_4,\pi]} \left|E\left(e^{j\omega}\right)W\left(e^{j\omega}\right)\right|,$$

where $E\left(e^{j\omega}\right)=\left[H\left(e^{j\omega}\right)-H_d\left(e^{j\omega}\right)\right]$ and $W\left(e^{j\omega}\right)$ is the weighting function. To ensure that the resulting filter meets the criteria, we need to choose $W\left(e^{j\omega}\right)$ such that

$\delta_1 W\left(e^{j\omega}\right)\Big|_{0\le\omega\le\omega_1}=\text{constant}$, $\delta_2 W\left(e^{j\omega}\right)\Big|_{\omega_2\le\omega\le\omega_3}=\text{constant}$, and $\delta_3 W\left(e^{j\omega}\right)\Big|_{\omega_4\le\omega\le\pi}=\text{constant}$.

Letting the constant equal 1,

$$W\left(e^{j\omega}\right)=\begin{cases}1/\delta_1, & 0\le|\omega|\le\omega_1\\ 1/\delta_2, & \omega_2\le|\omega|\le\omega_3\\ 1/\delta_3, & \omega_4\le|\omega|\le\pi.\end{cases}$$

Note that $W\left(e^{j\omega}\right)$ is undefined outside these bands, since the transitions are ignored.

(b) As shown in the graph, the filter $A\left(e^{j\omega}\right)$ has eight alternations. Since there can be at most $L-1$ alternations inside the bands, $L-1\ge 4$ interior alternations implies $L\ge 5$. Also, at least $L+2$ alternations are needed to satisfy the alternation theorem, so $L+2\le 8$ implies $L\le 6$, so $L=5$ or 6. Since the filter is Type I, $N=2L+1$, which implies $N=11$ or $N=13$. Thus the filter has at most 13 nonzero values in its impulse response.

(c) We have $A\left(e^{j\omega}\right)=\sum_{n=0}^{N-1}a_n e^{-j\omega n}$, which implies $B\left(e^{j\omega}\right)=k_1\left(\sum_{n=0}^{N-1}a_n e^{-j\omega n}\right)^2+k_2=\sum_{m=0}^{2(N-1)}b_m e^{-j\omega m}$

(squaring a polynomial doubles the order). Thus $L_B=\dfrac{2(N-1)+1-1}{2}=2L_A$, which is twice the order of the Chebyshev polynomial for $A\left(e^{j\omega}\right)$. Since $L_A\ge 5$ we have $L_B\ge 10$, and $B\left(e^{j\omega}\right)$ needs at least $L_B+2=12$ alternations to satisfy the alternation theorem. The figure, however, shows that $B\left(e^{j\omega}\right)$ has only eleven alternations, so it does not satisfy the alternation theorem.

7.37. (a) Expanding the sum to see things more clearly, we get

$$\begin{aligned} H_c(s) &= \sum_{k=1}^{r} \frac{A_k}{(s-s_0)^k} + G_c(s) \\ &= \frac{A_1}{s-s_0} + \frac{A_2}{(s-s_0)^2} + \cdots + \frac{A_r}{(s-s_0)^r} + G_c(s) \end{aligned}$$

Now multiplying both sides by $(s-s_0)^r$ we get

$$(s-s_0)^r H_c(s) = A_1(s-s_0)^{r-1} + A_2(s-s_0)^{r-2} + \cdots + A_r + (s-s_0)^r G_c(s)$$

Evaluating both sides of the equal sign at $s = s_0$ gives us

$$A_r = (s-s_0)^r H_c(s)\,|_{s=s_0}$$

Note that $(s-s_0)^r G_c(s) = 0$ when $s = s_0$ because $G_c(s)$ has at most one pole at $s = s_0$.

Similarly, by taking the first derivative and evaluating at $s = s_0$ we get

$$\begin{aligned} \frac{d}{ds}[(s-s_0)^r H_c(s)] &= \sum_{k=1}^{r}(r-k)A_k(s-s_0)^{(r-k-1)} + \frac{d}{ds}[(s-s_0)^r G_c(s)] \\ &= (r-1)A_1(s-s_0)^{r-2} + (r-2)A_2(s-s_0)^{r-3} + \cdots + A_{r-1} + 0 + \frac{d}{ds}[(s-s_0)^r G_c(s)] \\ A_{r-1} &= \frac{d}{ds}[(s-s_0)^r H_c(s)]\,|_{s=s_0} \end{aligned}$$

This idea can be continued. By taking the $(r-k)$-th derivative and evaluating at $s = s_0$ we get the the general form

$$A_k = \frac{1}{(r-k)!}\left(\frac{d^{r-k}}{ds^{r-k}}[(s-s_0)^r H_c(s)]\,|_{s=s_0}\right)$$

(b) Using the following transform pair from a lookup table,

$$\frac{t^{k-1}}{(k-1)!}e^{-\alpha t}u(t) \longrightarrow \frac{1}{(s+\alpha)^k}, \quad \mathcal{R}e\{s\} > -\alpha$$

we get

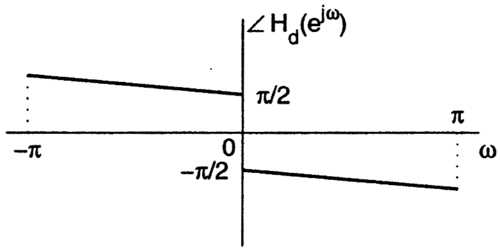
$$\begin{aligned} h_c(t) &= \mathcal{L}^{-1}\{H_c(s)\} \\ &= \mathcal{L}^{-1}\left\{\sum_{k=1}^{r}\frac{A_k}{(s-s_0)^k} + G_c(s)\right\} \\ &= \sum_{k=1}^{r} A_k \frac{t^{k-1}}{(k-1)!}e^{s_0 t}u(t) + g_c(t) \end{aligned}$$

7.38. (a)

$$H_d(e^{j\omega}) = [1 - 2u(\omega)]e^{j(\pi/2-\tau\omega)} \quad \text{for} -\pi < \omega < \pi$$

$$|H_d(e^{j\omega})| = 1, \quad \forall\omega$$

$$\angle H_d(e^{j\omega}) = \begin{cases} \frac{\pi}{2} - \tau\omega, & -\pi < \omega < 0 \\ -\frac{\pi}{2} - \tau\omega, & 0 < \omega < \pi \end{cases}$$



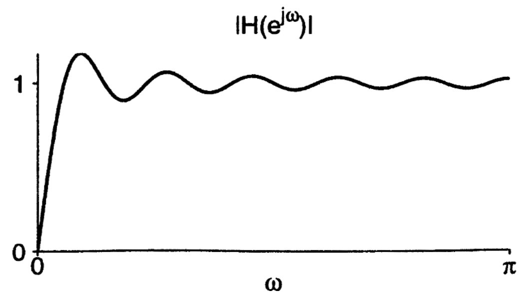
(b) A Hilbert transformer of this nature requires the filter to have a zero at $z = 0$ which introduces the 180^o phase difference at that point. A zero at $z = 0$ means that the sum of the filter coefficients equals zero. Thus, only Types III and IV fulfill the requirements.

(c)

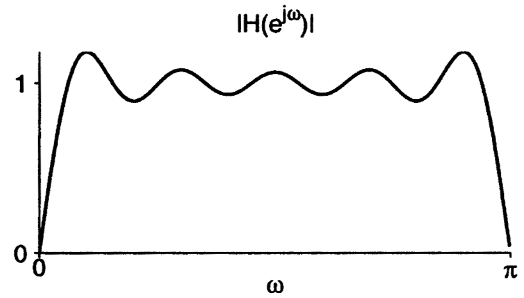
$$\begin{aligned} H_d(e^{j\omega}) &= [1 - 2u(\omega)]e^{j(\pi/2-\omega\tau)} \\ h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{0} e^{j(\pi/2-\omega\tau)}e^{j\omega n}d\omega - \frac{1}{2\pi}\int_{0}^{\pi} e^{j(\pi/2-\omega\tau)}e^{j\omega n}d\omega \\ &= \frac{e^{j\frac{\pi}{2}}}{2\pi}\int_{-\pi}^{0} e^{j\omega(n-\tau)}d\omega - \frac{e^{j\frac{\pi}{2}}}{2\pi}\int_{0}^{\pi} e^{j\omega(n-\tau)}d\omega \\ &= \begin{cases} \frac{1-\cos[\pi(n-\tau)]}{\pi(n-\tau)}, & n \neq \tau \\ 0, & n = \tau \end{cases} \\ &= \begin{cases} \frac{2}{\pi}\frac{\sin^2[\pi(n-\tau)/2]}{(n-\tau)}, & n \neq \tau \\ 0, & n = \tau \end{cases} \end{aligned}$$

For the windowed FIR system to be linear phase it must be antisymmetric about $\frac{M}{2}$. Since the ideal Hilbert transformer $h_d[n]$ is symmetric about $n = \tau$ we should choose $\tau = \frac{M}{2}$.

(d) The delay is $M/2 = 21/2 = 10.5$ samples. It is therefore a Type IV system. Notice the mandatory zero at $\omega = 0$.



(e) The delay is $M/2 = 20/2 = 10$ samples. It is therefore a Type III system. Notice the mandatory zeros at $\omega = 0$ and π.



7.39. (a) It is well known that convolving two rectangular windows results in a triangular window. Specifically, to get the $(M+1)$ point Bartlett window for M even, we can convolve the following rectangular windows.

$$\begin{aligned}
r_1[n] &= \begin{cases} \sqrt{\frac{2}{M}}, & n = 0, \ldots, \frac{M}{2} - 1 \\ 0, & \text{otherwise} \end{cases} \\
r_2[n] &= r_1[n-1]
\end{aligned}$$

Using the known transform of a rectangular window we have

$$\begin{aligned}
W_{R_1}(e^{j\omega}) &= \sqrt{\frac{2}{M}} \frac{\sin(\omega M/4)}{\sin(\omega/2)} e^{-j\omega\left(\frac{M}{4} - \frac{1}{2}\right)} \\
W_{R_2}(e^{j\omega}) &= \sqrt{\frac{2}{M}} \frac{\sin(\omega M/4)}{\sin(\omega/2)} e^{-j\omega\left(\frac{M}{4} + \frac{1}{2}\right)} \\
W_B(e^{j\omega}) &= W_{R_1}(e^{j\omega}) W_{R_2}(e^{j\omega}) \\
&= \frac{2}{M} \left(\frac{\sin(\omega M/4)}{\sin(\omega/2)} \right)^2 e^{-j\omega M/2}
\end{aligned}$$

Note: The Bartlett window as defined in the text is zero at $n = 0$ and $n = M$. These points are included in the $M + 1$ points.

For M odd, the Bartlett window is the convolution of

$$\begin{aligned}
r_3[n] &= \begin{cases} \sqrt{\frac{2}{M}}, & n = 0, \ldots, \frac{M-1}{2} \\ 0, & \text{otherwise} \end{cases} \\
r_4[n] &= \begin{cases} \sqrt{\frac{2}{M}}, & n = 1, \ldots, \frac{M-1}{2} \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

In the frequency domain we have

$$\begin{aligned}
W_{R_3}(e^{j\omega}) &= \sqrt{\frac{2}{M}} \frac{\sin(\omega(M+1)/4)}{\sin(\omega/2)} e^{-j\omega\left(\frac{M-1}{4}\right)} \\
W_{R_4}(e^{j\omega}) &= \sqrt{\frac{2}{M}} \frac{\sin(\omega(M-1)/4)}{\sin(\omega/2)} e^{-j\omega\left(\frac{M-3}{4} + 1\right)} \\
W_B(e^{j\omega}) &= W_{R_3}(e^{j\omega}) W_{R_4}(e^{j\omega}) \\
&= \frac{2}{M} \left(\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} \right) \left(\frac{\sin[\omega(M-1)/2]}{\sin(\omega/2)} \right) e^{-j\omega M/2}
\end{aligned}$$

(b)

$$w[n] = \left[A + B\cos\left(\frac{2\pi n}{M}\right) + C\cos\left(\frac{4\pi n}{M}\right)\right] w_R[n]$$

$$\begin{aligned} W(e^{j\omega}) &= \left\{2\pi A\delta(\omega) + B\pi\left[\delta\left(\omega + \frac{2\pi}{M}\right) + \delta\left(\omega - \frac{2\pi}{M}\right)\right] + C\pi\left[\delta\left(\omega + \frac{4\pi}{M}\right) + \delta\left(\omega - \frac{4\pi}{M}\right)\right]\right\} \\ &\quad \frac{\otimes}{2\pi}\left\{\frac{\sin(\omega(M+1)/2))}{\sin(\omega/2)} e^{-j\omega M/2}\right\} \end{aligned}$$

where $\otimes$ denotes periodic convolution.

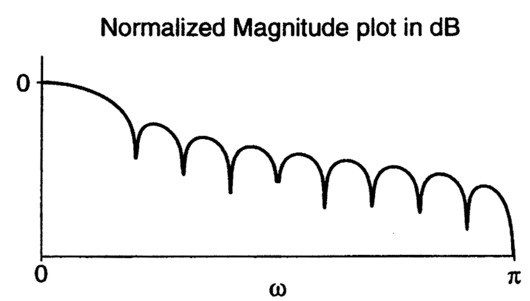
(c) For the Hanning window $A = 0.5$, $B = -0.5$, and $C = 0$.

$$\begin{aligned} w_{\text{Hanning}}[n] &= \left[0.5 - 0.5\cos\left(\frac{2\pi n}{M}\right)\right] w_r[n] \\ W_{\text{Hanning}}(e^{j\omega}) &= 0.5W_R(e^{j\omega}) - 0.25W_R(e^{j\omega}) \otimes \left[\delta\left(\omega + \frac{2\pi}{M}\right) + \delta\left(\omega - \frac{2\pi}{M}\right)\right] \\ &= 0.5W_R(e^{j\omega}) - 0.25\left[W_R(e^{j(\omega + \frac{2\pi}{M})}) + W_R(e^{j(\omega - \frac{2\pi}{M})})\right] \end{aligned}$$

where

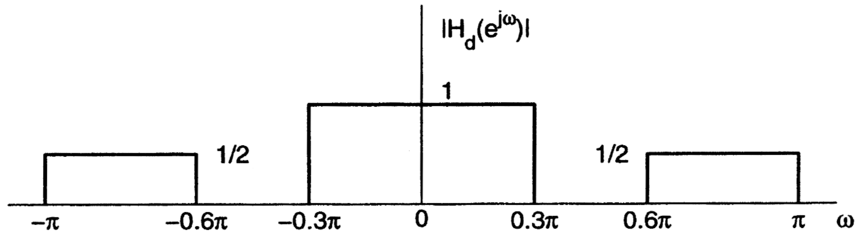
$$W_R(e^{j\omega}) = \frac{\sin(\omega(M+1)/2))}{\sin(\omega/2)} e^{-j\omega M/2}$$

Below is a normalized sketch of the magnitude response in dB.



7.40. (a) The delay is $\frac{M}{2} = 24$.

(b)



This can be viewed as the sum of two lowpass filters, one of which has been shifted in frequency (modulation in time-domain) to $\omega = \pi$. The linear phase factor adds a delay.

$$h_d[n] = \frac{\sin(0.3\pi(n-24))}{\pi(n-24)} + \frac{1}{2}(-1)^{(n-24)}\frac{\sin(0.4\pi(n-24))}{\pi(n-24)}$$

(c) To find the ripple values, which are all the same in this case since it is a Kaiser window design, we first need to determine A. Since we know β and A are related by

$$\beta = 3.68 = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0, & A < 21 \end{cases}$$

we can solve for A in the following manner:

1. We know $\beta = 3.68$. Therefore, from the formulas above, we see that $A \geq 21$.

2. If we assume $A > 50$ we find,

$$\begin{aligned} 3.68 &= 0.1102(A-8.7) \\ A &= 42.1 \end{aligned}$$

But, this contradicts our assumption that $A > 50$. Thus, $21 \le A \le 50$.

3. With $21 \le A \le 50$ we find,

$$\begin{aligned} 3.68 &= 0.5842(A-21)^{0.4} + 0.07886(A-21) \\ A &= 42.4256 \end{aligned}$$

With A, we can now calculate δ.

$$\begin{aligned} \delta &= 10^{-A/20} \\ &= 10^{-42.4256/20} \\ &= 0.0076 \end{aligned}$$

The discontinuity of 1 in the first passband creates a ripple of δ. The discontinuity of 1/2 in t second passband creates a ripple of $\delta/2$. The total ripple is $3\delta/2 = 0.0114$ and we therefore hav

$$\delta_1 = \delta_2 = \delta_3 = 0.0114$$

Now using the relationship between M, A, and $\Delta\omega$

$$\begin{aligned} M &= \frac{A-8}{2.285\Delta\omega} \\ \Delta\omega &= \frac{42.4256-8}{2.285(48)} = 0.3139 \approx 0.1\pi \end{aligned}$$

Putting it all together with the information about $H_d(e^{j\omega})$ we arrive at our final answer.

$$\begin{aligned} 0.9886 \le |H(e^{j\omega})| &\le 1.0114, & 0 \le \omega \le 0.25\pi \\ |H(e^{j\omega})| &\le 0.0114, & 0.35\pi \le \omega \le 0.55\pi \\ 0.4886 \le |H(e^{j\omega})| &\le 0.5114, & 0.65\pi \le \omega \le \pi \end{aligned}$$

With A, we can now calculate δ.

$$\begin{aligned} \delta &= 10^{-A/20} \\ &= 10^{-42.4256/20} \\ &= 0.0076 \end{aligned}$$

The discontinuity of 1 in the first passband creates a ripple of δ. The discontinuity of 1/2 in the second passband creates a ripple of $\delta/2$. The total ripple is $3\delta/2 = 0.0114$ and we therefore have

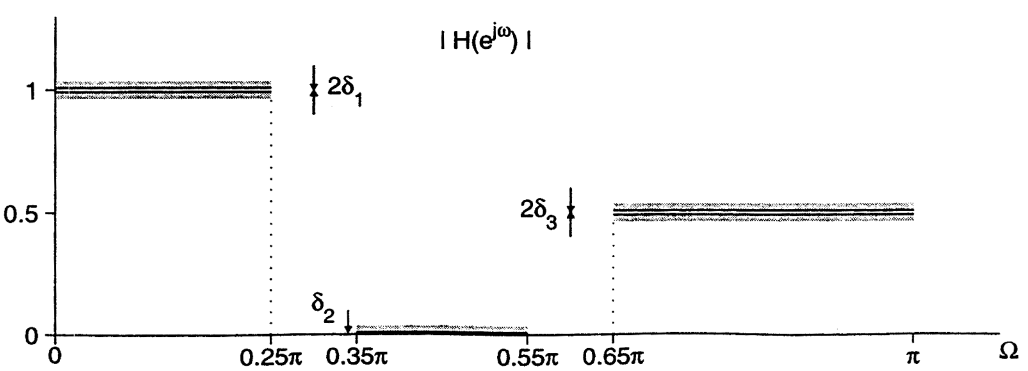
$$\delta_1 = \delta_2 = \delta_3 = 0.0114$$

Now using the relationship between M, A, and $\Delta\omega$

$$\begin{aligned} M &= \frac{A-8}{2.285\Delta\omega} \\ \Delta\omega &= \frac{42.4256-8}{2.285(48)} = 0.3139 \approx 0.1\pi \end{aligned}$$

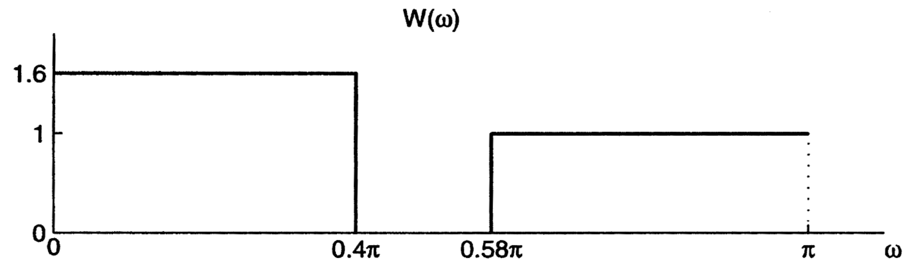
Putting it all together with the information about $H_d(e^{j\omega})$ we arrive at our final answer.

$$\begin{aligned} 0.9886 \le |H(e^{j\omega})| &\le 1.0114, & 0 \le \omega \le 0.25\pi \\ |H(e^{j\omega})| &\le 0.0114, & 0.35\pi \le \omega \le 0.55\pi \\ 0.4886 \le |H(e^{j\omega})| &\le 0.5114, & 0.65\pi \le \omega \le \pi \end{aligned}$$

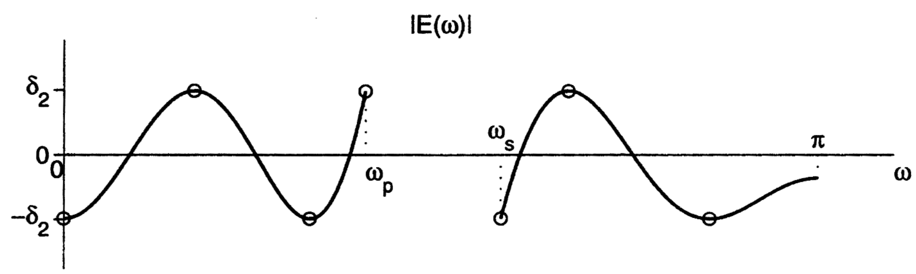


7.41. (a) Since $H(e^{j0}) \neq 0$ and $H(e^{j\pi}) \neq 0$, this must be a Type I filter.

(b) With the weighting in the stopband equal to 1, the weighting in the passband is $\frac{\delta_2}{\delta_1}$.



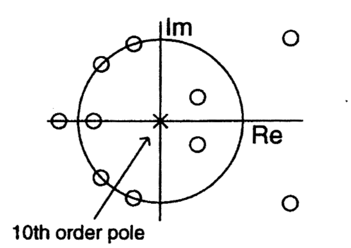
(c)



(d) An optimal (in the Parks-McClellan sense) Type I lowpass filter can have either $L+2$ or $L+3$ alternations. The second case is true only when an alternation occurs at all band edges. Since this filter does not have an alternation at $\omega = \pi$ it should only have $L+2$ alternations. From the figure, we see that there are 7 alternations so $L = 5$. Thus, the filter length is $2L + 1 = 11$ samples long.

(e) Since the filter is 11 samples long, it has a delay of 5 samples.

(f) Note the zeroes off the unit circle are implied by the dips in the frequency response at the indicated frequencies.



7.42. (a) The most straightforward way to find $h_d[n]$ is to recognize that $H_d(e^{j\omega})$ is simply the (periodic) convolution of two ideal lowpass filters with cutoff frequency $\omega_c = \pi/4$. That is,

$$H_d(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{lpf}(e^{j\theta})H_{lpf}(e^{j(\omega-\theta)})d\theta$$

where

$$H_{lpf}(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

Therefore, in the time domain, $h_d[n]$ is $(h_{lpf}[n])^2$, or

$$\begin{aligned} h_d[n] &= \left(\frac{\sin(\pi n/4)}{\pi n}\right)^2 \\ &= \frac{\sin^2(\pi n/4)}{\pi^2 n^2} \end{aligned}$$

(b) $h[n]$ must have even symmetry around $(N-1)/2$. $h[n]$ is a type-I FIR generalized linear phase system, since N is an odd integer, and $H(e^{j\omega}) \neq 0$ for $\omega = 0$. Type-I FIR generalized linear phase systems have even symmetry around $(N-1)/2$.

(c) Shifting the filter $h_d[n]$ by $(N-1)/2$ and applying a rectangular window will result in a causal $h[n]$ that minimizes the integral squared error ϵ. Consequently,

$$h[n] = \frac{\sin^2\left[\frac{\pi}{4}(n-\frac{N-1}{2})\right]}{\pi^2(n-\frac{N-1}{2})^2} w[n]$$

where

$$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

(d) The integral squared error ϵ

$$\epsilon = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|A(e^{j\omega}) - H_d(e^{j\omega})\right|^2 d\omega$$

can be reformulated, using Parseval's theorem, to

$$\epsilon = \sum_{-\infty}^{\infty} |a[n] - h_d[n]|^2$$

Since

$$a[n] = \begin{cases} h_d[n], & -\frac{N-1}{2} \le n \le \frac{N-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \epsilon &= \sum_{-\infty}^{-(N-1)/2-1} |a[n] - h_d[n]|^2 + \sum_{-(N-1)/2}^{(N-1)/2} |a[n] - h_d[n]|^2 + \sum_{(N-1)/2+1}^{\infty} |a[n] - h_d[n]|^2 \\ &= \sum_{-\infty}^{-(N-1)/2-1} |h_d[n]|^2 + 0 + \sum_{(N-1)/2+1}^{\infty} |h_d[n]|^2 \end{aligned}$$

By symmetry,

$$\epsilon = 2 \sum_{(N-1)/2+1}^{\infty} |h_d[n]|^2$$

7.43. (a) A Type-I lowpass filter that is optimal in the Parks-McClellan can have either $L+2$ or $L+3$ alternations. The second case is true only when an alternation occurs at all band edges. Since this filter does not have an alternation at $\omega = 0$ it only has $L+2$ alternations. From the figure we see there are 9 alternations so $L = 7$. Thus, $M = 2L = 2(7) = 14$.

(b) We have

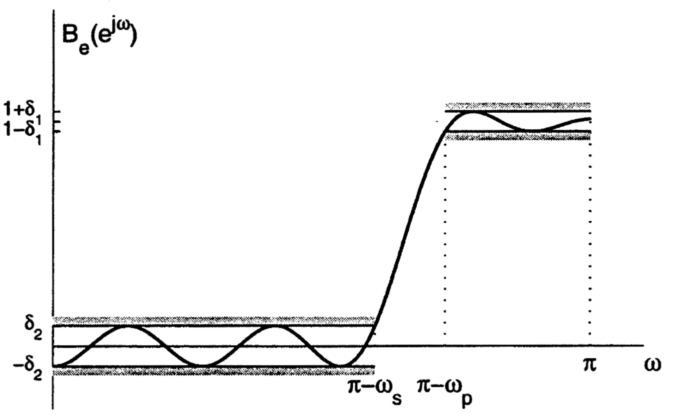
$$\begin{aligned} h_{HP}[n] &= -e^{j\pi n} h_{LP}[n] \\ H_{HP}(e^{j\omega}) &= -H_{LP}(e^{j(\omega-\pi)}) \\ &= -A_e(e^{j(\omega-\pi)})e^{-j(\omega-\pi)\frac{M}{2}} \\ &= A_e(e^{j(\omega-\pi)})e^{-j\omega\frac{M}{2}} \\ &= B_e(e^{j\omega})e^{-j\omega\frac{M}{2}} \end{aligned}$$

where

$$B_e(e^{j\omega}) = A_e(e^{j(\omega-\pi)})$$

The fact that $M = 14$ is used to simplify the exponential term in the third line above.

(c)



(d) *The assertion is correct.* The original amplitude function was optimal in the Parks-McClellan sense. The method used to create the new filter did not change the filter length, transition width, or relative ripple sizes. All it did was slide the frequency response along the frequency axis creating a new error function $E'(\omega) = E(\omega - \pi)$. Since translation does not change the Chebyshev error $(\max|E(\omega)|)$ the new filter is still optimal.

7.44. For this filter, $N = 3$, so the polynomial order L is

$$L = \frac{N-1}{2} = 1$$

Note that $h[n]$ must be a type-I FIR generalized linear phase filter, since it consists of three samples, and $H(e^{j\omega}) \neq 0$ for $\omega = 0$. $h[n]$ can therefore be written in the form

$$h[n] = a\delta[n] + b\delta[n-1] + a\delta[n-2]$$

Taking the DTFT of both sides gives

$$\begin{aligned} H(e^{j\omega}) &= a + be^{-j\omega} + ae^{-j2\omega} \\ &= e^{-j\omega}(ae^{j\omega} + b + ae^{-j\omega}) \\ &= e^{-j\omega}(b + 2a\cos w) \\ A(e^{j\omega}) &= b + 2a\cos w \end{aligned}$$

The filter must have at least $L+2 = 3$ alternations, but no more than $L+3 = 4$ alternations to satisfy the alternation theorem, and therefore be optimal in the minimax sense. Four alternations can be obtained if all four band edges are alternation frequencies such that the frequency response overshoots at $\omega = 0$, undershoots at $\omega = \frac{\pi}{3}$, overshoots at $\omega = \frac{\pi}{2}$, and undershoots at $\omega = \pi$.

Let the error in the passband and the stopband be δ_p and δ_s. Then,

$$\begin{aligned} A(e^{j\omega})\,|_{\omega=0} &= 1 + \delta_p \\ A(e^{j\omega})\,|_{\omega=\pi/3} &= 1 - \delta_p \\ A(e^{j\omega})\,|_{\omega=\pi/2} &= \delta_s \\ A(e^{j\omega})\,|_{\omega=\pi} &= -\delta_s \end{aligned}$$

Using $A(e^{j\omega}) = b + 2a\cos w$,

$$\begin{aligned} A(e^{j\omega})\,|_{\omega=0} &= b + 2a \\ A(e^{j\omega})\,|_{\omega=\pi/3} &= b + a \\ A(e^{j\omega})\,|_{\omega=\pi/2} &= b \\ A(e^{j\omega})\,|_{\omega=\pi} &= b - 2a \end{aligned}$$

Solving these systems of equations for a and b gives

$$\begin{aligned} a &= \frac{2}{5} \\ b &= \frac{2}{5} \end{aligned}$$

Thus, the optimal (in the minimax sense) causal 3-point lowpass filter with the desired passband and stopband edge frequencies is

$$h[n] = \frac{2}{5}\delta[n] + \frac{2}{5}\delta[n-1] + \frac{2}{5}\delta[n-2]$$

7.45. *True.* Since filter C is a stable IIR filter it has poles in the left half plane. The bilinear transform maps the left half plane to the inside of the unit circle. Thus, the discrete filter B has to have poles and is therefore an IIR filter.

7.46. *No.* The resulting discrete-time filter would not have a constant group delay. The bilinear transformation maps the entire $j\Omega$ axis in the s-plane to one revolution of the unit circle in the z-plane. Consequently, the linear phase of the continuous-time filter will get nonlinearly warped via the bilinar transform, resulting in a nonlinear phase for the discrete-time filter. Thus, the group delay of the discrete-time filter will not be a constant.

8.1. We sample a periodic continuous-time signal with a sampling rate:

$$F_s = \frac{\Omega s}{2\pi} = \frac{1}{T} = \frac{6}{10^{-3}}\text{Hz}$$

(a) The sampled signal is given by:

$$x[n] = x_c(nT)$$

Expressed as a Discrete Fourier Series:

$$x[n] = \sum_{k=-9}^{9} a_k e^{j\frac{2\pi}{6}kn}$$

We note that, in accordance with the discussion of Section 8.1, the sampled signal is represented by the summation of harmonically-related complex exponentials. The fundamental frequency of this set of exponentials is $2\pi/N$, where $N = 6$.
Therefore, the sequence $x[n]$ is periodic with period 6.

(b) For any bandlimited continuous-time signal, the Nyquist Criterion may be stated from Eq. (4.14b) as:

$$F_s \geq 2F_N,$$

where F_s is the sampling rate (Hz), and F_N corresponds to the highest frequency component in the signal (also Hz).
As evident by the finite Fourier series representation of $x_c(t)$, this continuous-time signal is, indeed, bandlimited with a maximum frequency of $F_n = \frac{9}{10^{-3}}$ Hz.
Therfore, by sampling at a rate of $F_s = \frac{6}{10^{-3}}$ Hz, the Nyquist Criterion is violated, and aliasing results.

(c) We use the analysis equation of Eq. (8.11):

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$$

From part (a), $\tilde{x}[n]$ is periodic with $N = 6$.
Substitution yields:

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{5}\left(\sum_{m=-9}^{9} a_m e^{j\frac{2\pi}{6}mn}\right) e^{-j\frac{2\pi}{6}kn} \\ &= \sum_{n=0}^{5}\sum_{m=-9}^{9} a_m e^{j(2\pi/6)(m-k)n}\end{aligned}$$

We reverse the order of the summations, and use the orthogonality relationship from Example 8.1:

$$\tilde{X}[k] = 6 \sum_{m=-9}^{9} a_m \sum_{r=-\infty}^{\infty} \delta[m-k+rN]$$

Taking the infinite summation to the outside, we recognize the convolution between a_m and shifted impulses (Recall $a_m = 0$ for $|m| > 9$). Thus,

$$\tilde{X}[k] = 6 \sum_{r=-\infty}^{\infty} a_{k-6r}$$

Note that from $\tilde{X}[k]$, the aliasing is apparent.

8.2. (a) Using the analysis equation of Eq. (8.11)

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

Since $\tilde{x}[n]$ is also periodic with period $3N$,

$$\begin{aligned}\tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] W_{3N}^{kn} \\ &= \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=N}^{2N-1} \tilde{x}[n] W_{3N}^{kn} + \sum_{n=2N}^{3N-1} \tilde{x}[n] W_{3N}^{kn}\end{aligned}$$

Performing a change of variables in the second and third summations of $\tilde{X}_3[k]$,

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_{3N}^{kn} + W_{3N}^{kN} \sum_{n=0}^{N-1} \tilde{x}[n+N] W_{3N}^{kn} + W_{3N}^{2kN} \sum_{n=0}^{N-1} \tilde{x}[n+2N] W_{3N}^{kn}$$

Since $\tilde{x}[n]$ is periodic with period N, and $W_{3N}^{kn} = W_N^{(\frac{k}{3})n}$,

$$\begin{aligned}\tilde{X}_3[k] &= \left(1 + e^{-j2\pi(\frac{k}{3})} + e^{-j2\pi(\frac{2k}{3})}\right) \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{(\frac{k}{3}n)} \\ &= \left(1 + e^{-j2\pi(\frac{k}{3})} + e^{-j2\pi(\frac{2k}{3})}\right) \tilde{X}[k] \\ &= \begin{cases} 3\tilde{X}[k/3], & k = 3\ell \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

(b) Using $N = 2$ and $\tilde{x}[n]$ as in Fig P8.2-1:

$$\begin{aligned}\tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \\ &= \sum_{n=0}^{1} \tilde{x}[n] e^{-j\frac{2\pi}{2}kn} \\ &= \tilde{x}[0] + \tilde{x}[1] e^{-j\pi k} \\ &= 1 + 2(-1)^k \\ &= \begin{cases} 3, & k = 0 \\ -1, & k = 1 \end{cases}\end{aligned}$$

Observe, from Fig. P8.2-1, that $\tilde{x}[n]$ is also periodic with period $3N = 6$:

$$\begin{aligned}\tilde{X}_3[k] &= \sum_{n=0}^{3N-1} \tilde{x}[n] W_{3N}^{kn} \\ &= \sum_{n=0}^{5} \tilde{x}[n] e^{-j\frac{\pi}{3}kn} \\ &= (1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k})(1 + 2(-1)^{\frac{k}{3}}) \\ &= (1 + e^{-j\frac{2\pi}{3}k} + e^{-j\frac{4\pi}{3}k})\tilde{X}[k/3] \\ &= \begin{cases} 9, & k = 0 \\ -3, & k = 3 \\ 0, & k = 1, 2, 4, 5. \end{cases}\end{aligned}$$

8.3. (a) The DFS coefficients will be real if $\tilde{x}[n]$ is even. Only signal B can be even (i.e., $\tilde{x}_B[n] = \tilde{x}_B[-n]$; if the origin is selected as the midpoint of either the nonzero block, or the zero block).

(b) The DFS coefficients will be imaginary if $\tilde{x}[n]$ is even. None of the sequences in Fig P8.3-1 can be odd.

(c) We use the analysis equation, Eq. (8.11) and the closed form expression for a geometric series. Assuming unit amplitudes and discarding DFS points which are zero:

$$\begin{aligned}
\tilde{X}_A[k] &= \sum_{n=0}^{3} e^{j\frac{2\pi}{8}kn} \\
&= \frac{1 - e^{j\frac{\pi}{4}4k}}{1 - e^{j\frac{\pi}{4}k}} \\
&= \frac{1 - (-1)^k}{1 - e^{j\frac{\pi}{4}k}} = 0, k = \pm 2, \pm 4, \ldots \\
\tilde{X}_B[k] &= \sum_{n=0}^{2} e^{j\frac{2\pi}{8}kn} \\
&= \frac{1 - e^{j\frac{\pi}{4}3k}}{1 - e^{j\frac{\pi}{4}k}} \\
\tilde{X}_C[k] &= \sum_{n=0}^{3} e^{j\frac{2\pi}{8}kn} - \sum_{n=4} 7e^{j\frac{2\pi}{8}kn} \\
&= \sum_{n=0}^{3} \left(e^{j\frac{\pi}{4}kn} - e^{j\frac{\pi}{4}k(n+4)} \right) \\
&= \left(1 - e^{j\pi k}\right) \frac{1 - e^{j\pi k}}{1 - e^{j\frac{\pi}{4}k}} \\
&= 0, \quad k = \pm 2, \pm 4, \ldots
\end{aligned}$$

8.4. A periodic sequence is constructed from the sequence:

$$x[n] = \alpha^n u[n]\ ,\ |\alpha| < 1$$

as follows:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n + rN]\ ,\ |\alpha| < 1$$

(a) The Fourier transform of $x[n]$:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \frac{1}{1-\alpha e^{-j\omega}}, \qquad |\alpha| < 1 \end{aligned}$$

(b) The DFS of $\tilde{x}[n]$:

$$\begin{aligned} \tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n]W_N^{kn} \\ &= \sum_{n=0}^{N-1}\sum_{r=-\infty}^{\infty} x[n+rN]W_N^{kn} \\ &= \sum_{n=0}^{N-1}\sum_{r=-\infty}^{\infty} \alpha^{n+rN}u[n+rN]W_N^{kn} \\ &= \sum_{n=0}^{N-1}\sum_{r=0}^{\infty} \alpha^{n+rN}W_N^{kn} \end{aligned}$$

Rearranging the summations gives:

$$\begin{aligned} \tilde{X}[k] &= \sum_{r=0}^{\infty} \alpha^{rN} \sum_{n=0}^{N-1} \alpha^n W_N^{kn} \\ &= \sum_{r=0}^{\infty} \alpha^{rN} \left(\frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j\frac{2\pi k}{N}}}\right)\ ,\ |\alpha| < 1 \\ &= \frac{1}{1-\alpha^N}\left(\frac{1-\alpha^N e^{-j2\pi k}}{1-\alpha e^{-j\frac{2\pi k}{N}}}\right)\ ,\ |\alpha| < 1 \\ \tilde{X}[k] &= \frac{1}{1-\alpha e^{-j(2\pi k/N)}}\ ,\ |\alpha| < 1 \end{aligned}$$

(c) Comparing the results of part (a) and part (b):

$$\tilde{X}[k] = X(e^{j\omega})\big|_{\omega=2\pi k/N}\,.$$

8.5. (a)

$$\begin{aligned} x[n] &= \delta[n] \\ X[k] &= \sum_{n=0}^{N-1} \delta[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= 1 \end{aligned}$$

(b)

$$\begin{aligned} x[n] &= \delta[n-n_0], \quad 0 \le n_0 \le (N-1) \\ X[k] &= \sum_{n=0}^{N-1} \delta[n-n_0] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= W_N^{kn_0} \end{aligned}$$

(c)

$$\begin{aligned} x[n] &= \begin{cases} 1, & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \\ X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{(N/2)-1} W_N^{2kn} \\ &= \frac{1-e^{-j2\pi k}}{1-e^{-j(\pi k/N)}} \\ X[k] &= \begin{cases} N/2, & k = 0, N/2 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

(d)

$$\begin{aligned} x[n] &= \begin{cases} 1, & 0 \le n \le ((N/2)-1) \\ 0, & N/2 \le n \le (N-1) \end{cases} \\ X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{(N/2)-1} W_N^{kn} \\ &= \frac{1-e^{-j\pi k}}{1-e^{-j(2\pi k)/N}} \\ X[k] &= \begin{cases} N/2, & k = 0 \\ \dfrac{2}{1-e^{-j(2\pi k/N)}}, & k \text{ odd} \\ 0, & k \text{ even}, 0 \le k \le (N-1) \end{cases} \end{aligned}$$

(e)

$$\begin{aligned} x[n] &= \begin{cases} a^n, & 0 \le n \le (N-1) \\ 0, & \text{otherwise} \end{cases} \\ X[k] &= \sum_{n=0}^{N-1} a^n W_N^{kn}, \quad 0 \le k \le (N-1) \\ &= \frac{1-a^N e^{-j2\pi k}}{1-ae^{-j(2\pi k)/N}} \\ X[k] &= \frac{1-a^N}{1-ae^{-j(2\pi k)/N}} \end{aligned}$$

8.6. Consider the finite-length sequence

$$x[n] = \begin{cases} e^{j\omega_0 n}, & 0 \le n \le (N-1) \\ 0, & \text{otherwise} \end{cases}$$

(a) The Fourier transform of $x[n]$:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} e^{j\omega_0 n} e^{-j\omega n} \\ X(e^{j\omega}) &= \frac{1 - e^{-j(\omega-\omega_0)N}}{1 - e^{-j(\omega-\omega_0)}} \\ &= \frac{e^{-j(\omega-\omega_0)(N/2)}}{e^{-j(\omega-\omega_0)/2}} \left(\frac{\sin[(\omega-\omega_0)(N/2)]}{\sin[(\omega-\omega_0)/2]} \right) \\ X(e^{j\omega}) &= e^{-j(\omega-\omega_0)((N-1)/2)} \left(\frac{\sin[(\omega-\omega_0)(N/2)]}{\sin[(\omega-\omega_0)/2]} \right) \end{aligned}$$

(b) N-point DFT:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad 0 \le k \le (N-1) \\ &= \sum_{n=0}^{N-1} e^{j\omega_0 n} W_N^{kn} \\ &= \frac{1 - e^{-j((2\pi k/N)-\omega_0)N}}{1 - e^{-j((2\pi k/N)-\omega_0)}} \\ &= e^{-j\left(\frac{2\pi k}{N}-\omega_0\right)\left(\frac{N-1}{2}\right)} \frac{\sin\left[\left(\frac{2\pi k}{N}-\omega_0\right)\frac{N}{2}\right]}{\sin\left[\left(\frac{2\pi k}{N}-\omega_0\right)/2\right]} \end{aligned}$$

Note that $X[k] = X(e^{j\omega})|_{\omega=(2\pi k)/N}$

(c) Suppose $\omega_0 = (2\pi k_0)/N$, where k_0 is an integer:

$$\begin{aligned} X[k] &= \frac{1 - e^{-j(k-k_0)2\pi}}{1 - e^{-j(k-k_0)(2\pi)/N}} \\ &= e^{-j(2\pi/N)(k-k_0)((N-1)/2)} \frac{\sin \pi(k-k_0)}{\sin(\pi(k-k_0)/N)} \end{aligned}$$

8.7. We have a six-point uniform sequence, $x[n]$, which is nonzero for $0 \le n \le 5$. We sample the Z-transform of $x[n]$ at four equally-spaced points on the unit circle.

$$X[k] = X(z)|_{z=e^{(2\pi k/4)}}$$

We seek the sequence $x_1[n]$ which is the inverse DFT of $X[k]$. Recall the definition of the Z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since $x[n]$ is zero for all n outside $0 \le n \le 5$, we may replace the infinite summation with a finite summation. Furthermore, after substituting $z = e^{j(2\pi k/4)}$, we obtain

$$X[k] = \sum_{n=0}^{5} x[n]W_4^{kn}, \qquad 0 \le k \le 4$$

Note that we have taken a 4-point DFT, as specified by the sampling of the Z-transform; however, the original sequence was of length 6. As a result, we can expect some aliasing when we return to the time domain via the inverse DFT.

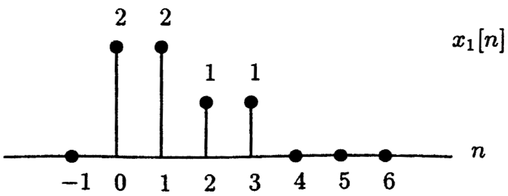
Performing the DFT,

$$X[k] = W_4^{0k} + W_4^{k} + W_4^{2k} + W_4^{3k} + W_4^{4k} + W_4^{5k}, \qquad 0 \le k \le 4$$

Taking the inverse DFT by inspection, we note that there are six impulses (one for each value of n above). However,

$$W_4^{4k} = W_4^{0k} \text{ and } W_4^{5k} = W_4^{k},$$

so two points are aliased. The resulting time-domain signal is



8.8. Fourier transform of $x[n] = (1/2)^n u[n]$:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n e^{-j\omega n} \\ &= \frac{1}{1-\frac{1}{2}e^{-j\omega}} \end{aligned}$$

Now, sample the frequency spectra of $x[n]$:

$$Y[k] = X(e^{j\omega})|_{\omega=2\pi k/10}, \qquad 0 \le k \le 9$$

We have the 10-pt DFT:

$$\begin{aligned} Y[k] &= \frac{1}{1-\frac{1}{2}e^{-j(2\pi k/10)}}, \qquad 0 \le k \le 9 \\ &= \sum_{n=0}^{9} y[n]W_{10}^{kn} \end{aligned}$$

Recall:

$$\left(\frac{1}{2}\right)^n \xrightarrow[\text{N−pt}]{\text{DFT}} \frac{1-(\frac{1}{2})^N}{1-\frac{1}{2}e^{-j(2\pi k/N)}}$$

So, we may infer:

$$y[n] = \frac{(\frac{1}{2})^n}{1-(\frac{1}{2})^{10}}, \qquad 0 \le n \le 9$$

8.9. Given a 20-pt finite-duration sequence $x[n]$:

(a) We wish to obtain $X(e^{j\omega})|_{\omega=4\pi/5}$ using the smallest DFT possible. A possible size of the DFT is evident by the periodicity of $e^{j\omega}|_{\omega=4\pi/5}$. Suppose we choose the size of the DFT to be $M = 5$. The data sequence is 20 points long, so we use the time-aliasing technique derived in the previous problem. Specifically, we alias $x[n]$ as:

$$x_1[n] = \sum_{r=-\infty}^{\infty} x[n + 5r]$$

This aliased version of $x[n]$ is periodic with period 5 now. The 5-pt DFT is computed. The desired value occurs at a frequency corresponding to:

$$\frac{2\pi k}{N} = \frac{4\pi}{5}$$

For $N = 5$, $k = 2$, so the desired value may be obtained as $X[k]|_{k=2}$.

(b) Next, we wish to obtain $X(e^{j\omega})|_{\omega=10\pi/27}$.
The smallest DFT is of size $L = 27$. Since the DFT is larger than the data block size, we pad $x[n]$ with 7 zeros as follows:

$$x_2[n] = \begin{cases} x[n], & 0 \le n \le 19 \\ 0, & 20 \le n \le 26 \end{cases}$$

We take the 27-pt DFT, and the desired value corresponds to $X[k]$ evaluated at $k = 5$.

8.10. From Fig P8.10-1, the two 8-pt sequences are related through a circular shift. Specifically,

$$x_2[n] = x_1[((n-4))_8]$$

From property 5 in Table 8.2,

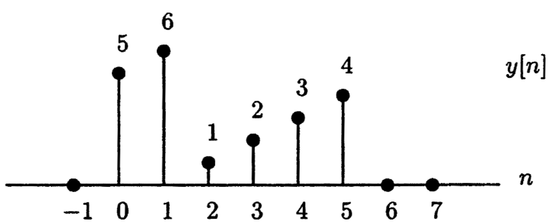
$$\text{DFT}\{x_1[((n-4))_8]\} = W_8^{4k} X_1[k]$$

Thus,

$$\begin{aligned} X_2[k] &= W_8^{4k} X_1[k] \\ &= e^{-j\pi k} X_1[k] \\ X_2[k] &= (-1)^k X_1[k] \end{aligned}$$

8.11. We wish to perform the circular convolution between two 6-pt sequences. Since $x_2[n]$ is just a shifted impulse, the circular-convolution coincides with a circular shift of $x_1[n]$ by two points.

$$
\begin{aligned}
y[n] &= x_1[n] \, \textcircled{6} \, x_2[n] \\
&= x_1[n] \, \textcircled{6} \, \delta[n-2] \\
&= x_1[((n-2))_6]
\end{aligned}
$$



8.12. (a)

$$x[n] = \cos(\frac{\pi n}{2}), \quad 0 \le n \le 3$$

transforms to

$$X[k] = \sum_{n=0}^{3} \cos(\frac{\pi n}{2}) W_4^{kn}, \quad 0 \le k \le 3$$

The cosine term contributes only two non-zero values to the summation, giving:

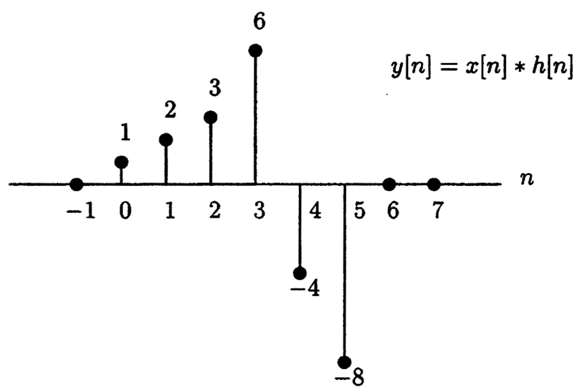
$$\begin{aligned} X[k] &= 1 - e^{-j\pi k}, \quad 0 \le k \le 3 \\ &= 1 - W_4^{2k} \end{aligned}$$

(b)

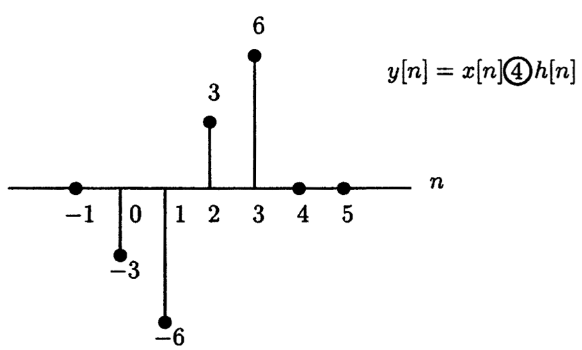
$$h[n] = 2^n, \quad 0 \le n \le 3$$

$$\begin{aligned} H[k] &= \sum_{n=0}^{3} 2^n W_4^{kn}, \quad 0 \le k \le 3 \\ &= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k} \end{aligned}$$

(c) Remember, circular convolution equals linear convolution plus aliasing. We need $N \ge 3+4-1 = 6$ to avoid aliasing. Since $N = 4$,we expect to get aliasing here. First, find $y[n] = x[n] * h[n]$:



For this problem, aliasing means the last three points ($n = 4, 5, 6$) will wrap-around on top of the first three points, giving $y[n] = x[n] \textcircled{4} h[n]$:



(d) Using the DFT values we calculated in parts (a) and (b):

$$\begin{aligned} Y[k] &= X[k]H[k] \\ &= 1 + 2W_4^k + 4W_4^{2k} + 8W_4^{3k} - W_4^{2k} - 2W_4^{3k} - 4W_4^{4k} - 8W_4^{5k} \end{aligned}$$

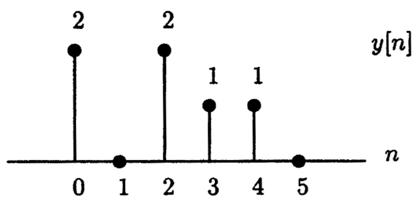
Since $W_4^{4k} = W_4^{0k}$ and $W_4^{5k} = W_4^k$

$$Y[k] = -3 - 6W_4^k + 3W_4^{2k} + 6W_4^{3k}, \quad 0 \le k \le 3$$

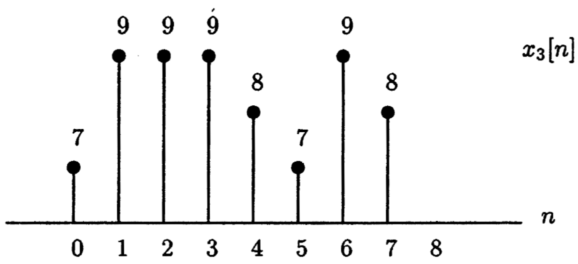
Taking the inverse DFT:

$$y[n] = -3\delta[n] - 6\delta[n-1] + 3\delta[n-2] + 6\delta[n-3], \quad 0 \le n \le 3$$

8.13. Using the properties of the DFT, we get $y[n] = x[((n-2))_5]$, that is $y[n]$ is equal to $x[n]$ circularly shifted by 2. We get:

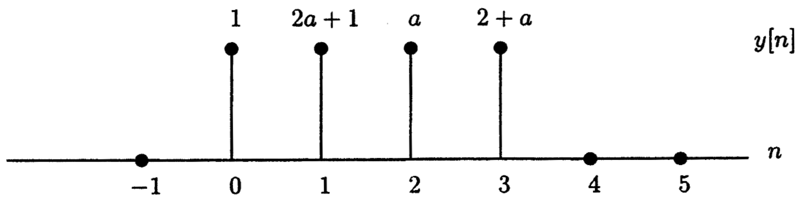


8.14. $x_3[n]$ is the linear convolution of $x_1[n]$ and $x_2[n]$ time-aliased to $N = 8$. Carrying out the 8-point circular convolution, we get:



We thus conclude $x_3[2] = 9$.

8.15. $y[n]$ is the linear convolution of $x_1[n]$ and $x_2[n]$ time-aliased to $N = 4$. Carrying out the 4-point circular convolution, we get:



Matching the above sequence to the one given, we get $a = -1$, which is unique.

8.16. $X_1[k]$ is the 4-point DFT of $x[n]$ and $x_1[n]$ is the 4-point inverse DFT of $X_1[k]$, therefore $x_1[n]$ is $x[n]$ time aliased to $N = 4$. In other words, $x_1[n]$ is one period of $\tilde{x}[n] = x[((n))_4]$. We thus have:

$$4 = b + 1.$$

Therefore, $b = 3$. This is clearly unique.

8.17. Looking at the sequences, we see that $x_1[n] * x_2[n]$ is non-zero for $1 \leq n \leq 8$. The smallest N such that $x_1[n] \, N \, x_2[n] = x_1[n] * x_2[n]$ is therefore $N = 9$.

8.18. Taking the inverse DFT of $X_1[k]$ and using the properties of the DFT, we get:

$$x_1[n] = x[((n+3))_5].$$

Therefore:

$$x_1[0] = x[3] = c.$$

We thus conclude that $c = 2$.

8.19. $x_1[n]$ and $x[n]$ are related by a circular shift as can be seen from the plots. Using the properties of the DFT and the relationship between $X_1[k]$ and $X[k]$, we have:

$$x_1[n] = x[((n - m))_6].$$

$m = 2$ works, clearly this choice is not unique, any $m = 2 + 6l$, where l is an integer, would work.

8.20.

$$X_1[k] = X[k]e^{+j(2\pi k 2/N)}.$$

Using the properties of the DFT, we get:

$$x_1[n] = x[((n+2))_N].$$

From the figures, we conclude that:

$$N = 5.$$

This choice of N is unique.

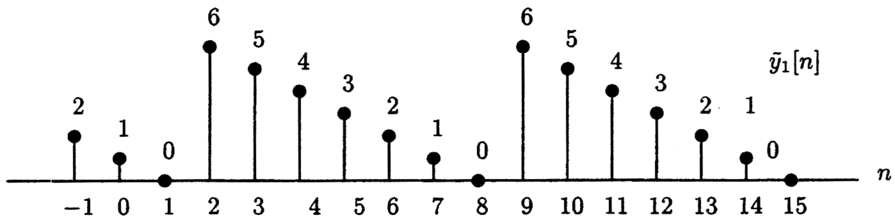
8.21. (a) We seek a sequence $\tilde{y}_1[n]$ such that

$$\tilde{Y}_1[k] = \tilde{X}_1[k]\tilde{X}_2[k]$$

From the discussion of Section 8.2.5, $\tilde{y}[n]$ is the result of the periodic convolution between $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$.

$$\tilde{y}_1[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$$

Since $\tilde{x}_2[n]$ is a periodic impulse, shifted by two, the resultant sequence will be a shifted (by two) replica of $\tilde{x}_1[n]$.



Using the analysis equation of Eq. (8.11), we may rigorously derive $\tilde{y}_1[n]$:

$$\begin{aligned}
\tilde{X}_1[k] &= \sum_{n=0}^{6} \tilde{x}_1[n]W_7^{kn} \\
&= 6 + 5W_7^{k} + 4W_7^{2k} + 3W_7^{3k} + 2W_7^{4k} + W_7^{5k} \\
\tilde{X}_2[k] &= \sum_{n=0}^{6} \tilde{x}_2[n]W_7^{kn} \\
&= W_7^{2k} \\
\tilde{Y}_1[k] &= \tilde{X}_1[k]\tilde{X}_2[k] \\
&= 6W_7^{2k} + 5W_7^{3k} + 4W_7^{4k} + 3W_7^{5k} + 2W_7^{6k} + W_7^{7k}
\end{aligned}$$

Noting that $W_7^{7k} = e^{j\frac{2\pi}{7}(7k)} = 1 = W_7^{0k}$, we use the synthesis equation of Eq. (8.12) to construct $\tilde{y}_1[n]$. The result is identical to the sequence depicted above.

(b) The DFS of the signal illustrated in Fig. P8.21-2.is given by:

$$\begin{aligned}
\tilde{X}_3[k] &= \sum_{n=0}^{6} \tilde{x}_3[n]W_7^{kn} \\
&= 1 + W_7^{4k}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\tilde{Y}_2[k] &= \tilde{X}_1[k]\tilde{X}_3[k] \\
&= \tilde{X}_1[k] + W_7^{4k}\tilde{X}_1[k]
\end{aligned}$$

Since the DFS is linear, the inverse DFS of $\tilde{Y}_2[k]$ is given by:

$$\tilde{y}_2[n] = \tilde{x}_1[n] + \tilde{x}_1[n-4].$$

8.22. For a finite-length sequence $x[n]$, with length equal to N, the periodic repetition of $x[-n]$ is represented by

$$x[((-n))_N] = x[((-n + \ell N))_N], \qquad \ell\text{: integer}$$

where the right side is justified since $x[n]$ (and $x[-n]$) is periodic with period N.

The above statement holds true for any choice of ℓ. Therefore, for $\ell = 1$:

$$x[((-n))_n] = x[((-n + N))_N]$$

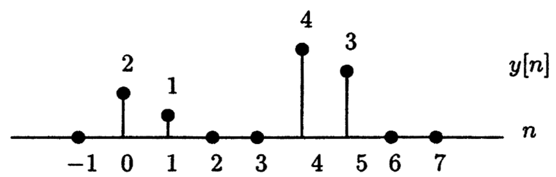
8.23. (a) When multiplying the DFT of a sequence by a complex exponential, the time-domain signal undergoes a circular shift.

For this case,

$$Y[k] = W_6^{4k} X[k], \quad 0 \le k \le 5$$

Therefore,

$$y[n] = x[((n-4))_6], \quad 0 \le n \le 5$$



(b) There are two ways to approach this problem. First, we attempt a solution by brute force.

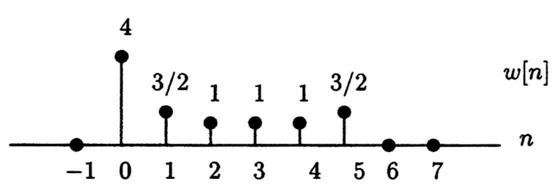
$$\begin{aligned}
X[k] &= 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}, \qquad W_6^k = e^{-j(2\pi k/6)} \text{ and } 0 \le k \le 5 \\
W[k] &= \mathcal{R}e\{X[k]\} \\
&= \frac{1}{2}(X[k] + X^*[k]) \\
&= \frac{1}{2}\left(4 + 3W_6^k + 2W_6^{2k} + W_6^{3k} + 4 + 3W_6^{-k} + 2W_6^{-2k} + W_6^{-3k}\right)
\end{aligned}$$

$$\begin{aligned}
W_N^k &= e^{-j(2\pi k/N)} \\
W_N^{-k} &= e^{j(2\pi k/N)} = e^{-j(2\pi/N)(N-k)} = W_N^{N-k} \\
W[k] &= 4 + \frac{3}{2}\left[W_6^k + W_6^{6-k}\right] + \left[W_6^{2k} + W_6^{6-2k}\right] + \frac{1}{2}\left[W_6^{3k} + W_6^{6-3k}\right], \qquad 0 \le k \le 5
\end{aligned}$$

So,

$$\begin{aligned}
w[n] &= 4\delta[n] + \frac{3}{2}\Big(\delta[n-1] + \delta[n-5]\Big) + \delta[n-2] + \delta[n-4] \\
&\quad + \frac{1}{2}\Big(\delta[n-3] + \delta[n-3]\Big) \\
w[n] &= 4\delta[n] + \frac{3}{2}\delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \frac{3}{2}\delta[n-5], \qquad 0 \le n \le 5
\end{aligned}$$

Sketching $w[n]$:

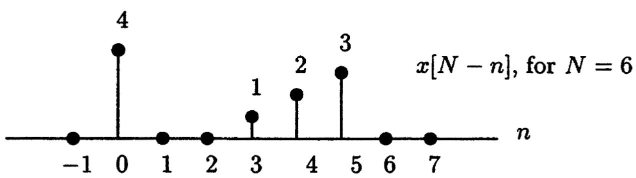


As an alternate approach, suppose we use the properties of the DFT as listed in Table 8.2.

$$\begin{aligned} W[k] &= \mathcal{R}e\{X[k]\} \\ &= \frac{X[k] + X^*[k]}{2} \\ w[n] &= \frac{1}{2}\,\text{IDFT}\{X[k]\} + \frac{1}{2}\,\text{IDFT}\{X^*[k]\} \\ &= \frac{1}{2}\left(x[n] + x^*[((-n))_N]\right) \end{aligned}$$

For $0 \leq n \leq N-1$ and $x[n]$ real:

$$w[n] = \frac{1}{2}\left(x[n] + x[N-n]\right)$$



So, we observe that $w[n]$ results as above.

(c) The DFT is decimated by two. By taking alternate points of the DFT output, we have half as many points. The influence of this action in the time domain is, as expected, the appearance of aliasing. For the case of decimation by two, we shall find that an additional replica of x[n] surfaces, since the sequence is now periodic with period 3.

From part (b):

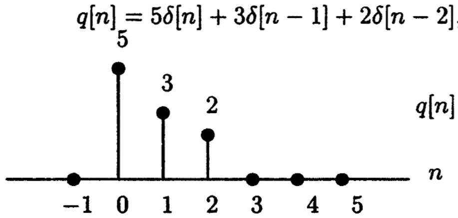
$$X[k] = 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}, \qquad 0 \leq k \leq 5$$

Let $Q[k] = X[2k]$,

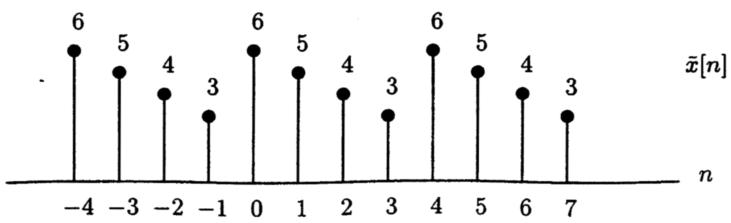
$$Q[k] = 4 + 3W_3^k + 2W_3^{2k} + W_3^{3k}, \qquad 0 \leq k \leq 2$$

Noting that $W_3^{3k} = W_3^{0k}$

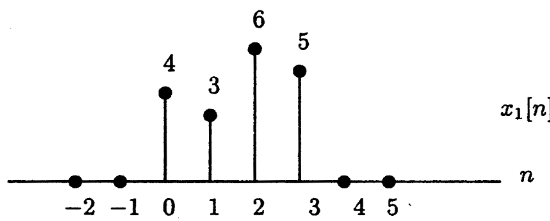
$$q[n] = 5\delta[n] + 3\delta[n-1] + 2\delta[n-2], \qquad 0 \leq n \leq 2$$



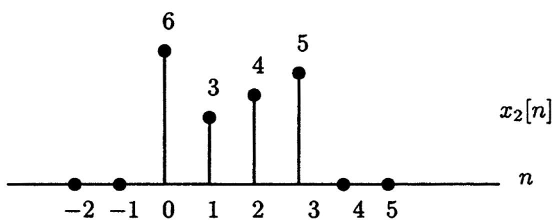
8.24. We may approach this problem in two ways. First, the notion of modulo arithmetic may be simplified if we utilize the implied periodic extension. That is, we redraw the original signal as if it were periodic with period $N = 4$. A few periods are sufficient:



To obtain $x_1[n] = x[((n-2))_4]$, we shift by two (to the right) and only keep those points which lie in the original domain of the signal (i.e. $0 \leq n \leq 3$):



To obtain $x_2[n] = x[((-n))_4]$, we fold the pseudo-periodic version of $x[n]$ over the origin (time-reversal), and again we set all points outside $0 \leq n \leq 3$ equal to zero. Hence,



Note that $x[((0))_4] = x[0]$, etc.

In the second approach, we work with the given signal. The signal is confined to $0 \leq n \leq 3$; therefore, the circular nature must be maintained by picturing the signal on the circumference of a cylinder.

8.25. No. Recall that the DFT merely samples the frequency spectra. Therefore, the fact the $Im\{X[k]\} = 0$ for $0 \le k \le (N-1)$ does not guarantee that the imaginary part of the continuous frequency spectra is also zero.

For example, consider a signal which consists of an impulse centered at $n = 1$.

$$x[n] = \delta[n-1], \quad 0 \le n \le 1$$

The Fourier transform is:

$$\begin{aligned} X(e^{j\omega}) &= e^{-j\omega} \\ Re\{X(e^{j\omega})\} &= \cos(\omega) \\ Im\{X(e^{j\omega})\} &= -\sin(\omega) \end{aligned}$$

Note that neither is zero for all $0 \le \omega \le 2$. Now, suppose we take the 2-pt DFT:

$$\begin{aligned} X[k] &= W_2^k, \quad 0 \le k \le 1 \\ &= \begin{cases} 1, & k = 0 \\ -1, & k = 1 \end{cases} \end{aligned}$$

So, $Im\{X[k]\} = 0, \quad \forall k$. However, $Im\{X(e^{j\omega})\} \neq 0$.

Note also that the size of the DFT plays a large role. For instance, consider taking the 3-pt DFT of

$$\begin{aligned} x[n] &= \delta[n-1], \quad 0 \le n \le 2 \\ X[k] &= W_3^k, \quad 0 \le k \le 2 \\ &= \begin{cases} 1, & k = 0 \\ e^{-j(2\pi/3)}, & k = 1 \\ e^{-j(4\pi/3)}, & k = 2 \end{cases} \end{aligned}$$

Now, $Im\{X[k]\} \neq 0$, for $k = 1$ or $k = 2$.

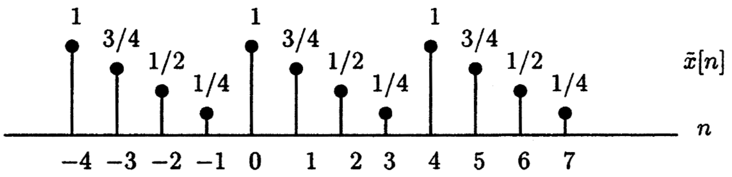
8.26. Both sequences $x[n]$ and $y[n]$ are of finite-length ($N = 4$).

Hence, no aliasing takes place. From Section 8.6.2, multiplication of the DFT of a sequence by a complex exponential corresponds to a circular shift of the time-domain sequence.

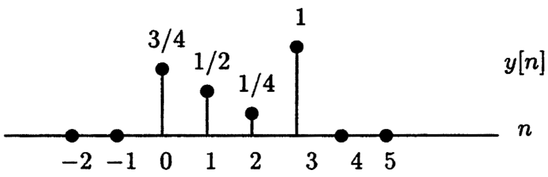
Given $Y[k] = W_4^{3k} X[k]$, we have

$$y[n] = x[((n-3))_4]$$

We use the technique suggested in problem 8.28. That is, we temporarily extend the sequence such that a periodic sequence with period 4 is formed.



Now, we shift by three (to the right), and set all values outside $0 \leq n \leq 3$ to zero.



8.27

A. We know

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi}{N}n} e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\frac{\pi}{N}n(1+2k)}$$

Let $\omega_k = \dfrac{\pi(1+2k)}{N}$. Then

$$\tilde{X}[k] = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n} = X\left(e^{j\omega_k}\right)$$

$$= X\left(e^{j\left(\frac{\pi+2\pi k}{N}\right)}\right), \quad k = 0, 1, \ldots, N-1.$$

B. The frequencies of sampling are given by

$$\omega_k = \frac{\pi(1+2k)}{N}, \quad k = 0, 1, \ldots, N-1.$$

C. Given the modified $\tilde{X}[k]$, we can use the inverse transform to find $\tilde{x}[n]$. To get $x[n]$ from $\tilde{x}[n]$ it is a simple point-by-point multiplication given by

$$x[n] = e^{j\frac{\pi}{N}n} \tilde{x}[n].$$

8.28

A. Using the analysis equation

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn} \\ &= \sum_{n=0}^{5} x[n] W_6^{kn} \\ &= 6W_6^0 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k}. \end{aligned}$$

B.

$$\begin{aligned} W[k] &= W_6^{-2k} X[k] \\ &= 6W_6^{-2k} + 5W_6^{-k} + 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k}. \end{aligned}$$

Using the fact that $W_6^k = e^{-j\frac{2\pi k}{6}}$,

$$\begin{aligned} W_6^{-2k} &= e^{j\frac{4\pi k}{6}} = e^{j\frac{4\pi k}{6}} \times e^{-j2\pi k} \text{ (since } e^{-j2\pi k} = 1) \\ &= e^{-j\frac{8\pi k}{6}} = W_6^{4k}, \end{aligned}$$

and similarly

$$W_6^{-k} = W_6^{5k}.$$

Then

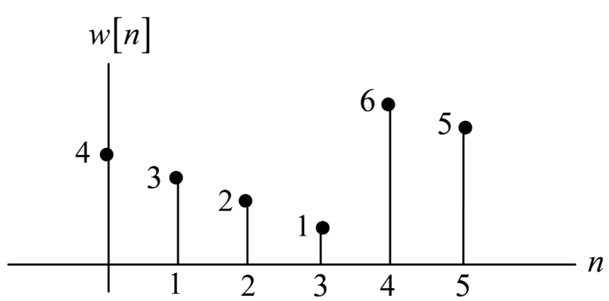
$$W[k] = 4 + 3W_6^k + 2W_6^{2k} + W_6^{3k} + 6W_6^{4k} + 5W_6^{5k}.$$

Using the synthesis equation,

$$w[n] = \frac{1}{6}\sum_{k=0}^{5} W[k] W_6^{-kn}.$$

We could go ahead and solve the problem in this "brute force" method, but notice that each $\delta[n-k] \xrightarrow{DFT} W_N^k$. Then,

$$w[n] = 4\delta[n] + 3\delta[n-1] + 2\delta[n-2] + \delta[n-3] + 6\delta[n-4] + 5\delta[n-5].$$



Notice that multiplying by W_6^{-2k} in frequency has the effect of a shift of 2 in time, but modulo 6.

C. One way to do this is to compute the linear convolution and then add copies of it shifted by N (6 in this case). Another method is to use the DFT, find the product $H[k]X[k]$, and then take an inverse DFT. We know

$$X[k] = 6W_6^0 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k}$$

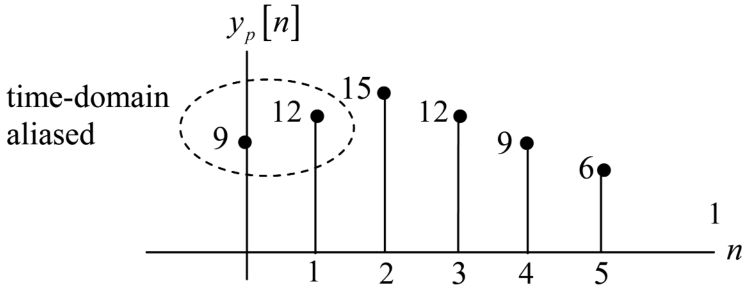
$$H[k] = 1 + W_6^k + W_6^{2k}$$

Then

$$\begin{aligned} Y_p[k] &= 6 + 5W_6^k + 4W_6^{2k} + 3W_6^{3k} + 2W_6^{4k} + W_6^{5k} \\ &\quad + 6W_6^k + 5W_6^{2k} + 4W_6^{3k} + 3W_6^{4k} + 2W_6^{5k} + W_6^{6k} \\ &\quad + 6W_6^{2k} + 5W_6^{3k} + 4W_6^{4k} + 3W_6^{5k} + 2W_6^{6k} + W_6^{7k} \\ &= 9 + 12W_6^k + 15W_6^{2k} + 12W_6^{3k} + 9W_6^{4k} + 6W_6^{5k}, \end{aligned}$$

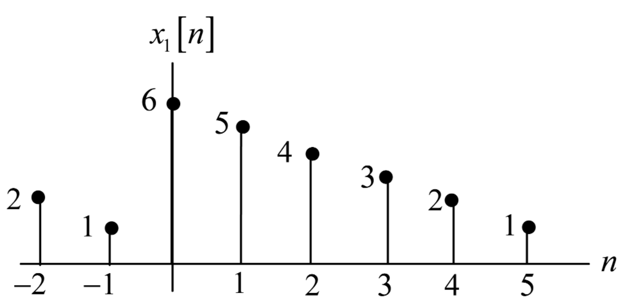
where we have used $W_6^{6k} = 1$ and $W_6^{7k} = W_6^k$. Now we have

$$y_p[n] = 9\delta[n] + 12\delta[n-1] + 15\delta[n-2] + 12\delta[n-3] + 9\delta[n-4] + 6\delta[n-5].$$



D. To ensure that no time-domain aliasing occurs in the output, N should be large enough to accommodate the length of the linear convolution. That is, $N \geq 6 + 3 - 1 = 8$

E.

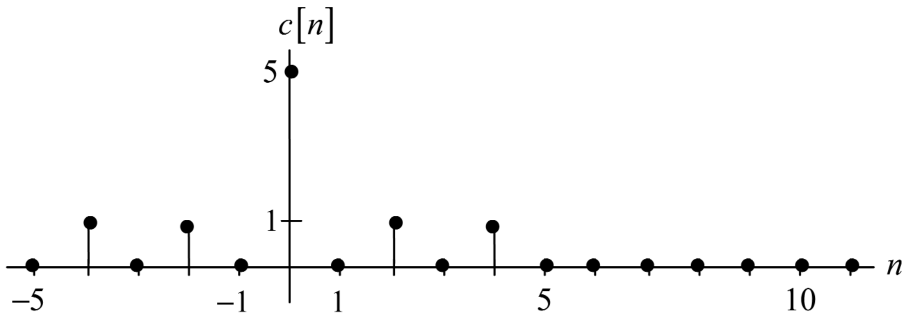


This new input when convolved with $h[n]$ will give the circular convolution found in C.

We merely extend $x[n]$ as a periodic signal with period 6 samples.

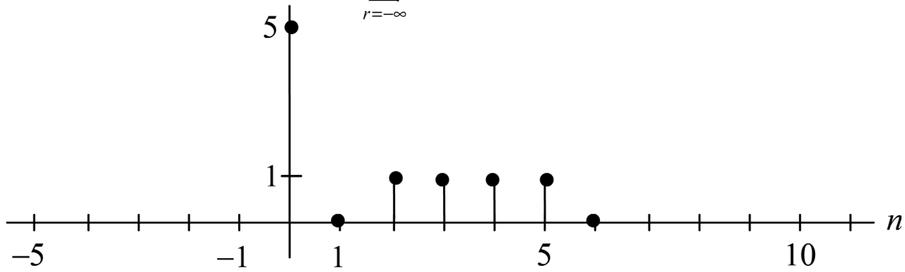
8.29

A.



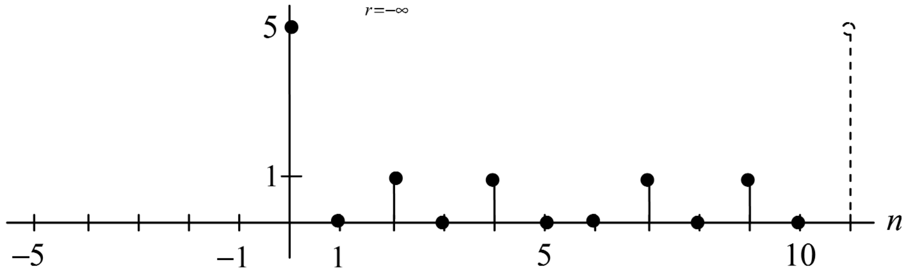
B.

$$c_7[n] = \sum_{r=-\infty}^{\infty} c[n - r7], \quad n = 0, 1, \ldots, 6$$

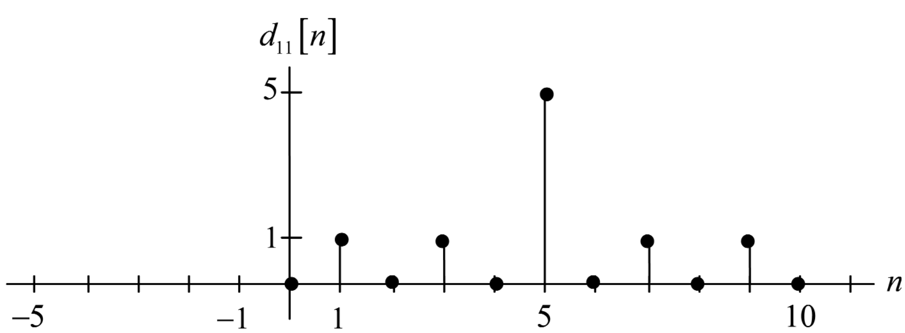


C.

$$c_{11}[n] = \sum_{r=-\infty}^{\infty} c[n - r11], \quad n = 0, 1, \ldots, 10$$

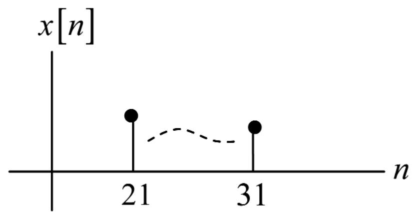


D. $W_{11}^{5k} \Rightarrow$ rotate right by 5 .

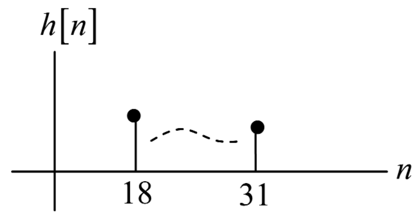


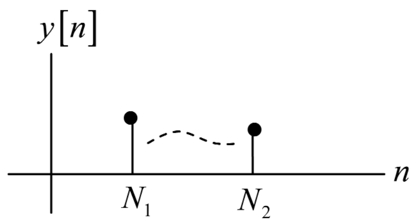
8.30

A.



*





$$N_1 = 21 + 18 = 39$$
$$N_2 = 31 + 31 = 62.$$

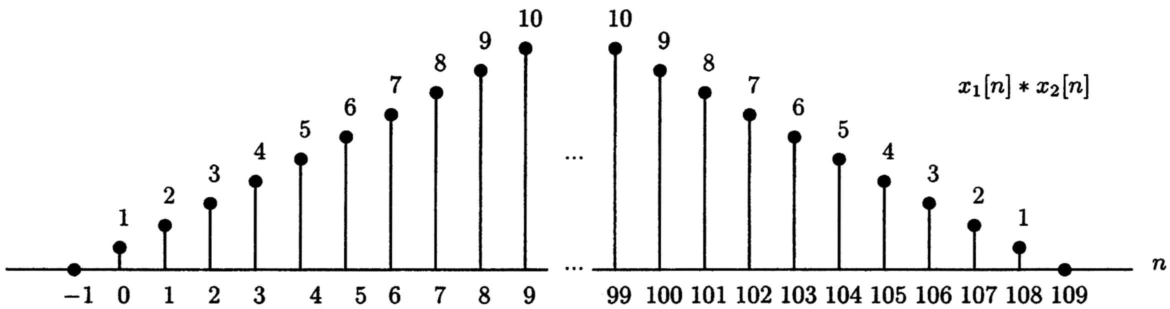
B. The sequence $y_1[n]$ is the 32-point circular convolution of $x_1[n]$ with $h_1[n]$. That is,

$$\begin{aligned} y_1[n] &= \sum_{r=-\infty}^{\infty} y[n + r32] \\ &= y[n+32], \quad n = 0,1,\ldots 31, \end{aligned}$$

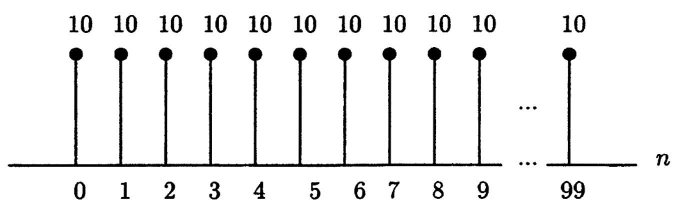
since $y[n+32]$ is the only one that fits in $0 \le n \le 31$.

C. If we add zeros at the ends too, we can get $y_1[n] = y[n]$ if $N > 62$.

8.31. (a) The linear convolution, $x_1[n] * x_2[n]$ is a sequence of length $100 + 10 - 1 = 109$.

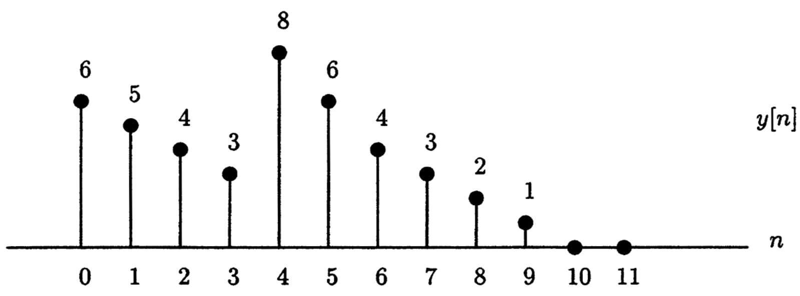


(b) The circular convolution, $x_1[n]$ 100 $x_2[n]$, can be obtained by aliasing the first 9 points of the linear convolution above:



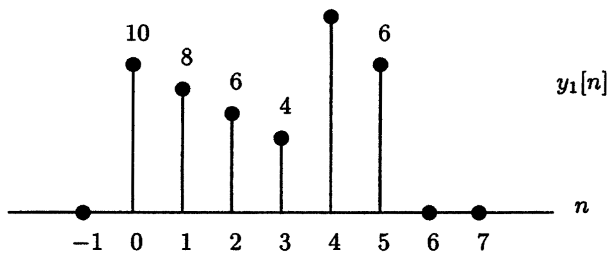
(c) Since $N \geq 109$, the circular convolution $x_1[n]$ 110 $x_2[n]$ will be equivalent to the linear convolution of part (a).

8.32. Circular convolution equals linear convolution plus aliasing. First, we find $y[n] = x_1[n] * x_2[n]$:



Note that $y[n]$ is a ten point sequence ($N = 6 + 5 - 1$).

(a) For $N = 6$, the last four non-zero point ($6 \leq n \leq 9$) will alias to the first four points, giving us $y_1[n] = x_1[n] \, 6 \, x_2[n]$



(b) For $N = 10, N \geq 6 + 5 - 1$, so no aliasing occurs, and circular convolution is identical to linear convolution.

8.33. We have $x[n]$ for $0 \le n \le P$.

We desire to compute $X(z)|_{z=e^{-j(2\pi k/N)}}$ using one N-pt DFT.

(a) Suppose $N > P$ (the DFT size is larger than the data segment). The technique used in this case is often referred to as zero-padding. By appending zeros to a small data block, a larger DFT may be used. Thus the frequency spectra may be more finely sampled. It is a common misconception to believe that zero-padding enhances spectral resolution. The addition of a larger block of data to a larger DFT would enhance this quality.

So, we append $N_z = N - P$ zeros to the end of the sequence as follows:

$$x'[n] = \begin{cases} x[n], & 0 \le n \le (P-1) \\ 0, & P \le n \le N \end{cases}$$

$$x[n] \longrightarrow \boxed{N\text{-pt DFT}} \longrightarrow X[k] \quad (0 \le k \le N)$$

(b) Suppose $N > P$, consider taking a DFT which is smaller than the data block. Of course, some aliasing is expected. Perhaps we could introduce time aliasing to offset the effects.

Consider the N-pt inverse DFT of $X[k]$,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad 0 \le n \le (N-1)$$

Suppose $X[k]$ was obtained as the result of an infinite summation of complex exponents:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{\infty} x[m]e^{-j(2\pi k/N)m}\right)W_N^{-kn}$$

Rearrange to get:

$$x[n] = \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{N}\sum_{k=0}^{N-1} e^{-j(2\pi/N)(m-n)k}\right)$$

Using the orthogonality relationship of Example 8.1:

$$\begin{aligned} x[n] &= \sum_{m=-\infty}^{\infty} x[m]\sum_{r=-\infty}^{\infty}\delta[m-n+rN] \\ x[n] &= \sum_{r=-\infty}^{\infty} x[n-rN] \end{aligned}$$

So, we should alias $x[n]$ as above. Then we take the N-pt DFT to get $X[k]$.

8.34

The z-transform $X_1(z)$ of $x_1[n]$ is given by

$$X_1(z) = \sum_{n=0}^{N-1} x_1[n] z^{-n}.$$

At $z = \frac{1}{2} e^{-j\frac{2\pi k}{N}}$ we have

$$X_1(z)\Big|_{z=\frac{1}{2}e^{-j\frac{2\pi k}{N}}} = \sum_{n=0}^{N-1} x_1[n] \left(\tfrac{1}{2}\right)^{-n} e^{j\frac{2\pi kn}{N}}, \quad k = 0, \ldots, N-1.$$

Now $X_2[k]$ is given by

$$\begin{aligned} X_2[k] &= \sum_{n=0}^{N-1} x_2[n] e^{-j\frac{2\pi kn}{N}} \\ &= \sum_{n=0}^{N-1} x_2\left[((-n))_N\right] e^{j\frac{2\pi kn}{N}}, \quad k = 0, \ldots, N-1. \end{aligned}$$

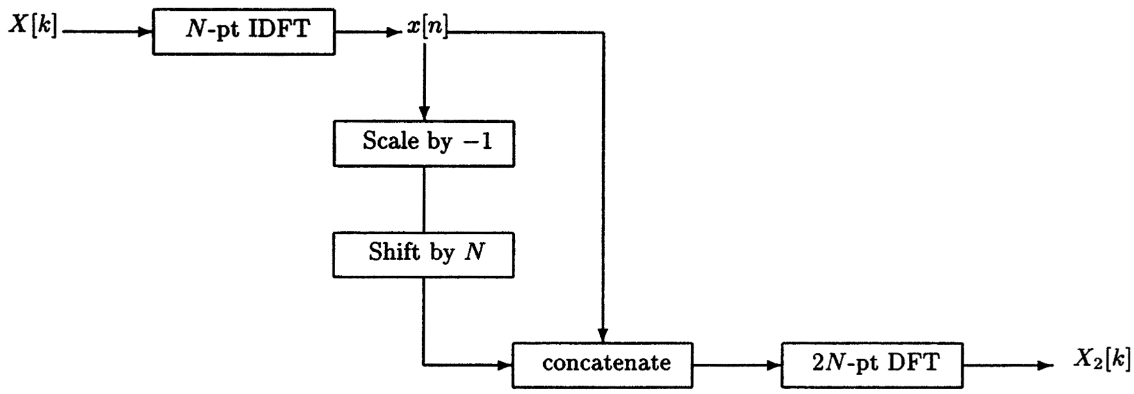
Then if $X_2[k] = X_1(z)\Big|_{z=\frac{1}{2}e^{-j\frac{2\pi k}{N}}}, \quad k = 0, \ldots, N$ we have

$$x_1[n] \left(\tfrac{1}{2}\right)^{-n} = x_2\left[((-n))_N\right], \quad n = 0, \ldots, N-1.$$

8.35. (a) Since

$$x_2[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ -x[n-N], & N \le n \le 2N-1 \\ 0, & \text{otherwise} \end{cases}$$

If $X[k]$ is known, $x_2[n]$ can be constructed by :

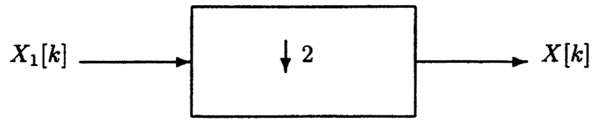


(b) To obtain $X[k]$ from $X_1[k]$, we might try to take the inverse DFT (2N-pt) of $X_1[k]$, then take the N-pt DFT of $x_1[n]$ to get $X[k]$.

However, the above approach is highly inefficient. A more reasonable approach may be achieved if we examine the DFT analysis equations involved. First,

$$\begin{aligned} X_1[k] &= \sum_{n=0}^{2N-1} x_1[n] W_{2N}^{kn}, \qquad 0 \le k \le (2N-1) \\ &= \sum_{n=0}^{N-1} x[n] W_{2N}^{kn} \\ &= \sum_{n=0}^{N-1} x[n] W_N^{(k/2)n}, \quad 0 \le k \le (N-1) \\ X_1[k] &= X[k/2], \quad 0 \le k \le (N-1) \end{aligned}$$

Thus, an easier way to obtain $X[k]$ from $X_1[k]$ is simply to decimate $X_1[k]$ by two.



8.36

Given

$$x[0]=1, x[1]=0, x[2]=2, x[3]=2, x[4]=b, x[5]=1,$$

we have

$$X\left(e^{j\omega}\right)=1+2e^{-j2\omega}+2e^{-j3\omega}+be^{-j4\omega}+e^{-j5\omega}.$$

Define

$$X_1[k]=X\left(e^{j\omega}\right)\Big|_{\omega=\frac{\pi}{2}k}, \quad k=0,1,2,3.$$

Sampling $X\left(e^{j\omega}\right)$ with $N=4$ causes time-domain aliasing of $x[n]$. That is,

$$x_1[0]=x[0]+x[4], x_1[1]=x[1]+x[5], x_1[2]=x[2], x_1[3]=x[3],$$

or

$$x_1[0]=1+b, x_1[1]=1, x_1[2]=2, x_1[3]=2.$$

We are given $x_1[0]=4$, so $b=3$.

Note that this can also be solved by direct calculation.

8.37. (a) Overlap add:

If we divide the input into sections of length L, each section will have an output length:

$$L + 100 - 1 = L + 99$$

Thus, the required length is

$$L = 256 - 99 = 157$$

If we had 63 sections, $63 \times 157 = 9891$, there will be a remainder of 109 points. Hence, we must pad the remaining data to 256 and use another DFT.

Therefore, we require 64 DFTs and 64 IDFTs. Since $h[n]$ also requires a DFT, the total:

65 DFTs and 64 IDFTs

(b) Overlap save:

We require 99 zeros to be padded in from of the sequence. The first 99 points of the output of each section will be discarded. Thus the length after padding is 10099 points. The length of each section overlap is $256 - 99 = 157 = L$.

We require $65 \times 157 = 10205$ to get all 10099 points. Because $h[n]$ also requires a DFT:

66 DFTs and 65 IDFTs

(c) Ignoring the transients at the beginning and end of the direct convolution, each output point requires 100 multiplies and 99 adds.

overlap add:

$$\begin{aligned} \#\text{ mult} &= 129(1024) = 132096 \\ \#\text{ add} &= 129(2048) = 264192 \end{aligned}$$

overlap save:

$$\begin{aligned} \#\text{ mult} &= 131(1024) = 134144 \\ \#\text{ add} &= 131(2048) = 268288 \end{aligned}$$

direct convolution:

$$\begin{aligned} \#\text{ mult} &= 100(10000) = 1000000 \\ \#\text{ add} &= 99(10000) = 990000 \end{aligned}$$

8.38. We have the finite-length sequence:

$$x[n] = 2\delta[n] + \delta[n-1] + \delta[n-3]$$

(i) Suppose we perform the 5-pt DFT:

$$X[k] = 2 + W_5^k + W_5^{3k}, \quad 0 \le k \le 5$$

where $W_5^k = e^{-j(\frac{2\pi}{5})k}$.

(ii) Now, we square the DFT of $x[n]$:

$$\begin{aligned} Y[k] &= X^2[k] \\ &= 2 + 2W_5^k + 2W_5^{3k} \\ &\quad + 2W_5^k + W_5^{2k} + W_5^{5k} \\ &\quad + 2W_5^{3k} + W_5^{4k} + W_5^{6k}, \quad 0 \le k \le 5 \end{aligned}$$

Using the fact $W_5^{5k} = W_5^0 = 1$ and $W_5^{6k} = W_5^k$

$$Y[k] = 3 + 5W_5^k + W_5^{2k} + 4W_5^{3k} + W_5^{4k}, \quad 0 \le k \le 5$$

(a) By inspection,

$$y[n] = 3\delta[n] + 5\delta[n-1] + \delta[n-2] + 4\delta[n-3] + \delta[n-4], \quad 0 \le n \le 5$$

(b) This procedure performs the autocorrelation of a real sequence. Using the properties of the DFT, an alternative method may be achieved with convolution:

$$y[n] = \text{IDFT}\{X^2[k]\} = x[n] * x[n]$$

The IDFT and DFT suggest that the convolution is circular. Hence, to ensure there is no aliasing, the size of the DFT must be $N \ge 2M - 1$ where M is the length of $x[n]$. Since $M = 3, N \ge 5$.

8.39. (a) Since $x[n]$ is 50 points long, and $h[n]$ is 10 points long, the linear convolution $y[n] = x[n] * h[n]$ must be $50 + 10 - 1 = 59$ pts long.

(b) Circular convolution = linear convolutin + aliasing.
If we let $y[n] = x[n] * h[n]$, a more mathematical statement of the above is given by

$$x[n] \; N \; h[n] = \sum_{r=-\infty}^{\infty} y[n + rN], \qquad 0 \le n \le (N-1)$$

For $N = 50$,

$$x[n] \; 50 \; h[n] = y[n] + y[n+50], \qquad 0 \le n \le 49$$

We are given: $x[n] \; 50 \; h[n] = 10$
Hence,

$$y[n] + y[n+50] = 10, \qquad 0 \le n \le 49$$

Also, $y[n] = 5, \quad 0 \le n \le 4$.
Using the above information:

$$\begin{array}{llcl}
n = 0 & y[0] + y[50] & = & 10 \\
\vdots & y[50] & = & 5 \\
n = 4 & y[4] + y[54] & = & 10 \\
 & y[54] & = & 5 \\
n = 5 & y[5] + y[55] & = & 10 \\
\vdots & y[55] & = & ? \\
n = 8 & y[8] + y[58] & = & 10 \\
 & y[58] & = & ? \\
n = 9 & y[9] & = & 10 \\
\vdots & & & \\
n = 49 & y[49] & = & 10
\end{array}$$

To conclude, we can determine $y[n]$ for $9 \le n \le 55$ only. (Note that $y[n]$ for $0 \le n \le 4$ is given.)

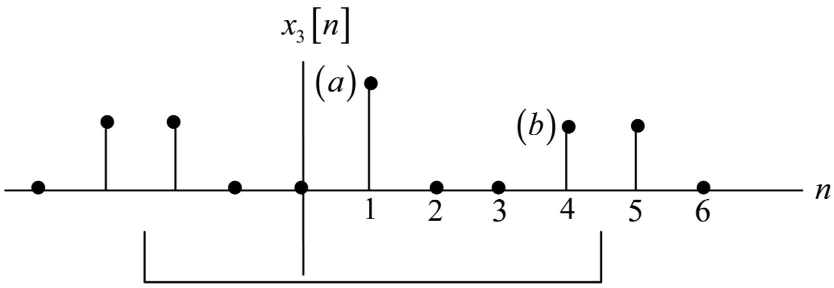
8.40. (i) This corresponds to $x_i[n] = x_i^*[((-n))_N]$, where $N = 5$. Note that this is only true for $x_2[n]$.

(ii) $X_i(e^{jw})$ has linear phase corresponds to $x_i[n]$ having some internal symmetry, this is only true for $x_1[n]$.

(iii) The DFT has linear phase corresponds to $\tilde{x}_i[n]$ (the periodic sequence obtained from $x_i[n]$) being symmetric, this is true for $x_1[n]$ and $x_2[n]$ only.

8.41

A DFT $X[k]$ will exhibit generalized linear phase if the periodic extension of the signal $x[n]$ has even or odd symmetry. Of the three given signals, only $x_3[n]$ has the required symmetry.



The figure shows even symmetry about $n=1$. Therefore for this signal, $\alpha=1$.

For a direct demonstration, we can calculate $X_3[k]$. That is,

$$\begin{aligned}
X_3[k] &= \sum_{n=0}^{6} x[n] e^{-j2\pi\frac{nk}{7}} \\
&= ae^{-j2\pi\frac{k}{7}} + be^{-j2\pi\frac{4k}{7}} + be^{-j2\pi\frac{5k}{7}} \\
&= ae^{-j2\pi\frac{k}{7}} + be^{-j2\pi\frac{4k}{7}} + be^{j2\pi\frac{2k}{7}} \\
&= \left(a + be^{-j2\pi\frac{3k}{7}} + be^{j2\pi\frac{3k}{7}}\right)e^{-j2\pi\frac{k}{7}} \\
&= \left(a + 2b\cos(3k/7)\right)e^{-j2\pi\frac{k}{7}},
\end{aligned}$$

which is in the required form with $\alpha=1$.

8.42

Initially we have

$$x[0]=1+j3, x[1]=0, x[2]=2-j, x[3]=0,$$
$$x[4]=-1-j3, x[5]=0, x[6]=-1+j3, x[7]=0.$$

Let $X[k]$ represent the DFT of $x[n]$. Then if $\hat{X}[k]=X[k+1]$ we have $\hat{x}[n]=x[n]e^{-j\frac{\pi}{4}n}$. That is,

$$\hat{x}[0]=1+j3, \hat{x}[1]=0, \hat{x}[2]=-1-j2, \hat{x}[3]=0,$$
$$\hat{x}[4]=1+j3, \hat{x}[5]=0, \hat{x}[6]=-3-j, \hat{x}[7]=0.$$

If we compress $\hat{X}[k]$ by a factor of $M=2$ we obtain $Y[k]=\hat{X}[2k], \quad k=0,1,2,3$. That is, $Y[k]=X[2k+1], \quad k=0,1,2,3$. The inverse DFT $y[n]$ of $Y[k]$ is the quantity we seek. Compressing $\hat{X}[k]$ in the frequency domain will cause aliasing of $\hat{x}[n]$ in the time domain. We have

$$\begin{aligned}
y[n] &= \tfrac{1}{4}\sum_{k=0}^{3} Y[k]e^{j2\pi\frac{kn}{4}} \\
&= \tfrac{1}{4}\sum_{k=0}^{3} \hat{X}[2k]e^{j2\pi\frac{kn}{4}} \\
&= \tfrac{1}{4}\sum_{k=0}^{3}\left(\sum_{m=0}^{7}\hat{x}[m]e^{-j2\pi\frac{(2k)m}{8}}\right)e^{j2\pi\frac{kn}{4}} \\
&= \sum_{m=0}^{7}\hat{x}[m]\tfrac{1}{4}\sum_{k=0}^{3}e^{j2\pi\frac{(n-m)k}{4}} \\
&= \sum_{m=0}^{7}\hat{x}[m]\sum_{r=-\infty}^{\infty}\delta[n-m-4r], \quad n=0,1,2,3.
\end{aligned}$$

The last expression is the convolution of $\hat{x}[n]$ with a train of impulses spaced every four samples. Taking into account that $\hat{x}[n]$ has a length of eight samples, we have

$$y[n]=\hat{x}[n]+\hat{x}[n+1], \quad n=0,1,2,3.$$

That is,

$$y[0]=2+j6, y[1]=0, y[2]=-4-j3, y[3]=0.$$

Then

$$y_r[0]=2, y_r[1]=0, y_r[2]=-4, y_r[3]=0$$

and

$$y_i[0]=6, y_i[1]=0, y_i[2]=-3, y_i[3]=0.$$

Note that this problem can also be solved by direct calculation.

8.43

A. $R_{xx}[m] = x[n] * x[-n]\big|_{n=m} = 0$ for $m < -1023$, $m > 1023$.

$$\left|X_N[k]\right|^2 = X_N^*[k]X_N[k] = N\text{-point DFT}\left\{x\left[((-n))_N\right] * x[n]\right\}$$

$$g_N[m] = N\text{-point IDFT}\left\{N\text{-point DFT}\left\{x\left[((-n))_N\right] * x[n]\right\}\right\}$$

$$= \begin{cases} x\left[((-n))_N\right] * x[n]\big|_{n=m}, & 0 \le m \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

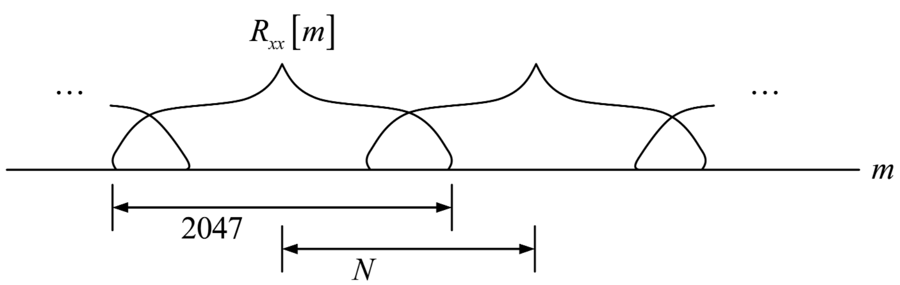
For $N = 2047$ and $0 \le m \le N-1$, $g_N[m] = x\left[((-n))_{2047}\right] * x[n]\big|_{n=m}$. For smaller N, circular convolution introduces time aliasing. To obtain $R_{xx}[m]$ use

$$R_{xx}[m] = \begin{cases} g_{2047}[m], & 0 \le m < 1024 \\ g_{2047}[2047+m], & -1024 < m \le -1. \end{cases}$$

B. For $0 \le m \le N-1$, $g_N[m] = x\left[((-n))_N\right] * x[n]\big|_{n=m}$. We would now like to use a variant of the earlier technique but for smaller N. For general evan N our "post-processing" step is

$$\hat{R}_{xx}[m] = \begin{cases} g_N[m], & 0 \le m < \dfrac{N+1}{2} \\ g_N[N+m], & -\dfrac{(N+1)}{2} < m \le -1. \end{cases}$$

If we want $\hat{R}_{xx}[m] = R_{xx}[m]$ for $|m| \le 10$, we need to ensure that the time aliasing from circular convolution does not affect $g_N[m]$ for $0 \le m \le 10$ and for $N-11 \le m \le N-1$.



For the lowest possible N, $N = 1024$, we have only $g_N[0]$ unaffected by aliasing. For $N = 1025$, $g_N[0]$, $g_N[1]$, and $g_N[1024]$ are unaffected, etc. Keeping this trent in mind we pick $N = 1034$. Our post-processing step becomes

$$R_{xx}[m] = \begin{cases} g_{1034}[m], & 0 \le m \le 10 \\ g_{1034}[1034+m], & -10 \le m \le -1. \end{cases}$$

8.44. Problem 5 in Fall2005 midterm exam.

Problem

In Figure 1, $x[n]$ is a finite sequence of length 1024. The sequence $R[k]$ is obtained by taking the 1024-point DFT of $x[n]$ and compressing the result by 2.

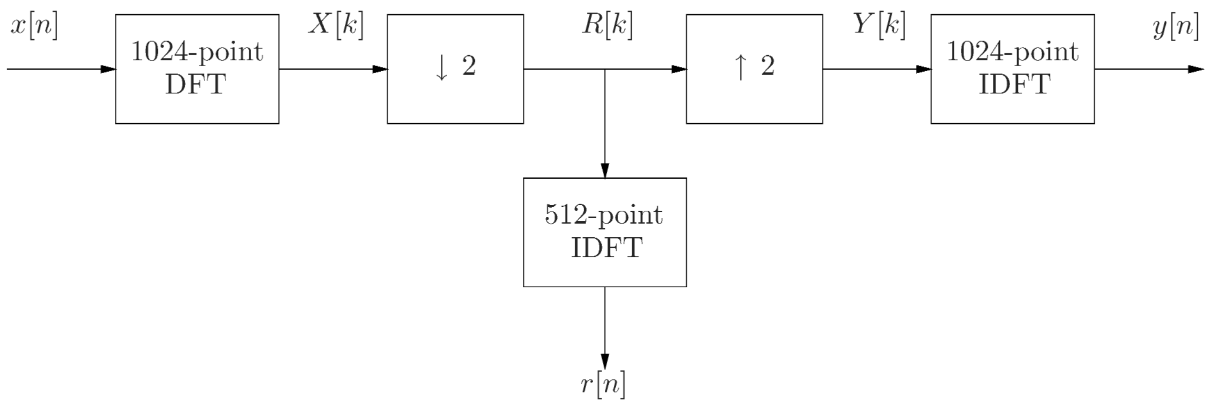


Figure 1:

(a) Choose the most accurate statement for $r[n]$, the 512-point inverse DFT of $R[k]$. Justify your choice in a few concise sentences.

A. $r[n] = x[n],\ 0 \le n \le 511$

B. $r[n] = x[2n],\ 0 \le n \le 511$

C. $r[n] = x[n] + x[n+512],\ 0 \le n \le 511$

D. $r[n] = x[n] + x[-n+512],\ 0 \le n \le 511$

E. $r[n] = x[n] + x[1023-n],\ 0 \le n \le 511$

In all cases $r[n] = 0$ outside $0 \le n \le 511$.

(b) The sequence $Y[k]$ is obtained by expanding $R[k]$ by 2. Choose the most accurate statement for $y[n]$, the 1024-point inverse DFT of $Y[k]$. Justify your choice in a few concise sentences.

A. $y[n] = \begin{cases} \frac{1}{2}\left(x[n] + x[n+512]\right), & 0 \le n \le 511 \\ \frac{1}{2}\left(x[n] + x[n-512]\right), & 512 \le n \le 1023 \end{cases}$

B. $y[n] = \begin{cases} x[n], & 0 \le n \le 511 \\ x[n-512], & 512 \le n \le 1023 \end{cases}$

C. $y[n] = \begin{cases} x[n], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

D. $y[n] = \begin{cases} x[2n], & 0 \le n \le 511 \\ x[2(n-512)], & 512 \le n \le 1023 \end{cases}$

E. $y[n] = \frac{1}{2}\left(x[n] + x[1023-n]\right),\ 0 \le n \le 1023$

In all cases $y[n] = 0$ outside $0 \le n \le 1023$.

Solution from Fall05 Midterm

Answer: C

Compressing the 1024-point DFT $X[k]$ by 2 undersamples the DTFT $X(e^{j\omega})$. Undersampling in the frequency domain corresponds to aliasing in the time domain. In this specific case, the second half of $x[n]$ is folded onto the first half, as described by statement C.

Answer: A

We can first think about how expanding by 2 in the time domain affects the DFT. Expanding a time sequence $x[n]$ by 2 compresses the DTFT $X(e^{j\omega})$ by 2 in frequency. As a result, the $2N$-point DFT of the expanded sequence samples *two* periods of $X(e^{j\omega})$ and equals two copies of the N-point DFT $X[k]$.

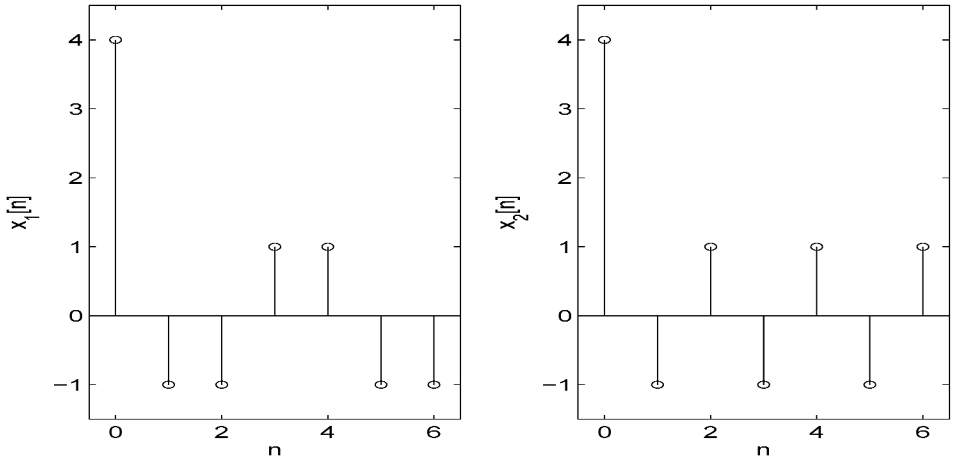
By duality, expanding the DFT $R[k]$ by 2 corresponds to repeating $r[n]$ back-to-back, with an additional scaling by $\frac{1}{2}$. Thus statement A is correct.

Alternatively, $Y[k] = \frac{1}{2}\left(1 + (-1)^k\right) X[k]$. Modulating the DFT by $(-1)^k$ corresponds to a circular time shift of $N/2 = 512$.

8.45. Problem 3 in Spring2005 final exam.

Problem

Below are two finite-length sequences $x_1[n]$ and $x_2[n]$ of length 7. $X_i(e^{j\omega})$ denotes the DTFT of $x_i[n]$, and $X_i[k]$ denotes the seven-point DFT of $x_i[n]$.



For each of the sequences $x_1[n]$ and $x_2[n]$, indicate whether each one of the following properties holds:

(a) $X_i(e^{j\omega})$ can be written in the form

$$X_i(e^{j\omega}) = A_i(\omega)e^{j\alpha_i\omega}, \quad \text{for } \omega \in (-\pi, \pi),$$

where $A_i(\omega)$ is real and α_i is a constant.

(b) $X_i[k]$ can be written in the form

$$X_i[k] = B_i[k]e^{j\beta_i k},$$

where $B_i[k]$ is real and β_i is a constant.

Solution from Spring2005

For the case of a length-7 sequence, we have generalized linear phase if and only if there is even or odd symmetry about the $n = 3$ sample. Neither sequence has this property.

Linear phase in the DFT arises if the *periodic extension* of the signal has even or odd symmetry. The periodic extension of $x_1[n]$ has the required symmetry but the periodic extension of $x_2[n]$ does not.

8.46. $x_2[n]$ is $x_1[n]$ time aliased to have only N samples. Since

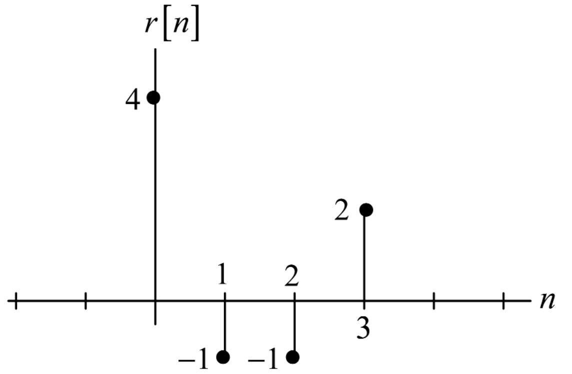
$$x_1[n] = (\frac{1}{3})^n u[n],$$

We get:

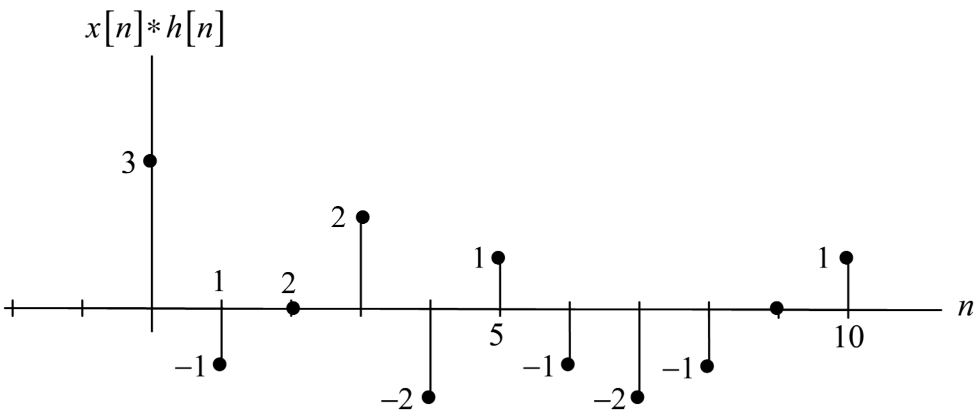
$$x_2[n] = \begin{cases} \frac{(\frac{1}{3})^n}{1-(\frac{1}{3})^N} & , \quad n = 0, ..., N-1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

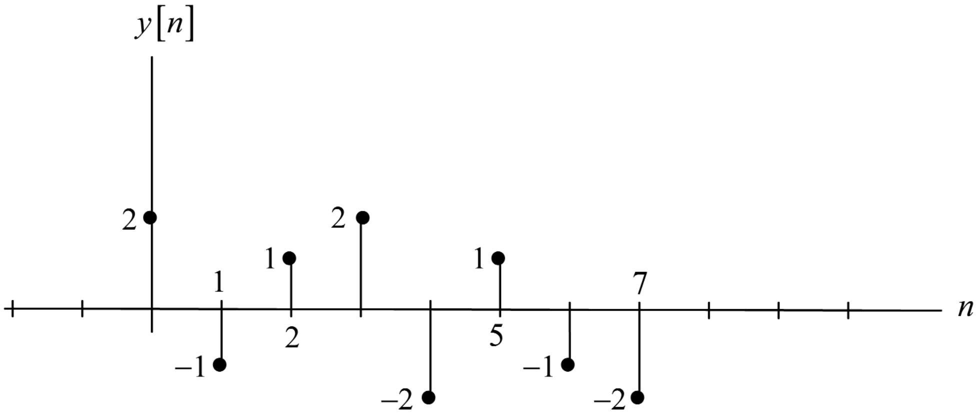
8.47

A. $R[k]$ is the four-point DFT of $x[n]$. Inverting $R[k]$ creates time aliasing.

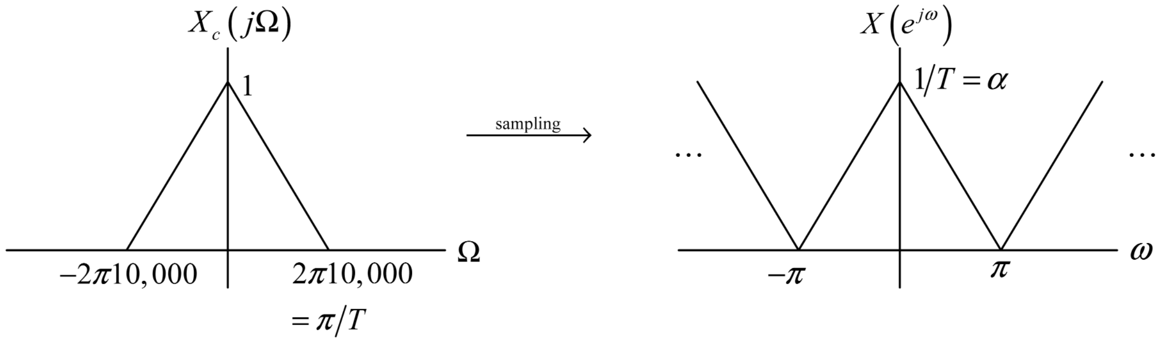


B. The product of the DFTs corresponds to circular convolution, i.e., linear convolution followed by time aliasing.

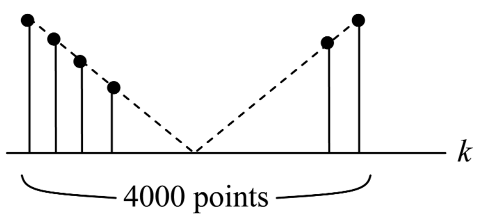




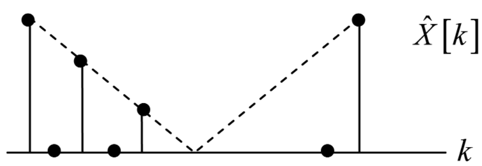
8.48



Desired DFT ($X[k]$):



But Eq. (2) is equivalent to the expansion by two of the 2000-point DFT of $x_c(nT)$. That is,



$$\hat{X}[k] = \begin{cases} X[k], & k \text{ even} \\ 0, & k \text{ odd.} \end{cases}$$

8.49

A. Given

$$c_{xy}[n] = y[-n] * x[n],$$

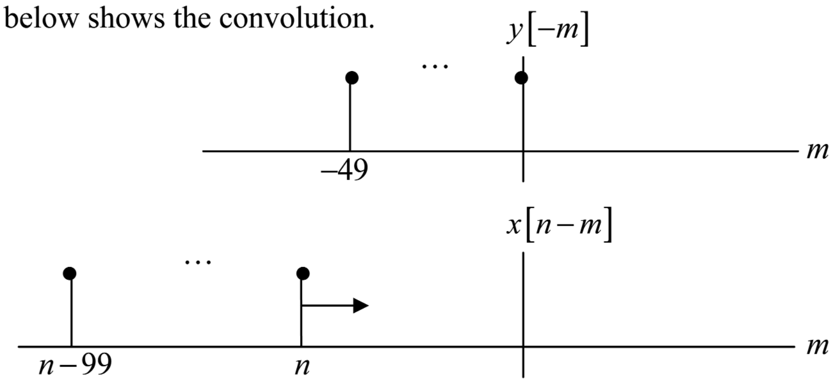
we have

$$C_{xy}(e^{j\omega}) = F[y[-n]]X(e^{j\omega}).$$

Now $F[y[-n]] = Y(e^{-j\omega})$, and $Y(e^{-j\omega}) = Y^*(e^{j\omega})$ since $y[n]$ is real. Substituting gives

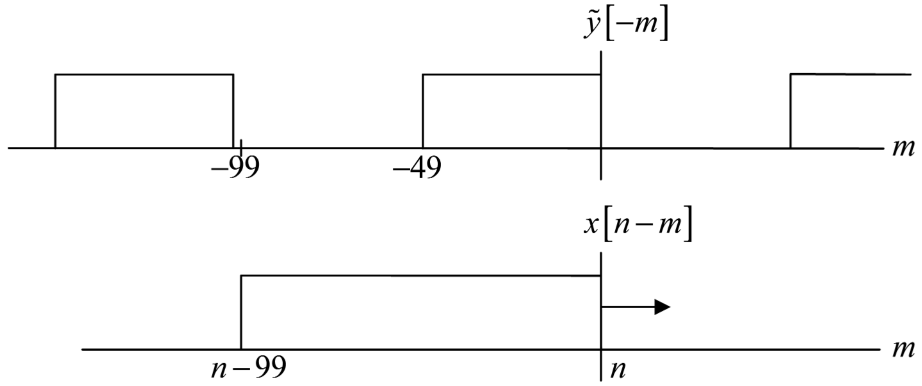
$$C_{xy}(e^{j\omega}) = Y^*(e^{j\omega})X(e^{j\omega}).$$

B. Given that $x[n]$ has a length of $N = 100$ samples and $y[n]$ has a length of $L = 50$ samples, then $y[-n]$ will also have a length of $L = 50$ samples and $c_{xy}[n] = y[-n] * x[n]$ will have a length of $N + L - 1 = 149$ samples. The construction shown below shows the convolution.



The construction shows that $c_{xy}[n]$ can be nonzero only over the range $N_1 \le n \le N_2$, where $N_1 = -49$ and $N_2 = 99$.

C. The prescribed procedure calculates the N-point circular convolution of $x[n]$ with $y[-n]$. The construction below shows the convolution.



Correct values of $c_{xy}[n]$ will be obtained for $0 \le n \le 20$ if $N \ge 100$.

8.50. We have a 10-point sequence, $x[n]$. We want a modified sequence, $x_1[n]$, such that the 10-pt. DFT of $x_1[n]$ corresponds to

$$X_1[k] = X(z)|_{z=\frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}}$$

Recall the definition of the Z-transform of $x[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Since $x[n]$ is of finite duration ($N = 10$), we assume:

$$x[n] = \begin{cases} \text{nonzero}, & 0 \le n \le 9 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$X(z) = \sum_{n=0}^{9} x[n]z^{-n}$$

Substituting in $z = \frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}$:

$$X(z)|_{z=\frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}} = \sum_{n=0}^{9} x[n]\left(\frac{1}{2}e^{j[(2\pi k/10)+(\pi/10)]}\right)^{-n}$$

We seek the signal $x_1[n]$, whose 10-pt. DFT is equivalent to the above expression. Recall the analysis equation for the DFT:

$$X_1[k] = \sum_{n=0}^{9} x_1[n]W_{10}^{kn}, \quad 0 \le k \le 9$$

Since $W_{10}^{kn} = e^{-j(2\pi/10)kn}$, by comparison

$$x_1[n] = x[n]\left(\frac{1}{2}e^{j(\pi/10)}\right)^{-n}$$

8.51. (a)

$$\frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{2\pi}{8}k9} = \frac{1}{8}\sum_{k=0}^{7} X[k]e^{j\frac{2\pi}{8}k} = x[1].$$

(b)

$$\begin{aligned}
V[k] &= X(z)|_{z=2e^{j(\frac{2\pi k+\pi}{8})}} \\
&= \sum_{n=-\infty}^{n=\infty} x[n]z^{-n}|_{z=2e^{j(\frac{2\pi k+\pi}{8})}} \\
&= \sum_{n=0}^{n=8} x[n]z^{-n}|_{z=2e^{j(\frac{2\pi k+\pi}{8})}} \\
&= \sum_{n=0}^{n=8} x[n](2e^{j\frac{\pi}{8}})^{-n}e^{-j\frac{2\pi k}{8}n} \\
&= \sum_{n=0}^{n=8} v[n]e^{-j\frac{2\pi k}{8}n}.
\end{aligned}$$

We thus conclude that

$$v[n] = x[n](2e^{j\frac{\pi}{8}})^{-n}.$$

(c)

$$\begin{aligned}
w[n] &= \frac{1}{4}\sum_{k=0}^{3} W[k]W_4^{-kn} \\
&= \frac{1}{4}\sum_{k=0}^{3}(X[k] + X[k+4])e^{+j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\sum_{k=0}^{3} X[k]e^{+j\frac{2\pi}{4}kn} + \frac{1}{4}\sum_{k=0}^{3} X[k+4]e^{+j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\sum_{k=0}^{3} X[k]e^{+j\frac{2\pi}{4}kn} + \frac{1}{4}\sum_{k=4}^{7} X[k]e^{+j\frac{2\pi}{4}kn} \\
&= \frac{1}{4}\sum_{k=0}^{7} X[k]e^{+j\frac{2\pi}{8}k2n} \\
&= 2x[2n].
\end{aligned}$$

We thus conclude that

$$w[n] = 2x[2n].$$

(d) Note that $Y[k]$ can be written as:

$$\begin{aligned}
Y[k] &= X[k] + (-1)^k X[k] \\
&= X[k] + W_8^{4k} X[k].
\end{aligned}$$

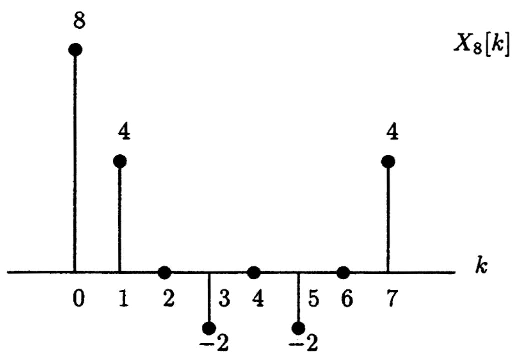
Using the DFT properties, we thus conclude that

$$y[n] = x[n] + x[((n-4))_8].$$

8.52. (a) Let $n = 0, ..., 7$, we can write $x[n]$ as:

$$\begin{aligned} x[n] &= 1 + \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}) - \frac{1}{4}(e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}) \\ &= 1 + \frac{1}{2}e^{j\frac{2\pi}{8}n} + \frac{1}{2}e^{j\frac{2\pi}{8}n7} - \frac{1}{4}e^{j\frac{2\pi}{8}n3} - \frac{1}{4}e^{j\frac{2\pi}{8}n5} \\ &= \frac{1}{8}(8 + 4e^{j\frac{2\pi}{8}n} + 4e^{j\frac{2\pi}{8}n7} - 2e^{j\frac{2\pi}{8}n3} - 2e^{j\frac{2\pi}{8}n5}) \\ &= \frac{1}{8}\sum_{k=0}^{7} X_8[k]e^{j\frac{2\pi k}{8}n} \end{aligned}$$

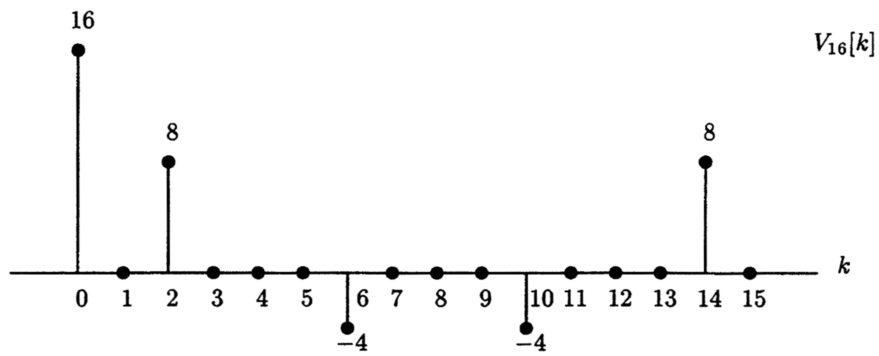
We thus get the following plot for $X_8[k]$:



(b) Now let $n = 0, ..., 15$, we can write $v[n]$ as:

$$\begin{aligned} v[n] &= 1 + \frac{1}{2}(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}) - \frac{1}{4}(e^{j\frac{3\pi}{4}n} + e^{-j\frac{3\pi}{4}n}) \\ &= 1 + \frac{1}{2}e^{j\frac{2\pi}{16}n2} + \frac{1}{2}e^{j\frac{2\pi}{16}n14} - \frac{1}{4}e^{j\frac{2\pi}{16}n6} - \frac{1}{4}e^{j\frac{2\pi}{16}n10} \\ &= \frac{1}{16}(16 + 8e^{j\frac{2\pi}{16}n2} + 8e^{j\frac{2\pi}{16}n14} - 4e^{j\frac{2\pi}{16}n6} - 4e^{j\frac{2\pi}{16}n10}) \\ &= \frac{1}{16}\sum_{k=0}^{15} V_{16}[k]e^{j\frac{2\pi k}{16}n} \end{aligned}$$

We thus get the following plot for $V_{16}[k]$:



(c)

$$|X_{16}[k]| = X(e^{j\omega})|_{\omega=\frac{2\pi}{16}k} \qquad 0 \le k \le 15$$

where $X(e^{j\omega})$ is the Fourier transform of $x[n]$.
Note that $x[n]$ can be expressed as:

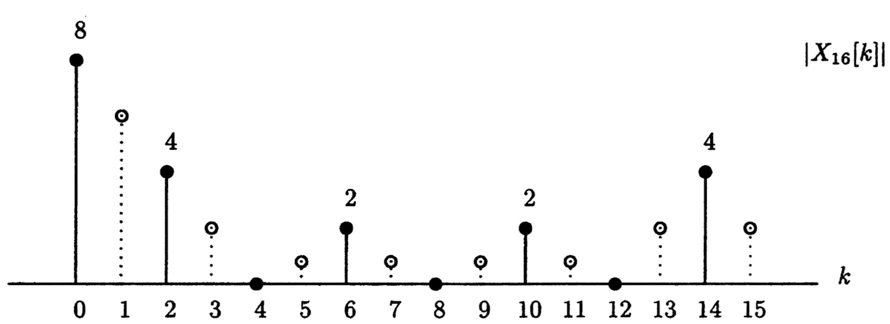
$$x[n] = y[n]w[n]$$

where:

$$y[n] = 1 + \cos(\frac{\pi n}{4}) - \frac{1}{2}\cos(\frac{3\pi n}{4})$$

and $w[n]$ is an eight-point rectangular window.
$|X_{16}[k]|$ will therefore have as its even points the sequence $|X_8[k]|$ .The odd points will correspond to the bandlimited interpolation between the even-point samples. The values that we can find exactly by inspection are thus:

$$|X_{16}[k]| = |X_8[k/2]| \qquad k = 0, 2, 4, ..., 14.$$



9.1. There are several possible approaches to this problem. Two are presented below.

Solution #1: Use the program to compute the DFT of $X[k]$, yielding the sequence $g[n]$.

$$g[n] = \sum_{k=0}^{N-1} X[k]e^{-j2\pi kn/N}$$

Then, compute

$$x[n] = \frac{1}{N}g[((N-n))_N]$$

for $n = 0, \dots, N-1$. We demonstrate that this solution produces the inverse DFT below.

$$\begin{aligned} x[n] &= \frac{1}{N}g[((N-n))_N] \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{-j2\pi k(N-n)/N} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} \end{aligned}$$

Solution #2: Take the complex conjugate of $X[k]$, and then compute its DFT using the program, yielding the sequence $f[n]$.

$$f[n] = \sum_{k=0}^{N-1} X^*[k]e^{-j2\pi kn/N}$$

Then, compute

$$x[n] = \frac{1}{N}f^*[n]$$

We demonstrate that this solution produces the inverse DFT below.

$$\begin{aligned} x[n] &= \frac{1}{N}f^*[n] \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N} \end{aligned}$$

9.2. Multiplying out the terms, we find that

$$(A - B)D + (C - D)A = AD - BD + AC - AD = AC - BD = X$$

$$(A - B)D + (C + D)B = AD - BD + BC + BD = AD + BC = Y$$

Thus, the algorithm is verified.

9.3. First, we derive a relationship between the $X_1(e^{j\omega})$ and $X(e^{j\omega})$ using the shift and time reversal properties of the DTFT.

$$\begin{aligned} x_1[n] &= x[32-n] \\ X_1(e^{j\omega}) &= X(e^{-j\omega})e^{-j32\omega} \end{aligned}$$

Looking at the figure we see that calculating $y[32]$ is just an application of the Goertzel algorithm with $k = 7$ and $N = 32$. Therefore,

$$\begin{aligned} y[32] &= X_1[7] \\ &= X_1(e^{j\omega})\Big|_{\omega=\frac{2\pi 7}{32}} \\ &= X(e^{-j\omega})e^{-j\omega 32}\Big|_{\omega=\frac{7\pi}{16}} \\ &= X(e^{-j\frac{7\pi}{16}})e^{-j(\frac{7\pi}{16})32} \\ &= X(e^{-j\frac{7\pi}{16}}) \end{aligned}$$

Note that if we put $x[n]$ through the system directly, we would be evaluating $X(z)$ at the conjugate location on the unit circle, i.e., at $\omega = +7\pi/16$.

9.4. The figure corresponds to the flow graph of a second-order recursive system implementing Goertzel's algorithm. This system finds $X[k]$ for $k = 7$, which corresponds to a frequency of

$$\omega_k = \frac{14\pi}{32} = \frac{7\pi}{16}$$

9.5 . $X(e^{j6\pi/8})$ **corresponds to the** $k = 3$ **index of a length** $N = 8$ **DFT. Using the flow graph of the second-order recursive system for Goertzel's algorithm,**

$$\begin{aligned} a &= 2\cos\left(\frac{2\pi k}{N}\right) \\ &= 2\cos\left(\frac{2\pi(3)}{8}\right) \\ &= -\sqrt{2} \\ b &= -W_N^k \\ &= -e^{-j6\pi/8} \\ &= \frac{1+j}{\sqrt{2}} \end{aligned}$$

9.6. (a) The "gain" along the emphasized path is $-W_N^2$.

(b) In general, there is only one path between each input sample and each output sample.

(c) $x[0]$ to $X[2]$: The gain is 1.
$x[1]$ to $X[2]$: The gain is W_N^2.
$x[2]$ to $X[2]$: The gain is $-W_N^0 = -1$.
$x[3]$ to $X[2]$: The gain is $-W_N^0 W_N^2 = -W_N^2$.
$x[4]$ to $X[2]$: The gain is $W_N^0 = 1$.
$x[5]$ to $X[2]$: The gain is $W_N^0 W_N^2 = W_N^2$.
$x[6]$ to $X[2]$: The gain is $-W_N^0 W_N^0 = -1$.
$x[7]$ to $X[2]$: The gain is $-W_N^0 W_N^0 W_N^2 = -W_N^2$, as in Part (a).
Now

$$\begin{aligned}
X[2] &= \sum_{n=0}^{7} x[n] W_8^{2n} \\
&= x[0] + x[1]W_8^2 + x[2]W_8^4 + x[3]W_8^6 + x[4]W_8^8 + x[5]W_8^{10} + x[6]W_8^{12} \\
&\quad + x[7]W_8^{14} \\
&= x[0] + x[1]W_8^2 + x[2](-1) + x[3](-W_8^2) + x[4](1) + x[5]W_8^2 \\
&\quad + x[6](-1) + x[7](-W_8^2)
\end{aligned}$$

Each input sample contributes the proper amount to the output DFT sample.

9.7. (a) The input should be placed into A[r] in bit-reversed order.

$$\begin{aligned} A[0] &= x[0] \\ A[1] &= x[4] \\ A[2] &= x[2] \\ A[3] &= x[6] \\ A[4] &= x[1] \\ A[5] &= x[5] \\ A[6] &= x[3] \\ A[7] &= x[7] \end{aligned}$$

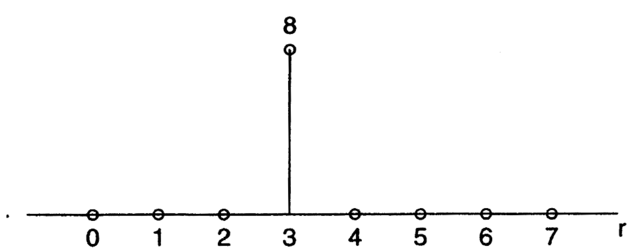
The output should then be extracted from D[r] in sequential order.

$$X[k] = D[k], \quad k = 0, \ldots, 7$$

(b) First, we find the DFT of $(-W_N)^n$ for $N = 8$.

$$\begin{aligned} X[k] &= \sum_{n=0}^{7} (-W_8)^n W_8^{nk} \\ &= \sum_{n=0}^{7} (-1)^n W_8^n W_8^{nk} \\ &= \sum_{n=0}^{7} (W_8^{-4})^n W_8^n W_8^{nk} \\ &= \sum_{n=0}^{7} W_8^{n(k-3)} \\ &= \frac{1 - W^{k-3}}{1 - W_8^{k-3}} \\ &= 8\delta[k-3] \end{aligned}$$

A sketch of D[r] wis provided below.



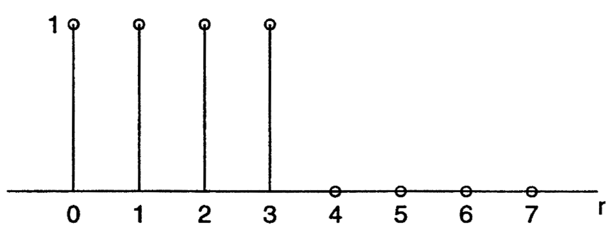
(c) First, the array $D[r]$ is expressed in terms of $C[r]$.

$$\begin{aligned} D[0] &= C[0] + C[4] \\ D[1] &= C[1] + C[5]W_8^1 \\ D[2] &= C[2] + C[6]W_8^2 \\ D[3] &= C[3] + C[7]W_8^3 \\ D[4] &= C[0] - C[4] \\ D[5] &= C[1] - C[5]W_8^1 \\ D[6] &= C[2] - C[6]W_8^2 \\ D[7] &= C[3] - C[7]W_8^3 \end{aligned}$$

Solving this system of equations for $C[r]$ gives

$$\begin{aligned}
C[0] &= (D[0] + D[4])/2 \\
C[1] &= (D[1] + D[5])/2 \\
C[2] &= (D[2] + D[6])/2 \\
C[3] &= (D[3] + D[7])/2 \\
C[4] &= (D[0] - D[4])/2 \\
C[5] &= (D[1] - D[5])W_8^{-1}/2 \\
C[6] &= (D[2] - D[6])W_8^{-2}/2 \\
C[7] &= (D[3] - D[7])W_8^{-3}/2
\end{aligned}$$

for $r = 0, 1, ..., 7$. A sketch of C[r] is provided below.



9.8. (a) In any stage, $N/2$ butterflies must be computed. In the mth stage, there are 2^{m-1} different coefficients.

(b) Looking at figure 9.10, we notice that the coefficients are

$$\begin{array}{ll} \text{1st stage:} & W_8^0 \\ \text{2nd stage:} & W_8^0, W_8^2 \\ \text{3rd stage:} & W_8^0, W_8^1, W_8^2, W_8^3 \end{array}$$

Here we have listed the *different* coefficients only. The values above correspond to the impulse response

$$h[n] = W_{2^m}^n u[n]$$

which can be generated by the recursion

$$y[n] = W_{2^m} y[n-1] + x[n]$$

Using this recursion, we only generate a sequence of length $L = 2^{m-1}$, which consists of the different coefficients. Then, the remaining $\frac{N}{2} - L$ coefficients are found by repeating these L coefficients.

(c) The difference equation from Part (b) is periodic, since

$$\begin{aligned} h[n] &= W_{2^m}^n u[n] \\ &= e^{-j2\pi n/2^m} u[n] \end{aligned}$$

has a period $R = 2^m$. Thus, the frequency of this oscillator is

$$\omega = \frac{2\pi}{2^m}$$

9.9. *Answer 3*

Decimation in Time:	**The figure is the basic butterfly with $r = 2$.**
Decimation in Frequency:	**The figure is the end of one butterfly and the start of a second with $r = 2$.**

9.10. This is an application of the causal version of the chirp transform with

$$\begin{array}{rcll} N & = & 20 & \text{The length of } x[n] \\ M & = & 10 & \text{The number of desired samples} \\ \omega_0 & = & \frac{2\pi}{7} & \text{The starting frequency} \\ \Delta\omega & = & \frac{2\pi}{21} & \text{The frequency spacing between samples} \end{array}$$

We therefore have

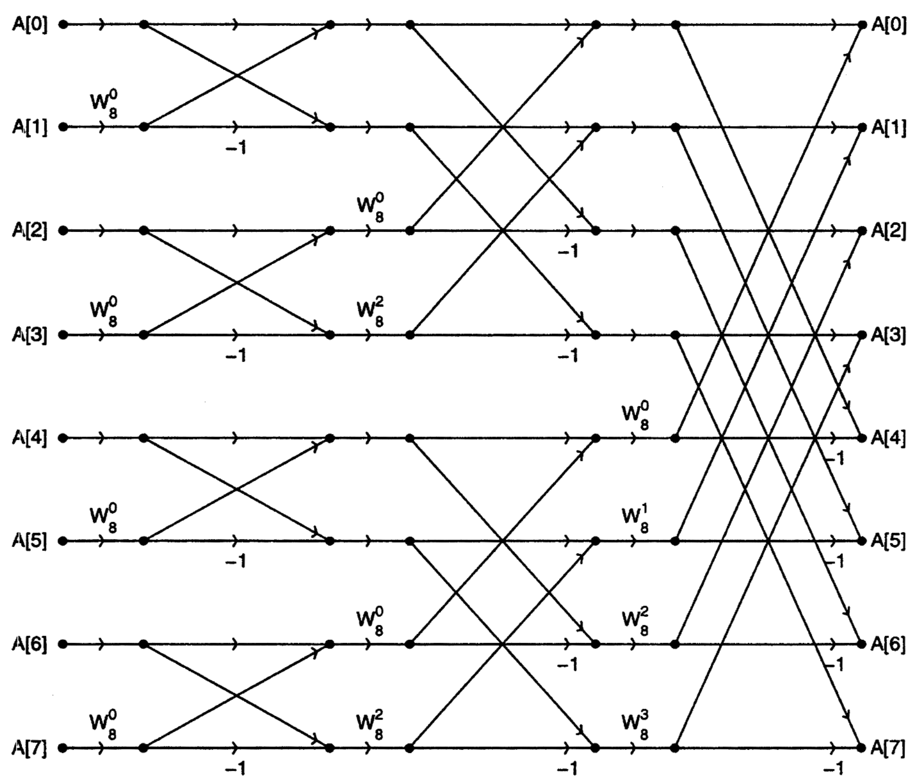
$$y[n+19] = X(e^{j\omega_n}), \quad n = 0, \ldots, 9$$

for $\omega_n = \omega_0 + n\Delta\omega$ or

$$y[n] = X(e^{j\omega_n}), \quad n = 19, \ldots, 28$$

for $\omega_n = \omega_0 + (n-19)\Delta\omega$.

9.11. In this problem, we are using butterfly flow graphs to compute a DFT. These computations are done in place, in an array of registers. An example flow graph for a $N = 8$, (or $v = \log_2 8 = 3$), decimation-in-time DFT is provided below.



(a) The difference between ℓ_1 and ℓ_0 can be found by using the figure above. For example, in the first stage, the array elements $A[4]$ and $A[5]$ comprise a butterfly. Thus, $\ell_1 - \ell_0 = 5 - 4 = 1$. This difference of 1 holds for all the other butterflies in the first stage. Looking at the other stages, we find

$$\text{stage m} = 1: \quad \ell_1 - \ell_0 = 1$$
$$\text{stage m} = 2: \quad \ell_1 - \ell_0 = 2$$
$$\text{stage m} = 3: \quad \ell_1 - \ell_0 = 4$$

From this we find that the difference, in general, is

$$\ell_1 - \ell_0 = 2^{m-1}, \qquad \text{for } m = 1, \ldots, v$$

(b) Again looking at the figure, we notice that for stage 1, there are 4 butterflies with the same twiddle factor. The ℓ_0 for these butterflies are 0, 2, 4, and 6, which we see differ by 2. For stage 2, there are two butterflies with the same twiddle factor. Consider the butterflies with the W_8^0 twiddle factor. The ℓ_0 for these two butterflies are 0 and 4, which differ by 4. Note that in the last stage, there are no butterflies with the same twiddle factor, as the four twiddle factors are unique. Thus, we found

$$\text{stage m} = 1: \quad \Delta\ell_0 = 2$$
$$\text{stage m} = 2: \quad \Delta\ell_0 = 4$$
$$\text{stage m} = 3: \quad \text{n/a}$$

From this, we can generalize the result

$$\Delta\ell_0 = 2^m, \qquad \text{for } m = 1, \ldots, v - 1$$

9.12. This is an application of the causal version of the chirp transform with

N	$=$	12	The length of $x[n]$
M	$=$	5	The number of desired samples
ω_0	$=$	$\frac{2\pi}{19}$	The starting frequency
$\Delta\omega$	$=$	$\frac{2\pi}{10}$	The distance in frequency between samples

Letting $W = e^{-j\Delta\omega}$ we must have

$$r[n] = e^{-j\omega_0 n} W^{n^2/2} = e^{-j\frac{2\pi}{19}n} e^{-j\frac{2\pi}{10}n^2/2}$$

9.13. Reversing the bits (denoted by $\rightarrow$) gives

$$\begin{array}{rcccccc}
0 & = & 0000 & \rightarrow & 0000 & = & 0 \\
1 & = & 0001 & \rightarrow & 1000 & = & 8 \\
2 & = & 0010 & \rightarrow & 0100 & = & 4 \\
3 & = & 0011 & \rightarrow & 1100 & = & 12 \\
4 & = & 0100 & \rightarrow & 0010 & = & 2 \\
5 & = & 0101 & \rightarrow & 1010 & = & 10 \\
6 & = & 0110 & \rightarrow & 0110 & = & 6 \\
7 & = & 0111 & \rightarrow & 1110 & = & 14 \\
8 & = & 1000 & \rightarrow & 0001 & = & 1 \\
9 & = & 1001 & \rightarrow & 1001 & = & 9 \\
10 & = & 1010 & \rightarrow & 0101 & = & 5 \\
11 & = & 1011 & \rightarrow & 1101 & = & 13 \\
12 & = & 1100 & \rightarrow & 0011 & = & 3 \\
13 & = & 1101 & \rightarrow & 1011 & = & 11 \\
14 & = & 1110 & \rightarrow & 0111 & = & 7 \\
15 & = & 1111 & \rightarrow & 1111 & = & 15
\end{array}$$

The new sample order is 0, 8, 4, 12, 2, 10, 6, 14, 1, 9, 5, 13, 3, 11, 7, 15.

9.14. *False.* It is possible by rearranging the order in which the nodes appear in the signal flow graph. However, the computation cannot be carried out in-place.

9.15. Only the $m = 1$ stage will have this form. No other stage of a $N = 16$ radix-2 decimation-in-frequency FFT will have a W_{16} term raised to an odd power.

9.16. The possible values of r for each of the four stages are

$$
\begin{aligned}
m &= 1, \qquad r = 0 \\
m &= 2, \qquad r = 0, 4 \\
m &= 3, \qquad r = 0, 2, 4, 6 \\
m &= 4, \qquad r = 0, 1, 2, 3, 4, 5, 6, 7
\end{aligned}
$$

9.17. Plugging in some values of N for the two programs, we find

N	Program A	Program B
2	4	20
4	16	80
8	64	240
16	256	640
32	1024	1600
64	4096	3840

Thus, we see that a sequence with length $N = 64$ is the shortest sequence for which Program B runs faster than Program A.

9.18. The possible values for r for each of the four stages are

$$\begin{aligned} m &= 1, \qquad r = 0 \\ m &= 2, \qquad r = 0, 4 \\ m &= 3, \qquad r = 0, 2, 4, 6 \\ m &= 4, \qquad r = 0, 1, 2, 3, 4, 5, 6, 7 \end{aligned}$$

where W_N^r is the twiddle factor for each stage. Since the particular butterfly shown has $r = 2$, the stages which have this butterfly are

$$m = 3, 4$$

9.19. The FFT is a decimation-in-time algorithm, since the decimation-in-frequency algorithm has only W_{32}^0 terms in the last stage.

9.20. If the $N_1 = 1021$ point DFT was calculated using the convolution sum directly it would take N_1^2 multiplications. If the $N_2 = 1024$ point DFT was calculated using the FFT it would take $N_2 \log_2 N_2$ multiplications. Assuming that the number of multiplications is proportional to the calculation time the ratio of the two times is

$$\frac{N_1^2}{N_2 \log_2 N_2} = \frac{1021^2}{1024 \log_2 1024} = 101.8 \approx 100$$

which would explain the results.

9.21. (a) Assume $x[n] = 0$, for $n < 0$ and $n > N-1$. From the figure, we see that

$$y_k[n] = x[n] + W_N^k y_k[n-1]$$

Starting with $n = 0$, and iterating this recursive equation, we find

$$\begin{aligned}
y_k[0] &= x[0] \\
y_k[1] &= x[1] + W_N^k x[0] \\
y_k[2] &= x[2] + W_N^k x[1] + W_N^{2k} x[0] \\
&\vdots \\
y_k[N] &= x[N] + W_N^k x[N-1] + \cdots + W_N^{k(N-1)} x[1] + W_N^{kN} x[0] \\
&= 0 + \sum_{\ell=0}^{N-1} W_N^{k(N-\ell)} x[\ell] \\
&= \sum_{\ell=0}^{N-1} W_N^{-k\ell} x[\ell] \\
&= \sum_{\ell=0}^{N-1} x[\ell] W_N^{(N-k)\ell} \\
&= X[N-k]
\end{aligned}$$

(b) Using the figure, we find the system function $Y_k(z)$.

$$\begin{aligned}
Y_k(z) &= X(z) \frac{1 - W_N^{-k} z^{-1}}{1 - 2z^{-1}\cos(\frac{2\pi k}{N}) + z^{-2}} \\
&= X(z) \frac{1 - W_N^{-k} z^{-1}}{(1 - W_N^{-k} z^{-1})(1 - W_N^k z^{-1})} \\
&= \frac{X(z)}{1 - W_N^k z^{-1}}
\end{aligned}$$

Therefore, $y_k[n] = x[n] + W_N^k y_k[n-1]$. This is the same difference equation as in part (a).

9.22

To compute the DTFT of $x[n]$ at a particular frequency point we need the impulse response of the LTI filter to be a complex exponential. If $a = W_N^{-k} = e^{j\frac{2\pi k}{N}}$, we can write

$$y[M] = \sum_{n=0}^{M} x[n] W_N^{-k(M-n)}.$$

We need the output sample to be equal to the DTFT of $x[n]$ evaluated at $\omega = 2\pi/60$, i.e.,

$$\sum_{n=0}^{M} x[n] W_N^{nk} W_N^{-Mk} = \sum_{n=0}^{89} x[n] W_{60}^{n}.$$

We can see that we need $M \ge 89$, otherwise samples of $x[n]$ will be disregarded in the computation. If M is chosen to be an integer multiple of N, then $W_N^{-Mk} = 1$ and we eliminate that term. All that remains is to choose k and N such that $\frac{k}{N} = \frac{1}{60}$.

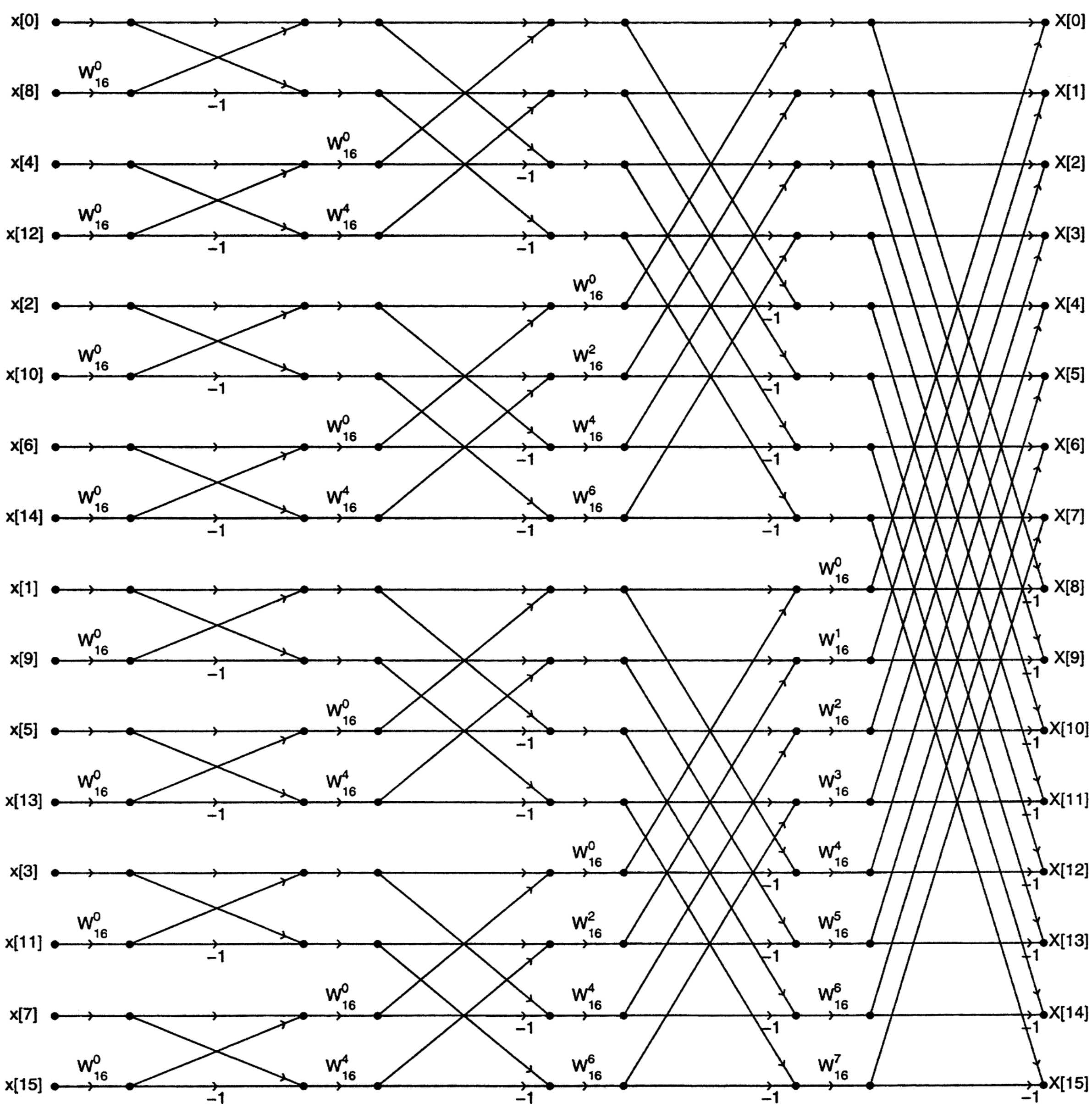
If $k = 1$, $N = 60$, and $M = 120$ we have

$$y[M] = \sum_{n=0}^{120} x[n] W_{60}^{n} W_{60}^{-120} = \sum_{n=0}^{89} x[n] W_{60}^{n},$$

so we can use $a = W_{60}^{-1} = e^{j\frac{2\pi}{60}}$ and $M = 120$.

In fact, any M that is a multiple of 60 and is greater than 89 will work with this choice of a, due to the periodicity of W_{60}^{n}.

9.23. The flow graph for 16 point radix-2 decimation-in-time FFT algorithm is shown below.



To determine the number of real multiplications and additions required to implement the flow graph, consider the number of real multiplications and additions introduced by each of the coefficients W_N^k:

$$
\begin{array}{lll}
W_{16}^0: & 0 \text{ real multiplications} + 0 \text{ real additions} & (W_{16}^0 = 1) \\
W_{16}^4: & 0 \text{ real multiplications} + 0 \text{ real additions} & (W_{16}^4(a+jb) = b - aj) \\
W_{16}^2: & 2 \text{ real multiplications} + 2 \text{ real additions} & (W_{16}^2(a+jb) = \frac{\sqrt{2}}{2}(a+b) + j\frac{\sqrt{2}}{2}(b-a)) \\
W_{16}^6: & 2 \text{ real multiplications} + 2 \text{ real additions} & \text{similarly} \\
W_{16}^1: & 4 \text{ real multiplications} + 2 \text{ real additions} & \\
W_{16}^3: & 4 \text{ real multiplications} + 2 \text{ real additions} & \\
W_{16}^5: & 4 \text{ real multiplications} + 2 \text{ real additions} & \\
W_{16}^7: & 4 \text{ real multiplications} + 2 \text{ real additions} &
\end{array}
$$

The contribution of all the W_N^k's on the flow graph is 28 real multiplications and 20 real additions. The butterflies contribute 0 real multiplications and 32 real additions per stage. Since there are four stages, the butterflies contribute 0 real multiplications and 128 real additions. In total, 28 real multiplications and 148 real additions are required to implement the flow graph.

9.24

Let $X[k]$ be the DFT of the N-point sequence $x[k]$.

1) Swap the real and imaginary parts of $X[k]$. This gives $Y[k] = jX^*[k]$.
2) Applying the FFT to $Y[k]$ gives

$$\begin{aligned} y[n] &= \sum_{k=0}^{N-1} Y[k] e^{-j2\pi\frac{kn}{N}} \\ &= \sum_{k=0}^{N-1} jX^*[k] e^{-j2\pi\frac{kn}{N}} \\ &= j\sum_{k=0}^{N-1} X^*[k] e^{-j2\pi\frac{kn}{N}}. \end{aligned}$$

3) Swap the real and imaginary part of $y[n]$. This gives $z[n] = jy^*[n]$. That is,

$$\begin{aligned} z[n] &= jy^*[n] \\ &= j\left(j\sum_{k=0}^{N-1} X^*[k] e^{-j2\pi\frac{kn}{N}} \right)^* \\ &= \sum_{k=0}^{N-1} X[k] e^{j2\pi\frac{kn}{N}}. \end{aligned}$$

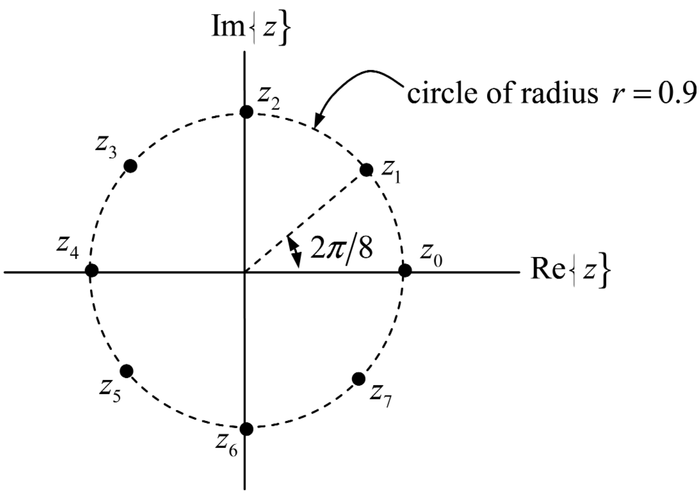
4) Scale by $\frac{1}{N}$. This gives

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi\frac{kn}{N}},$$

which is the IDFT of $X[k]$. The procedure works as claimed.

9.25

A.



B.

$$X(z_k) = \sum_{n=0}^{N-1} x[n] z_k^{-n} = \sum_{n=0}^{N-1} x[n] r^{-n} e^{-j\frac{2\pi}{N}kn} = \tilde{X}[k],$$

where $\tilde{x}[n] = x[n] r^{-n}, \quad 0 \le n \le N-1$.

C. 1. For $n = 0$ to $N-1$

$$\tilde{x}[n] = r^{-n} x[n]$$

2. $\tilde{X}[k] = \text{fft}\{\tilde{x}[n]\}$

3. Done.

9.26. Let

$$y[n] = e^{-j2\pi n/627}x[n]$$

Then

$$Y(e^{j\omega}) = X(e^{j(\omega + \frac{2\pi}{627})})$$

Let $y'[n] = \sum_{m=-\infty}^{\infty} y[n + 256m], \quad 0 \le n \le 255$, and let $Y'[k]$ be the 256 point DFT of $y'[n]$. Then

$$Y'[k] = X\left(e^{j\left(\frac{2\pi k}{256} + \frac{2\pi}{627}\right)}\right)$$

See problem 9.30 for a more in-depth analysis of this technique.

9.27. (a) The problem states that the effective frequency spacing, Δf, should be 50 Hz or less. This constrains N such that

$$\begin{aligned} \Delta f &= \frac{1}{NT} \leq 50 \\ N &\geq \frac{1}{50T} \\ &\geq 200 \end{aligned}$$

Since the sequence length L is 500, and N must be a power of 2, we might conclude that the minimum value for N is 512 for computing the desired samples of the z-transform.

However, we can compute the samples with N equal to 256 by using time aliasing. In this technique, we would zero pad $x[n]$ to a length of 512, then form the 256 point sequence

$$y[n] = \begin{cases} x[n] + x[n+256], & 0 \leq n \leq 255 \\ 0, & \text{otherwise} \end{cases}$$

We could then compute 256 samples of the z-transform of $y[n]$. The effective frequency spacing of these samples would be $1/(NT) \approx 39$ Hz which is lower than the 50 Hz specification.

Note that these samples also correspond to the *even*-indexed samples of a length 512 sampled z-transform of $x[n]$. Problem 9.30 discusses this technique of time aliasing in more detail.

(b) Let

$$y[n] = (1.25)^n x[n]$$

Then, using the modulation property of the z-transform, $Y(z) = X(0.8z)$ and so $Y[k] = X(0.8e^{j2\pi k/N})$.

9.28

The DFT of the 4-point sequence $x[0], x[1], x[2], x[3]$ is given by

$$X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{4}}, \quad k = 0,1,2,3,$$

where

$$X(e^{j\omega}) = \sum_{n=0}^{3} x[n]e^{-j\omega n}.$$

The 8-point DFT of the sequence $x[0], x[1], x[2], x[3], 0, 0, 0, 0$ is given by

$$\hat{X}[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{8}}, \quad k = 0,\ldots,7,$$

where $X(e^{j\omega})$ is as above. We see that

$$X[k] = \hat{X}[2k], \quad k = 0,1,2,3.$$

The cost of the system is the cost of the 8-point DFT, which is \$1.

9.29. (a) Note that we can write the even-indexed values of $X[k]$ as $X[2k]$ for $k = 0, \ldots, (N/2) - 1$. From the definition of the DFT, we find

$$\begin{aligned}
X[2k] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi(2k)n/N} \\
&= \sum_{n=0}^{N/2-1} x[n] e^{-j\frac{2\pi}{(N/2)}kn} \\
&\quad + \sum_{n=0}^{N/2-1} x[n+(N/2)] e^{-j\frac{2\pi}{(N/2)}kn} e^{-j\frac{2\pi}{(N/2)}(N/2)k} \\
&= \sum_{n=0}^{N/2-1} (x[n] + x[n+(N/2)]) e^{-j\frac{2\pi}{(N/2)}kn} \\
&= Y[k]
\end{aligned}$$

Thus, the algorithm produces the desired results.

(b) Taking the M-point DFT $Y[k]$, we find

$$\begin{aligned}
Y[k] &= \sum_{n=0}^{M-1} \sum_{r=-\infty}^{\infty} x[n+rM] e^{-j2\pi kn/M} \\
&= \sum_{r=-\infty}^{\infty} \sum_{n=0}^{M-1} x[n+rM] e^{-j2\pi k(n+rM)/M} e^{j2\pi(rM)k/M}
\end{aligned}$$

Let $l = n + rM$. This gives

$$\begin{aligned}
Y[k] &= \sum_{l=-\infty}^{\infty} x[l] e^{-j2\pi kl/M} \\
&= X(e^{j2\pi k/M})
\end{aligned}$$

Thus, the result from Part (a) is a special case of this result if we let $M = N/2$. In Part (a), there are only two r terms for which $y[n]$ is nonzero in the range $n = 0, \ldots, (N/2) - 1$.

(c) We can write the odd-indexed values of $X[k]$ as $X[2k+1]$ for $k = 0, \ldots, (N/2) - 1$. From the definition of the DFT, we find

$$\begin{aligned}
X[2k+1] &= \sum_{n=0}^{N-1} x[n] e^{-j2\pi(2k+1)n/N} \\
&= \sum_{n=0}^{N-1} x[n] e^{-j2\pi n/N} e^{-j2\pi(2k)n/N} \\
&= \sum_{n=0}^{(N/2)-1} x[n] e^{-j2\pi n/N} e^{-j\frac{2\pi}{(N/2)}kn} + \sum_{n=0}^{(N/2)-1} x[n+(N/2)] e^{-j2\pi[n+(N/2)]/N} e^{-j\frac{2\pi}{N/2}k[n+(N/2)]} \\
&= \sum_{n=0}^{(N/2)-1} \left[(x[n] - x[n+(N/2)]) e^{-j\frac{2\pi}{N}n} \right] e^{-j\frac{2\pi}{(N/2)}kn}
\end{aligned}$$

Let

$$y[n] = \begin{cases} (x[n] - x[n+(N/2)]) e^{-j(2\pi/N)n}, & 0 \leq n \leq (N/2) - 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $Y[k] = X[2k+1]$. Thus, The algorithm for computing the odd-indexed DFT values is as follows.

step 1: Form the sequence

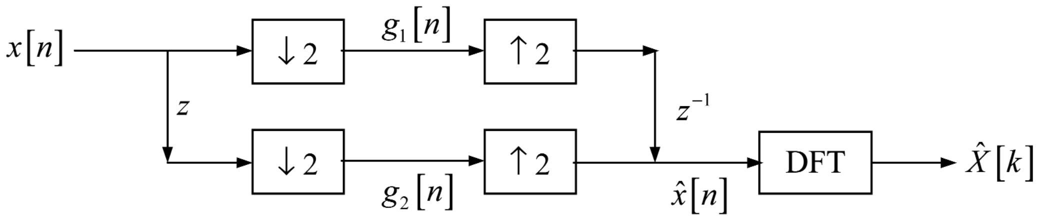
$$y[n] = \begin{cases} (x[n] - x[n+(N/2)]) e^{-j(2\pi/N)n}, & 0 \leq n \leq (N/2) - 1 \\ 0, & \text{otherwise} \end{cases}$$

step 2: Compute the $N/2$ point DFT of $y[n]$, yielding the sequence $Y[k]$.

step 3: The odd-indexed values of $X[k]$ are then $X[k] = Y[(k-1)/2]$, $k = 1, 3, \ldots, N-1$.

9.30

Swapping $g_1[n]$ and $g_2[n]$ corresponds to the following system:



$$G_1\left(e^{j\omega}\right) = \tfrac{1}{2}\left[X\left(e^{j\frac{\omega}{2}}\right) + X\left(e^{j\left(\frac{\omega}{2}-\pi\right)}\right)\right]$$

$$G_2\left(e^{j\omega}\right) = \tfrac{1}{2}\left[e^{j\frac{\omega}{2}}X\left(e^{j\frac{\omega}{2}}\right) + e^{j\left(\frac{\omega}{2}-\pi\right)}X\left(e^{j\left(\frac{\omega}{2}-\pi\right)}\right)\right]$$

$$= \frac{e^{j\frac{\omega}{2}}}{2}\left[X\left(e^{j\frac{\omega}{2}}\right) - X\left(e^{j\left(\frac{\omega}{2}-\pi\right)}\right)\right].$$

Then

$$\hat{X}\left(e^{j\omega}\right) = \frac{e^{-j\omega}}{2}\left[X\left(e^{j\omega}\right) + X\left(e^{j(\omega-\pi)}\right)\right] + \frac{e^{j\omega}}{2}\left[X\left(e^{j\omega}\right) - X\left(e^{j(\omega-\pi)}\right)\right]$$

$$= \tfrac{1}{2}\left[\left(e^{-j\omega} + e^{j\omega}\right)X\left(e^{j\omega}\right) + \left(e^{-j\omega} - e^{j\omega}\right)X\left(e^{j(\omega-\pi)}\right)\right].$$

This gives

$$\hat{X}[k] = \tfrac{1}{2}\left[\left(W_N^k + W_N^{-k}\right)X[k] + \left(W_N^k - W_N^{-k}\right)X\left[\left((k-N/2)\right)_N\right]\right],$$

where $W_N^k = e^{-j\frac{2\pi k}{N}}$.

9.31. (a) Setting up the butterfly's system of equations in matrix form gives

$$\begin{bmatrix} 1 & 1 \\ W_N^r & -W_N^r \end{bmatrix} \begin{bmatrix} X_{m-1}[p] \\ X_{m-1}[q] \end{bmatrix} = \begin{bmatrix} X_m[p] \\ X_m[q] \end{bmatrix}$$

Solving for

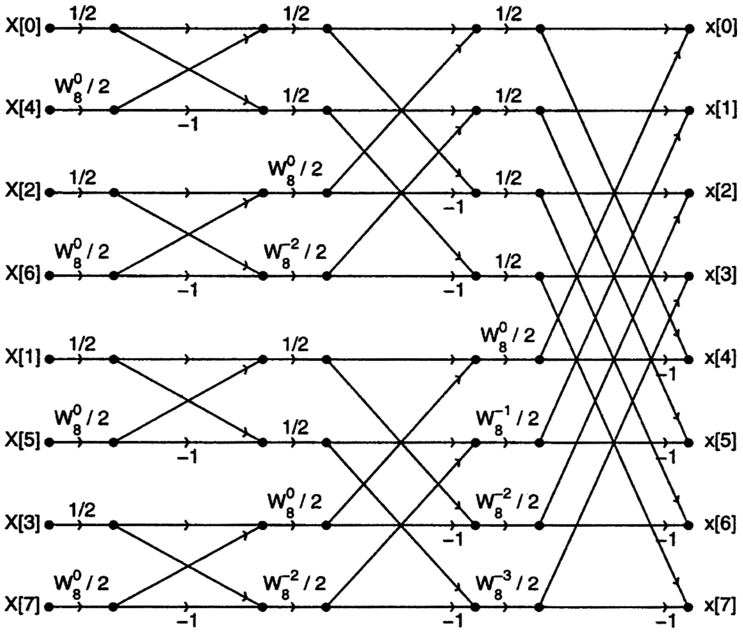
$$\begin{bmatrix} X_{m-1}[p] \\ X_{m-1}[q] \end{bmatrix}$$

gives

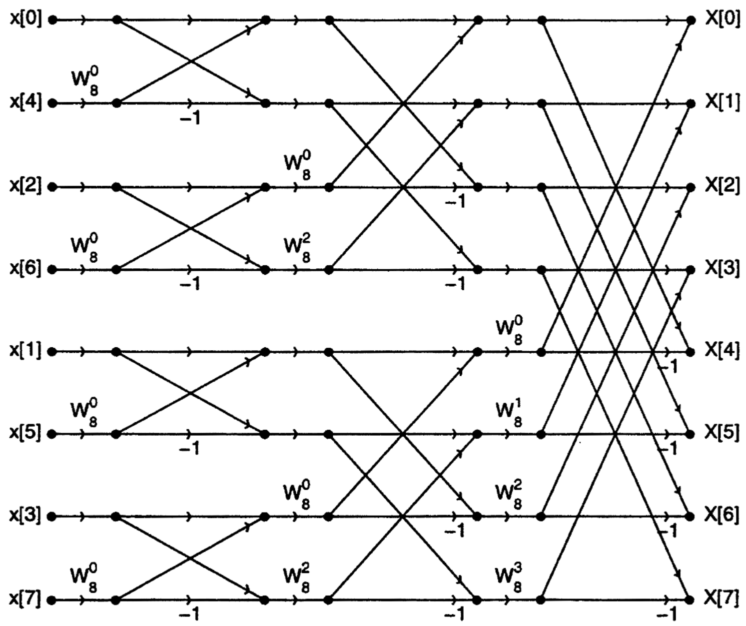
$$\begin{bmatrix} X_{m-1}[p] \\ X_{m-1}[q] \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}W_N^{-r} \\ \frac{1}{2} & -\frac{1}{2}W_N^{-r} \end{bmatrix} \begin{bmatrix} X_m[p] \\ X_m[q] \end{bmatrix}$$

which is consistent with Figure P9.6-2.

(b) The flow graph appears below.



(c) The modification is made by removing all factors of 1/2, changing all W_N^{-r} to W_N^r, and relabeling the input and the output, as shown in the flow graph below.



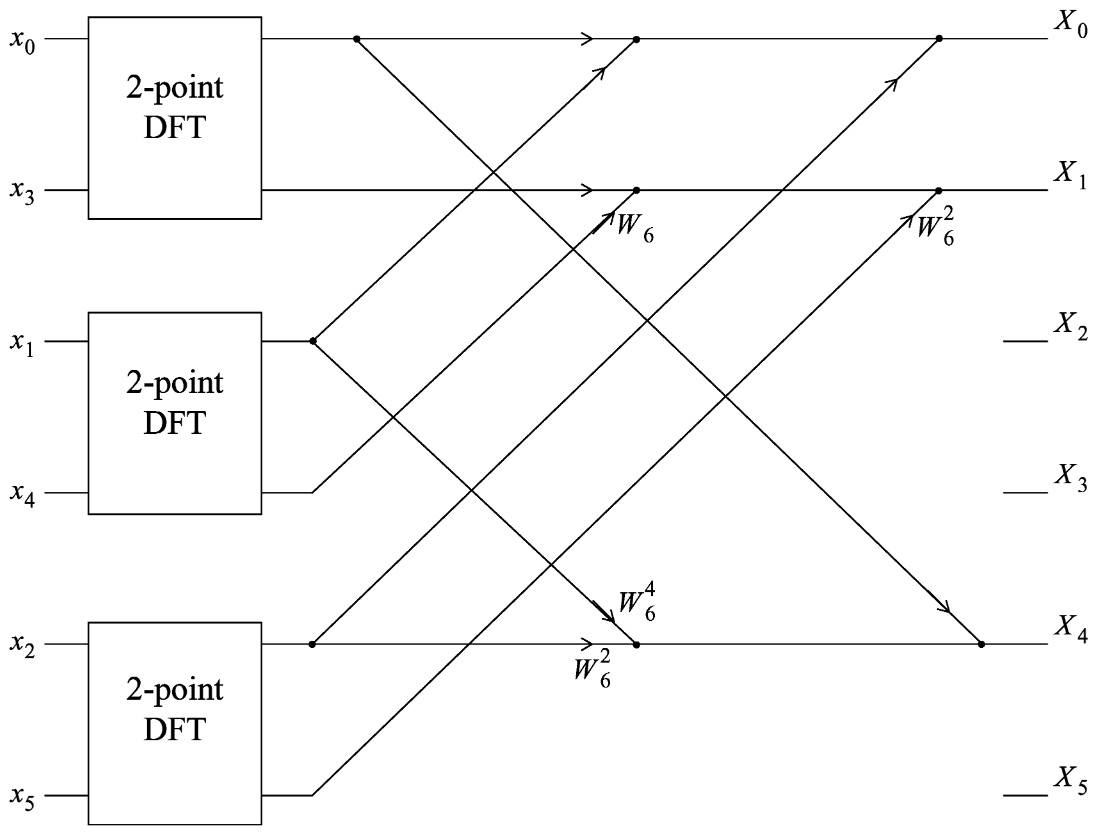
(d) *Yes.* In general, for each decimation-in-time FFT algorithm there exists a decimation-in-frequency FFT algorithm that corresponds to interchanging the input and output and reversing the direction of all the arrows in the flow graph.

9.32. Problem 2 in Spring 2003 Final exam.
Appears in: Spring04 PS8, Fall03 PS8.

Problem

We want to implement a 6-point decimation-in-time FFT using a mixed radix approach. One option is to first take three 2-point DFTs, and then use the results to compute the 6-point DFT. For this option:

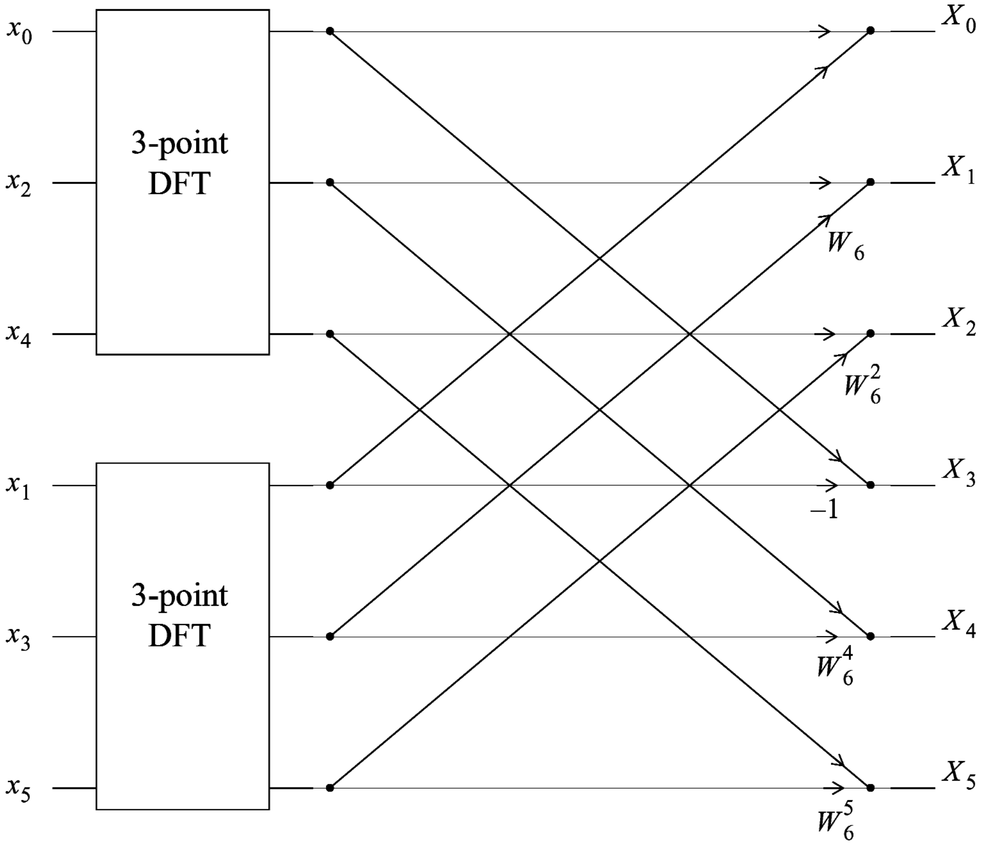
(a) Draw a flowgraph to show what a 2-point DFT calculates. Also, fill in the parts of the flowgraph below involved in calculating the DFT values X_0, X_1, and X_4. Use the definition $W_N = e^{-j2\pi/N}$. Note that due to this function's properties, there is no need to write W_N^p where $p \geq N$ because the function can be rewritten.



(b) How many complex multiplications does this option require? (Multiplying a number by -1 does not count as a complex multiplication.)

A second option is to start with two 3-point DFTs, and then use the results to compute the 6-point DFT.

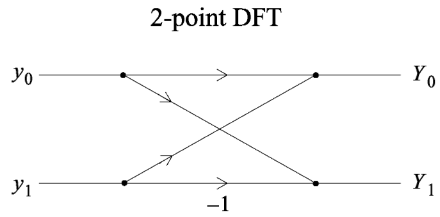
(c) Draw a flowgraph to show what a 3-point DFT calculates. Also, fill in all of the following flowgraph and briefly explain how you derived your implementation:

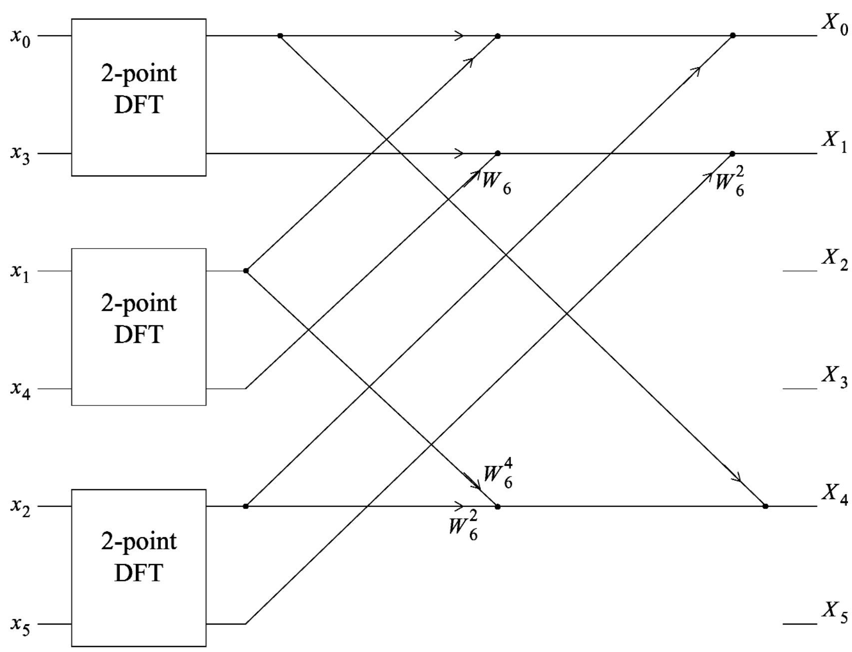


(d) How many complex multiplications does this option require?

Solution from Spring04 PS8

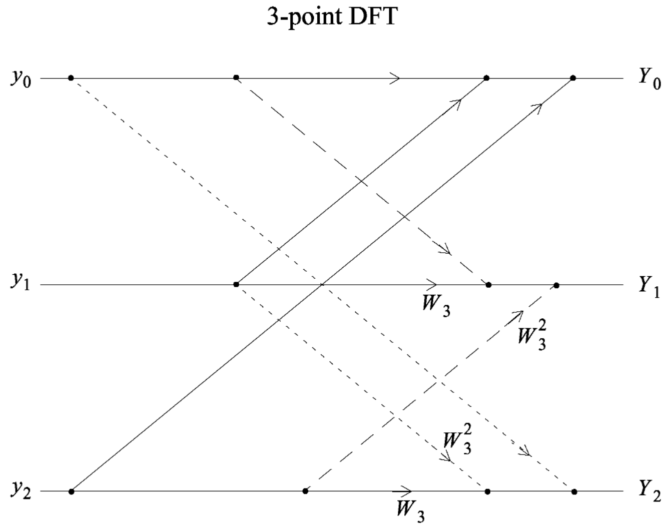
(a)

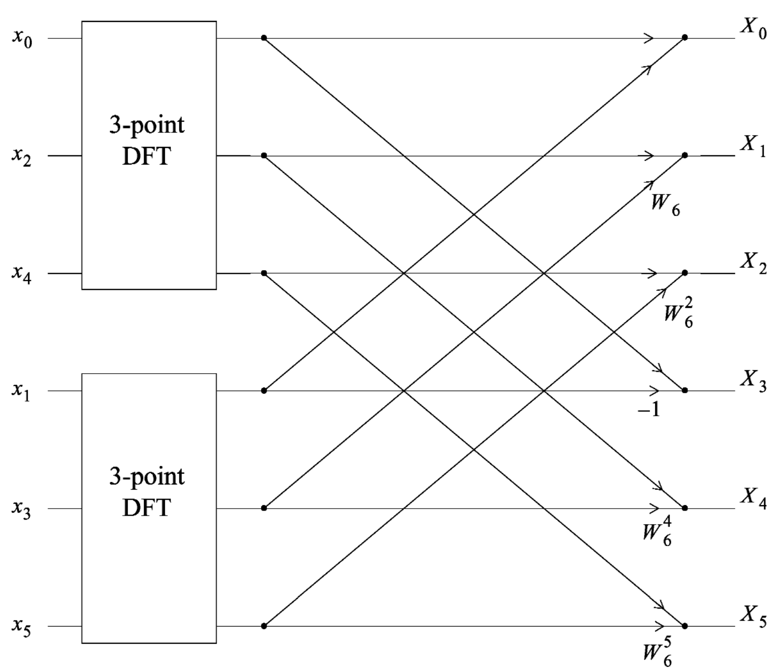




(b) 8 complex multiplications needed

(c)

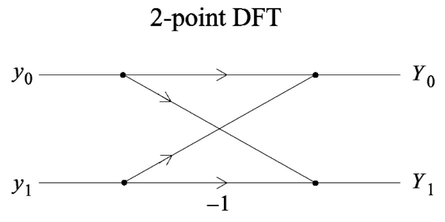


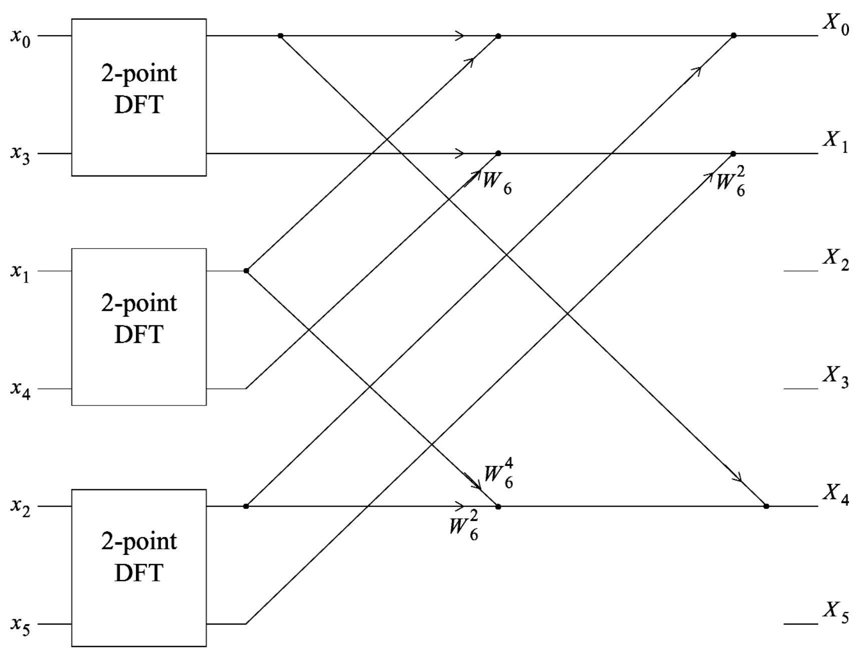


(d) 12 complex multiplications needed

Solution from Fall03 PS8

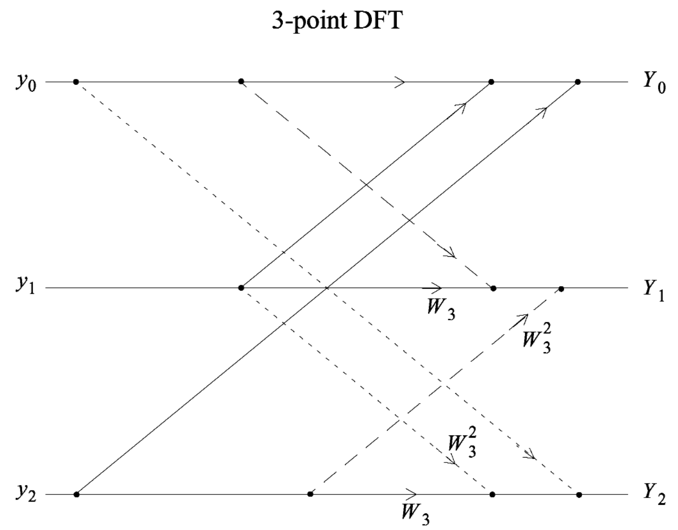
(a)

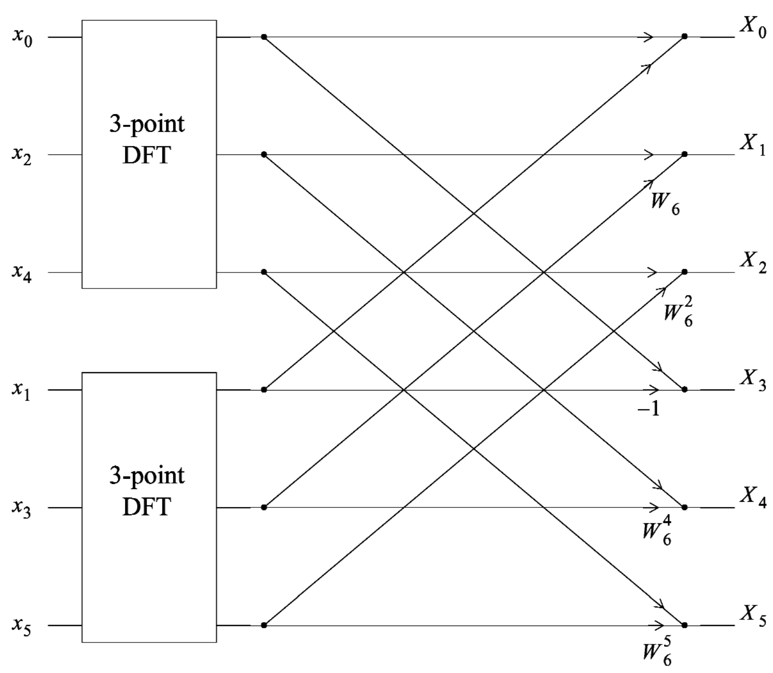




(b) 8 complex multiplications needed

(c)

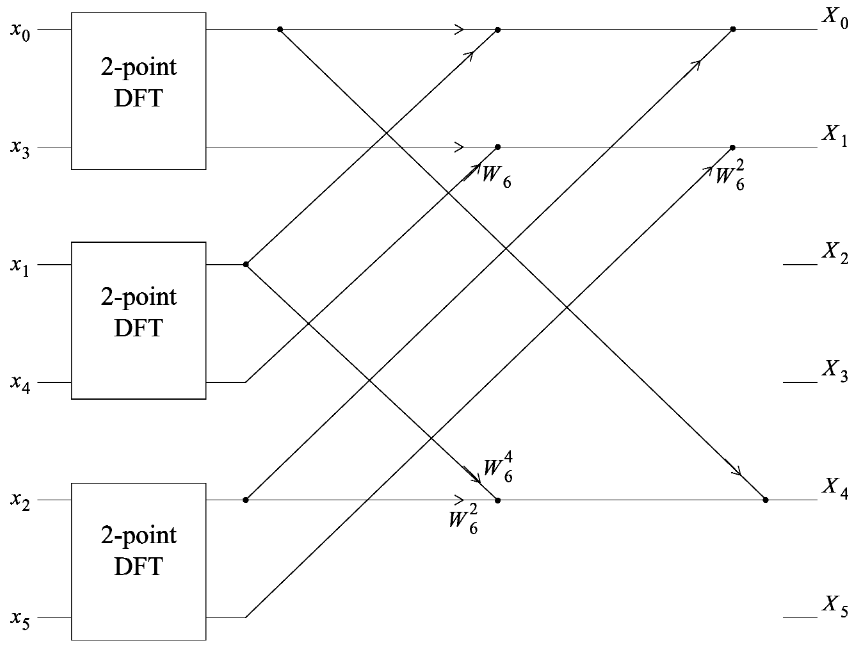




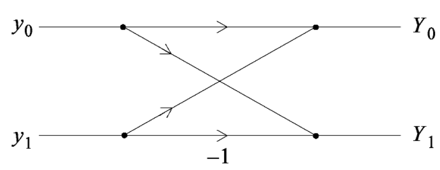
(d) 12 complex multiplications needed

Solution from Spring2003 Final

(a) The flowgraphs are:

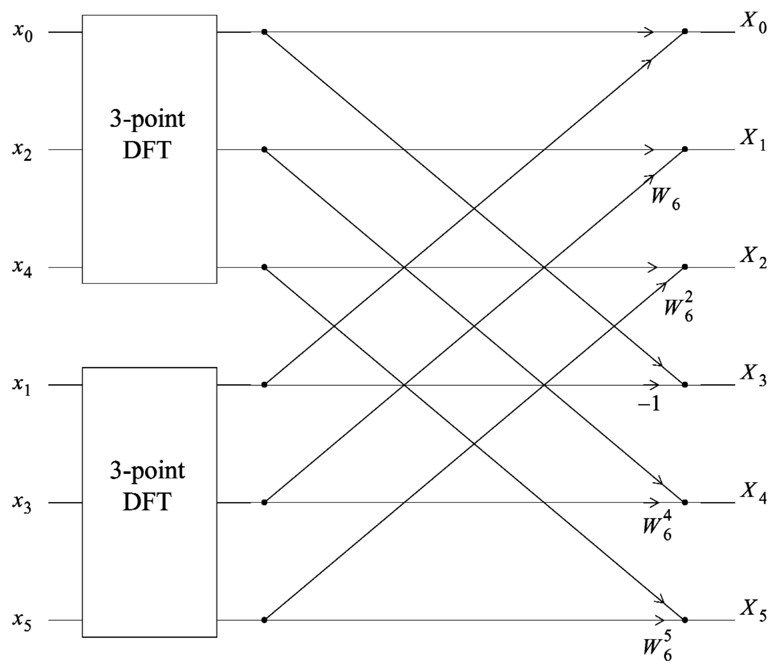


2-point DFT

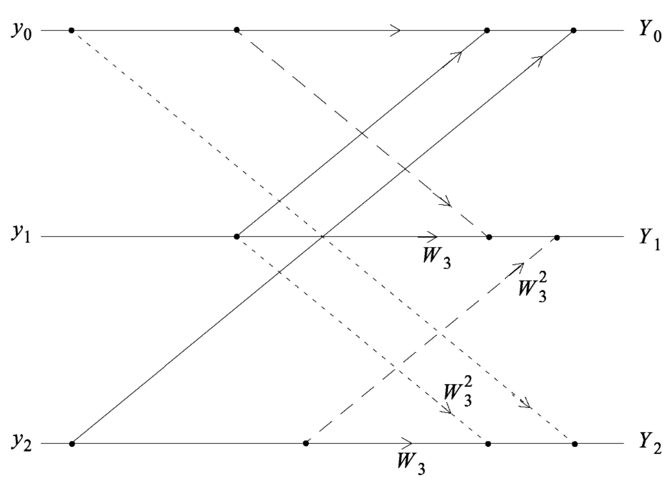


(b) 8 complex multiplications needed

(c) The flowgraphs are:



3-point DFT



(d) 12 complex multiplications needed

9.33. (a) Using the figure, it is observed that each output $Y[k]$ is a scaled version of $X[k]$. The scaling factor is $W[k]$, which is found to be

$$\begin{array}{lcllllllll} k & = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ W[k] & = & 1 & G & G & G^2 & G & G^2 & G^2 & G^3 \end{array}$$

Using this $W[k]$, $Y[k] = W[k]X[k]$.

(b) $W[k] = G^{p[k]}$, where $p[k] =$ the number of ones in the binary representation of index k.

(c) A procedure for finding $\hat{x}[n]$ is as follows.

step 1: Form $W'[k] = 1/W[k]$.

step 2: Take the inverse DFT of $W'[k]$, yielding $w'[n]$.

step 3: Let $\hat{x}[n]$ be the circular convolution of $x[n]$ and $w'[n]$.

If $\hat{x}[n]$ is input to the modified FFT algorithm, then the output will be $X[k]$, as shown below.

$$\begin{aligned} Y[k] &= W[k]\hat{X}[k] \\ &= W[k]X[k]W'[k] \\ &= X[k] \end{aligned}$$

9.34. Let z_k be the z-plane locations of the 25 points uniformly spaced on an arc of a circle of radius 0.5 from $-\pi/6$ to $2\pi/3$. Then

$$z_k = 0.5e^{j(\omega_0+k\Delta\omega)}, \quad k = 0, 1, \ldots, 24$$

where

$$\begin{aligned}\omega_0 &= -\frac{\pi}{6} \\ \Delta\omega &= \left(\frac{5\pi}{6}\right)\left(\frac{1}{24}\right) \\ &= \frac{5\pi}{144}\end{aligned}$$

From the definition of the z-transform,

$$X(z_k) = \sum_{n=0}^{N-1} x[n]z_k^{-n}$$

Plugging in z_k, and setting $W = e^{-j\Delta\omega}$,

$$X(z_k) = \sum_{n=0}^{N-1} x[n](0.5)^{-n}e^{-j\omega_0 n}W^{nk}$$

This is similar to the expression for $X(e^{jw})$ using the chirp transform algorithm. The only difference is the $(0.5)^{-n}$ term. Setting

$$g[n] = x[n](0.5)^{-n}e^{-j\omega_0 n}W^{n^2/2}$$

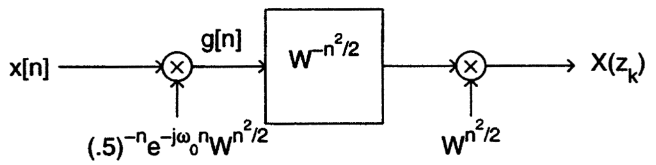
we get

$$X(z_k) = W^{k^2/2}\sum_{n=0}^{N-1} g[n]W^{-(k-n)^2/2}$$

using the result of the chirp transform algorithm. A procedure for computing $X(z)$ at the points z_k is then

- Multiply the sequence x[n] by the sequence $(0.5)^{-n}e^{-j\omega_0 n}W^{n^2/2}$ to form $g[n]$.
- Convolve $g[n]$ with the sequence $W^{-n^2/2}$.
- Multiply this result by the sequence $W^{n^2/2}$ to form $X(z_k)$.

A block diagram of this system appears below.



9.35.

$$
\begin{aligned}
Y[k] &= \sum_{n=0}^{2N-1} y[n] e^{-j(\frac{2\pi}{2N})kn} \\
&= \sum_{n=0}^{N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} + \sum_{n=N}^{2N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} \\
&= \sum_{n=0}^{N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} + \sum_{l=0}^{N-1} e^{-j(\pi/N)(l+N)^2} e^{-j(2\pi/N)(k/2)(l+N)} \\
&= \sum_{n=0}^{N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} + e^{-j\pi k} \sum_{l=0}^{N-1} e^{-j(\pi/N)(l^2+2Nl+N^2)} e^{-j(2\pi/N)(k/2)l} \\
&= \sum_{n=0}^{N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} + (-1)^k \sum_{l=0}^{N-1} e^{-j(\pi/N)l^2} e^{-j(2\pi/N)(k/2)l} \\
&= (1+(-1)^k) \sum_{n=0}^{N-1} e^{-j(\pi/N)n^2} e^{-j(2\pi/N)(k/2)n} \\
&= \begin{cases} 2X[k/2], & k \text{ even} \\ 0, & k \text{ odd} \end{cases}
\end{aligned}
$$

Thus,

$$
Y[k] = \begin{cases} 2\sqrt{N} e^{-j\pi/4} e^{j(\pi/N)k^2/4}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}
$$

9.36. (a) We offer two solutions to this problem.

Solution #1: Looking at the DFT of the sequence, we find

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{(N/2)-1} x[n]e^{-j2\pi kn/N} + \sum_{n=N/2}^{N-1} x[n]e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{(N/2)-1} x[n]e^{-j2\pi kn/N} + \sum_{r=0}^{(N/2)-1} x[r+(N/2)]e^{-j2\pi k[r+(N/2)]/N} \\
&= \sum_{n=0}^{(N/2)-1} x[n][1-(-1)^k]e^{-j2\pi kn/N} \\
&= 0, \quad k \text{ even}
\end{aligned}$$

Solution #2: Alternatively, we can use the circular shift property of the DFT to find

$$\begin{aligned}
X[k] &= -X[k]e^{-j(\frac{2\pi}{N})k(\frac{N}{2})} \\
&= -(-1)^k X[k] \\
&= (-1)^{k+1} X[k]
\end{aligned}$$

When k is even, we have $X[k] = -X[k]$ which can only be true if $X[k] = 0$.

(b) Evaluating the DFT at the odd-indexed samples gives us

$$\begin{aligned}
X[2k+1] &= \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)(2k+1)n} \\
&= \sum_{n=0}^{N/2-1} x[n]e^{-j2\pi n/N}e^{-j2\pi kn/(N/2)} + \sum_{n=N/2}^{N-1} x[n]e^{-j2\pi n/N}e^{-j2\pi kn/(N/2)} \\
&= \text{DFT}_{N/2}\left\{x[n]e^{-j(2\pi/N)n}\right\} + \sum_{l=0}^{N/2-1} x[l+(N/2)]e^{-j2\pi[l+(N/2)]/N}e^{-j2\pi k[l+(N/2)]/(N/2)} \\
&= \text{DFT}_{N/2}\left\{x[n]e^{-j(2\pi/N)n}\right\} + (-1)(-1)\sum_{l=0}^{N/2-1} x[l]e^{-j2\pi l/N}e^{-j2\pi kl/(N/2)} \\
&= \text{DFT}_{N/2}\left\{2x[n]e^{-j(2\pi/N)n}\right\}
\end{aligned}$$

for $k = 0, \ldots, N/2-1$. Thus, we can compute the odd-indexed DFT values using one $N/2$ point DFT plus a small amount of extra computation.

9.37. Problem 7 in Spring2005 final exam.

Problem

Consider a 1024-point sequence $x[n]$ constructed by interleaving two 512-point sequences $x_e[n]$ and $x_o[n]$. Specifically,

$$x[n] = \begin{cases} x_e[n/2], & \text{if } n = 0,\ 2,\ 4,\ \ldots,\ 1022; \\ x_o[(n-1)/2], & \text{if } n = 1,\ 3,\ 5,\ \ldots,\ 1023; \\ 0, & \text{for } n \text{ outside of the range } 0 \le n \le 1023. \end{cases}$$

Let $X[k]$ denote the 1024-point DFT of $x[n]$ and $X_e[k]$ and $X_o[k]$ denote the 512-point DFTs of $x_e[n]$ and $x_o[n]$, respectively.

Given $X[k]$ we would like to obtain $X_e[k]$ from $X[k]$ in a computationally efficient way where computation efficiency is measured in terms of the total number of complex multiplies and adds required.

One not-very-efficient approach is as follows:

$$X[k] \;\longrightarrow\; \boxed{\text{1024-point IDFT}} \;\longrightarrow\; \boxed{\downarrow 2} \;\longrightarrow\; \boxed{\text{512-point DFT}} \;\longrightarrow\; \hat{X}[k]$$

Specify the most efficient algorithm that you can (certainly more efficient than the block diagram above) to obtain $X_e[k]$ from $X[k]$.

Note: An intuitive verbal description and justification of your algorithm may receive partial credit. An analytical justification is required for full credit.

Solution from Spring05 final

The basic intuition is that since we want the DFT of the even samples and the decimation in time FFT algorithm computes the DFT of the even samples as an intermediate step, we can undo the last stage of a decimation in time FFT algorithm.

Let $N = 1024$. Using the IDFT equation for $x \leftrightarrow X$ and the fact that $x_e[n]$ is a downsampled version of $x[n]$,

$$\begin{aligned} x_e[n] &= x[2n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-k2n} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_{N/2}^{-kn} \\ &= \frac{1}{N}\sum_{k=0}^{(N/2)-1} \left(X[k] + X[k+N/2]\right) W_{N/2}^{-kn} \quad \text{since } W_{N/2}^{-kn} \text{ is a periodic function of } k \text{ with period } N/2. \end{aligned}$$

Comparing to the IDFT equation for $x_e \leftrightarrow X_e$

$$x_e[n] = \frac{1}{(N/2)}\sum_{k=0}^{(N/2)-1} X_e[k]W_{N/2}^{-kn},$$

we conclude

$$X_e[k] = \frac{1}{2}\left(X[k] + X[k+(N/2)]\right).$$

Thus an algorithm for finding $X_e[k]$ does not require any DFTs or IDFTs—just the $N/2$ additions and $N/2$ multiplications by $1/2$ above.

Another useful intuition is that since $x_e[n]$ is $x[n]$ downsampled by 2,

$$X_e(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega/2}) + X(e^{j(\omega/2+\pi)})\right).$$

Since the DFT is samples of the DTFT, a similar equation holds for the DFT.

9.38. (a) Since $x[n]$ is real, $x[n] = x^*[n]$, and $X[k]$ is conjugate symmetric.

$$\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x^*[n]e^{-j\frac{2\pi}{N}kn} \\
&= \left(\sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi}{N}kn}e^{-j\frac{2\pi}{N}Nn}\right)^* \\
&= X^*[N-k]
\end{aligned}$$

Hence, $X_R[k] = X_R[N-k]$ and $X_I[k] = -X_I[N-k]$.

(b) In Part (a) it was shown that the DFT of a real sequence $x[n]$ consists of a real part that has even symmetry, and an imaginary part that has odd symmetry. We use this fact in the DFT of the sequence $g[n]$ below.

$$\begin{aligned}
G[k] &= X_1[k] + jX_2[k] \\
&= (X_{1ER}[k] + jX_{1OI}[k]) + j(X_{2ER}[k] + jX_{2OI}[k]) \\
&= \underbrace{X_{1ER}[k] - X_{2OI}[k]}_{\text{real part}} + j\underbrace{(X_{1OI}[k] + X_{2ER}[k])}_{\text{imaginary part}}
\end{aligned}$$

In these expressions, the subscripts "E" and "O" denote even and odd symmetry, respectively, and the subscripts "R" and "I" denote real and imaginary parts, respectively.

Therefore, the even and real part of $G[k]$ is

$$G_{ER}[k] = X_{1ER}[k]$$

the odd and real part of $G[k]$ is

$$G_{OR}[k] = -X_{2OI}[k]$$

the even and imaginary part of $G[k]$ is

$$G_{EI}[k] = X_{2ER}[k]$$

and the odd and imaginary part of $G[k]$ is

$$G_{OI}[k] = X_{1OI}[k]$$

Having established these relationships, it is easy to come up with expressions for $X_1[k]$ and $X_2[k]$.

$$\begin{aligned}
X_1[k] &= X_{1ER}[k] + jX_{1OI}[k] \\
&= G_{ER}[k] + jG_{OI}[k] \\
X_2[k] &= X_{2ER}[k] + jX_{2OI}[k] \\
&= G_{EI}[k] - jG_{OR}[k]
\end{aligned}$$

(c) An $N = 2^\nu$ point FFT requires $(N/2)\log_2 N$ complex multiplications and $N\log_2 N$ complex additions. This is equivalent to $2N\log_2 N$ real multiplications and $3N\log_2 N$ real additions.

(i) The two N-point FFTs, $X_1[k]$ and $X_2[k]$, require a total of $4N\log_2 N$ real multiplications and $6N\log_2 N$ real additions.

(ii) Computing the N-point FFT, $G[k]$, requires $2N\log_2 N$ real multiplications and $3N\log_2 N$ real additions. Then, the computation of $G_{ER}[k]$, $G_{EI}[k]$, $G_{OI}[k]$, and $G_{OR}[k]$ from $G[k]$ requires approximately $4N$ real multiplications and $4N$ real additions. Then, the formation of $X_1[k]$ and $X_2[k]$ from $G_{ER}[k]$, $G_{EI}[k]$, $G_{OI}[k]$, and $G_{OR}[k]$ requires no real additions or multiplications. So this technique requires a total of approximately $2N\log_2 N + 4N$ real multiplications and $3N\log_2 N + 4N$ real additions.

(d) Starting with

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

and separating $x[n]$ into its even and odd numbered parts, we get

$$X[k] = \sum_{n \text{ even}} x[n]e^{-j2\pi kn/N} + \sum_{n \text{ odd}} x[n]e^{-j2\pi kn/N}$$

Substituting $n = 2\ell$ for n even, and $n = 2\ell + 1$ for n odd, gives

$$\begin{aligned}
X[k] &= \sum_{\ell=0}^{(N/2)-1} x[2\ell]e^{-j2\pi k\ell/(N/2)} + \sum_{\ell=0}^{(N/2)-1} x[2\ell+1]e^{-j2\pi k(2\ell+1)/N} \\
&= \sum_{\ell=0}^{(N/2)-1} x[2\ell]e^{-j2\pi k\ell/(N/2)} + e^{-j2\pi k/N}\sum_{\ell=0}^{(N/2)-1} x[2\ell+1]e^{-j2\pi k\ell/(N/2)} \\
&= \begin{cases} X_1[k] + e^{-j2\pi k/N}X_2[k], & 0 \le k < \frac{N}{2} \\ X_1[k-(N/2)] - e^{-j2\pi k/N}X_2[k-(N/2)], & \frac{N}{2} \le k < N \end{cases}
\end{aligned}$$

9.39. (a) The length of the sequence is $L + P - 1$.

(b) In evaluating $y[n]$ using the convolution sum, each nonzero value of $h[n]$ is multiplied once with every nonzero value of $x[n]$. This can be seen graphically using the flip and slide view of convolution. The total number of real multiplies is therefore LP.

(c) To compute $y[n] = h[n] * x[n]$ using the DFT, we use the procedure described below.

step 1: Compute N point DFTs of $x[n]$ and $h[n]$.
step 2: Multiply them together to get $Y[k] = H[k]X[k]$.
step 3: Compute the inverse DFT to get $y[n]$.

Since $y[n]$ has length $L + P - 1$, N must be greater than or equal to $L + P - 1$ so the circular convolution implied by step 2 is equivalent to linear convolution.

(d) For these signals, N is large enough so that circular convolution of $x[n]$ and $h[n]$ and the linear convolution of $x[n]$ and $h[n]$ produce the same result. Counting the number of complex multiplications for the procedure in part (b) we get

DFT of $x[n]$	$(N/2)\log_2 N$
DFT of $h[n]$	$(N/2)\log_2 N$
$Y[k] = X[k]H[k]$	N
Inverse DFT of $Y[k]$	$(N/2)\log_2 N$
	$(3N/2)\log_2 N + N$

Since there are 4 real multiplications for every complex multiplication we see that the procedure takes $6N\log_2 N + 4N$ real multiplications. Using the answer from part (a), we see that the direct method requires $(N/2)(N/2) = N^2/4$ real multiplications.

The following table shows that the smallest $N = 2^\nu$ for which the FFT method requires fewer multiplications than the direct method is 256.

N	Direct Method	FFT method
2	1	20
4	4	64
8	16	176
16	64	448
32	256	1088
64	1024	2560
128	4096	5888
256	16384	13312

9.40. (a) For each L point section, $P-1$ samples are discarded, leaving $L-P+1$ output samples. The complex multiplications are:

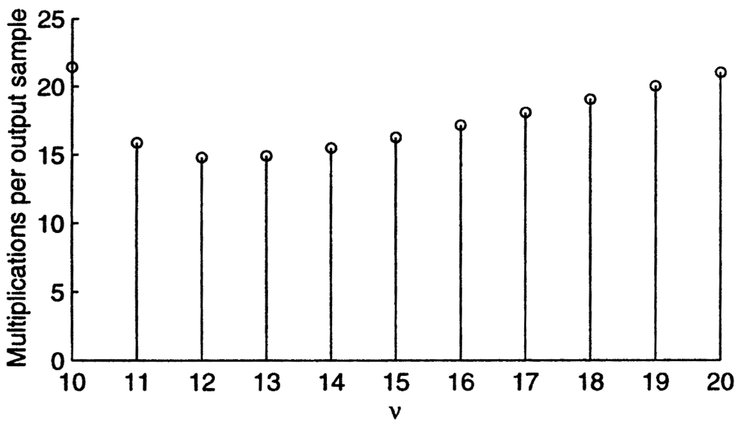
L point FFT of input:	$(L/2)\log_2 L = \nu 2^\nu/2$
Multiplication of filter and section DFT:	$L = 2^\nu$
L point inverse FFT:	$(L/2)\log_2 L = \nu 2^\nu/2$
Total per section:	$2^\nu(\nu+1)$

Therefore,

$$\frac{\text{Complex Multiplications}}{\text{Output Sample}} = \frac{2^\nu(\nu+1)}{2^\nu - P + 1}$$

Note we assume here that $H[k]$ has been precalculated.

(b) The figure below plots the number of complex multiplications per sample versus ν. For $\nu = 12$, the number of multiplies per sample reaches a minimum of 14.8. In comparison, direct evaluation of the convolution sum would require 500 complex multiplications per output sample.



Although $\nu = 9$ is the first valid choice for overlap-save method, it is not plotted since the value is so large (in the hundreds) it would obscure the graph.

(c)

$$\lim_{\nu\to\infty} \frac{2^\nu(\nu+1)}{2^\nu - P + 1} = \lim_{\nu\to\infty} \frac{\nu+1}{1+\frac{-P+1}{2^\nu}} = \nu$$

Thus, for $P = 500$ the direct method will be more efficient for $\nu > 500$.

(d) We want

$$\frac{2^\nu(\nu+1)}{2^\nu - P + 1} \le P.$$

Plugging in $P = L/2 = 2^{\nu-1}$ gives

$$\frac{2^\nu(\nu+1)}{2^\nu - 2^{\nu-1} + 1} \le 2^{\nu-1}.$$

As seen in the table below, the FFT will require fewer complex multiplications than the direct method when $\nu = 5$ or $P = 2^4 = 16$.

	Overlap/Save	Direct
ν	$\frac{2^\nu(\nu+1)}{2^\nu-2^{\nu-1}+1}$	$2^{\nu-1}$
1	2	1
2	4	2
3	6.4	4
4	8.9	8
5	11.3	16

9.41. This problem asks that we find eight equally spaced *inverse* DFT coefficients using the chirp transform algorithm. The book derives the algorithm for the forward DFT. However, with some minor tweaking, it is easy to formulate an inverse DFT. First, we start with the inverse DFT relation

$$\begin{aligned} x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N} \\ x[n_\ell] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi n_\ell k/N} \end{aligned}$$

Next, we define

$$\begin{aligned} \Delta n &= 1 \\ n_\ell &= n_0 + \ell\Delta n \end{aligned}$$

where $\ell = 0, \ldots, 7$. Substituting this into the equation above gives

$$x[n_\ell] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi n_0 k/N}e^{j2\pi\ell\Delta nk/N}$$

Defining

$$W = e^{-j2\pi\Delta n/N}$$

we find

$$x[n_\ell] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi n_0 k/N}W^{-\ell k}$$

Using the relation

$$\ell k = \frac{1}{2}[\ell^2 + k^2 - (k-\ell)^2]$$

we get

$$x[n_\ell] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi n_0 k/N}W^{-\ell^2/2}W^{-k^2/2}W^{(k-\ell)^2/2}$$

Let

$$G[k] = X[k]e^{j2\pi n_0 k/N}W^{-k^2/2}$$

Then,

$$x[n_\ell] = \frac{1}{N}W^{-\ell^2/2}\left(\sum_{k=0}^{N-1} G[k]W^{(k-\ell)^2/2}\right)$$

From this equation, it is clear that the inverse DFT can be computed using the chirp transform algorithm. All we need to do is replace n by k, change the sign of each of the exponential terms, and divide by a factor of N. Therefore,

$$\begin{aligned} m_1[k] &= e^{j2\pi k n_0/N} W^{-k^2/2} \\ m_2[k] &= W^{-k^2/2} \\ h[k] &= \frac{1}{N} W^{k^2/2} \end{aligned}$$

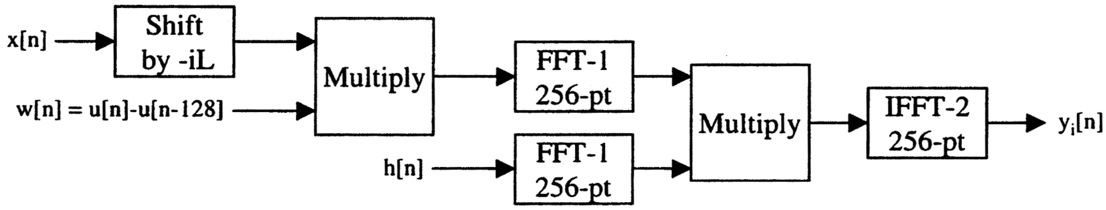
Using this system with $n_0 = 1020$, and $\ell = 0, \ldots, 7$ will result in a sequence $y[n]$ which will contain the desired samples, where

$$\begin{aligned} y[0] &= x[1020] \\ y[1] &= x[1021] \\ y[2] &= x[1022] \\ y[3] &= x[1023] \\ y[4] &= x[0] \\ y[5] &= x[1] \\ y[6] &= x[2] \\ y[7] &= x[3] \end{aligned}$$

9.42. First note that

$$
\begin{aligned}
x_i[n] &= \begin{cases} x[n], & iL \le n \le iL + 127, \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} x[n + iL], & 0 \le n \le 127, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

Using the above we can implement the system with the following block diagram.



The FFT size was chosen as the next power of 2 higher than the length of the linear convolution. This insures the circular convolution implied by multiplying DFTs corresponds to linear convolution as well.

$$
\begin{aligned}
N_{\text{Conv}} &= N_{x_i} + N_{\text{h}} - 1 \\
&= 128 + 64 - 1 \\
&= 191
\end{aligned}
$$

$$N_{\text{FFT}} = 256$$

10.1. (a) Using the relation

$$\Omega_k = \frac{2\pi k}{NT},$$

we find that the index $k = 150$ in $X[k]$ corresponds to a continuous time frequency of

$$\begin{aligned}\Omega_{150} &= \frac{2\pi(150)}{(1000)(10^{-4})} \\ &= 2\pi(1500) \text{ rad/s}\end{aligned}$$

(b) For this part, it is important to realize that the $k = 800$ index corresponds to a *negative* continuous-time frequency. Since the DFT is periodic in k with period N,

$$\begin{aligned}\Omega_{800} &= \frac{2\pi(800 - 1000)}{1000(10^{-4})} \\ &= -2\pi(2000) \text{ rad/s}\end{aligned}$$

10.2. Using the relation

$$\Omega_k = \frac{2\pi k}{NT}$$

or

$$f_k = \frac{k}{NT}$$

we find that the equivalent analog spacing between frequencies is

$$\Delta f = \frac{1}{NT}$$

Thus, in addition to the constraint that N is a power of 2, there are two conditions which must be met:

$$\begin{aligned} \tfrac{1}{T} &> 10,000 \text{ Hz} \quad \text{(to avoid aliasing)} \\ \tfrac{1}{NT} &< 5 \text{ Hz} \quad \text{(given)} \end{aligned}$$

These conditions can be expressed in the form

$$10,000 < \frac{1}{T} < 5N$$

The minimal $N = 2^{\nu}$ that satisfies the relationship is

$$N = 2048$$

for which

$$10,000 \text{ Hz} < \frac{1}{T} < 10,240 \text{ Hz}$$

Thus, $F_{\text{min}} = 10,000$ Hz, and $F_{\text{max}} = 10,240$ Hz.

10.3. (a) After windowing, we have

$$\begin{aligned} x[n] &= \cos(\Omega_0 T n) \\ &= \frac{1}{2}\left[e^{j\Omega_0 T n} + e^{-j\Omega_0 T n}\right] \\ &= \frac{1}{2}\left[e^{j\frac{2\pi}{N}(\frac{N\Omega_0 T}{2\pi})n} + e^{-j\frac{2\pi}{N}(\frac{N\Omega_0 T}{2\pi})n}\right] \end{aligned}$$

for $n = 0, \ldots, N-1$ and $x[n] = 0$ outside this range. Using the DFT properties we get

$$X[k] = \frac{N}{2}\delta[((k - \tfrac{N\Omega_0 T}{2\pi}))_N] + \frac{N}{2}\delta[((k + \tfrac{N\Omega_0 T}{2\pi}))_N]$$

If we choose

$$T = \frac{2\pi}{N\Omega_0}k_0$$

then

$$X[k] = \frac{N}{2}\delta[k - k_0] + \frac{N}{2}\delta[k - (N - k_0)],$$

which is nonzero for $X[k_0]$ and $X[N - k_0]$, but zero everywhere else.

(b) *No*, the choice for T is not unique since we can choose the integer k_0.

10.4. Since $x[n]$ is real, $X[k]$ must be conjugate symmetric.

$$\begin{aligned} X[k] &= X^*[((-k))_N] \\ X[((-k))_N] &= X^*[k] \end{aligned}$$

Therefore,

$$\begin{aligned} X[((-900))_{1000}] &= (1)^* \\ X[100] &= 1 \\ X[((-420))_{1000}] &= (5)^* \\ X[580] &= 5 \end{aligned}$$

Note that the $k = 900$ and $k = 580$ correspond to negative frequencies of Ω. Since the DFT is periodic in k with period N, we use $k = 900 - 1000 = -100$ and $k = 580 - 1000 = -420$, respectively, in the equations below.

Starting with

$$\Omega_k = \frac{2\pi k}{NT},$$

we find

$$\begin{aligned} \Omega_{-100} &= \frac{2\pi(-100)}{(1000)(1/10,000)} \\ &= -2\pi(1000) \text{ rad/s} \\ \Omega_{100} &= \frac{2\pi(100)}{(1000)(1/10,000)} \\ &= 2\pi(1000) \text{ rad/s} \\ \Omega_{-420} &= \frac{2\pi(-420)}{(1000)(1/10,000)} \\ &= -2\pi(4200) \text{ rad/s} \\ \Omega_{420} &= \frac{2\pi(420)}{(1000)(1/10,000)} \\ &= 2\pi(4200) \text{ rad/s} \end{aligned}$$

Consequently,

$$\begin{aligned} X_c(j\,\Omega)\,|_{\Omega=-2\pi(1000)} &= \frac{1}{10,000} \\ X_c(j\,\Omega)\,|_{\Omega=2\pi(1000)} &= \frac{1}{10,000} \\ X_c(j\,\Omega)\,|_{\Omega=-2\pi(4200)} &= \frac{1}{2,000} \\ X_c(j\,\Omega)\,|_{\Omega=2\pi(4200)} &= \frac{1}{2,000} \end{aligned}$$

Note that all expressions for $X_c(j\,\Omega)$ have been multiplied by the sampling period $T = 1/10,000$ because sampling the continuous-time signal $x_c(t)$ involves multiplication by $1/T$.

10.5. The Hamming window's mainlobe is $\Delta\omega_{ml} = \frac{8\pi}{L-1}$ radians wide. We want

$$\begin{aligned} \Delta\omega_{ml} &\leq \frac{\pi}{100} \\ \frac{8\pi}{L-1} &\leq \frac{\pi}{100} \\ L &\geq 801 \end{aligned}$$

Because the window length is constrained to be a power of 2, we see that

$$L_{\text{min}} = 1024$$

10.6. The rectangular window's mainlobe is

$$\Delta\omega_{ml} = \frac{4\pi}{L} = \frac{4\pi}{64}$$

radians wide. The difference in frequency between each cosine must be greater than this amount to be resolved. If they are not separated enough, the mainlobes from each cosine will overlap too much and only a single peak will be seen. The separation of the cosines for each signal is

$$\begin{aligned}\Delta\omega_1 &= \left|\frac{\pi}{4} - \frac{17\pi}{64}\right| = \frac{\pi}{64} \\ \Delta\omega_2 &= \left|\frac{\pi}{4} - \frac{21\pi}{64}\right| = \frac{5\pi}{64} \\ \Delta\omega_3 &= \left|\frac{\pi}{4} - \frac{21\pi}{64}\right| = \frac{5\pi}{64}\end{aligned}$$

Clearly, the cosines in $x_1[n]$ are too closely spaced in frequency to produce distinct peaks.

In $x_3[n]$, we have a small amplitude cosine which will be obscurred by the large sidelobes from the rectangular window. The peak will therefore not be visible.

The only signal from which we would expect to see two distinct peaks is $x_2[n]$.

10.7. The equivalent frequency spacing is

$$\Delta\Omega = \frac{2\pi}{NT} = \frac{2\pi}{(8192)(50\mu s)} = 15.34 \text{ rad/s}$$

or

$$\Delta f = \frac{\Delta\Omega}{2\pi} = 2.44 \text{ Hz}$$

10.8. The equivalent frequency spacing is

$$\Delta f = \frac{1}{NT}$$

Thus, the minimum DFT length N such that adjacent samples of $X[k]$ correspond to a frequency spacing of 5 Hz or less in the original continuous-time signal is

$$\begin{aligned}
\Delta f &\leq 5 \\
\frac{1}{NT} &\leq 5 \\
N &\geq \frac{1}{5T} \\
&\geq \frac{8000}{5} \\
&\geq 1600 \text{ samples}
\end{aligned}$$

10.9. Since $w[n]$ is the rectangular window and we are using $N = 36$ we have

$$\begin{aligned} X_r[k] &= \sum_{m=0}^{35} x[rR+m]e^{-j(2\pi/36)km} \\ &= \text{DFT}\{x[rR+n]\} \end{aligned}$$

Because $x[n]$ is zero outside the range $0 \le n \le 71$, $X_r[k]$ will be zero except when $r = 0$ or $r = 1$. When $r = 0$, the 36 points in the sum of the DFT only include the section

$$\cos(\pi n/6) = \frac{e^{j(\frac{2\pi}{36})3n} + e^{-j(\frac{2\pi}{36})3n}}{2}$$

of $x[n]$. Therefore, we can use the properties of the DFT to find

$$\begin{aligned} X_0[k] &= \tfrac{36}{2}\delta[((k-3))_{36}] + \tfrac{36}{2}\delta[((k+3))_{36}] \\ &= 18\delta[k-3] + 18\delta[k-33] \end{aligned}$$

When $r = 1$, the 36 points in the sum of the DFT only include the section

$$\cos(\pi n/2) = \frac{e^{j(\frac{2\pi}{36})9n} + e^{-j(\frac{2\pi}{36})9n}}{2}$$

of $x[n]$. Therefore, we can use the properties of the DFT to find

$$\begin{aligned} X_1[k] &= \tfrac{36}{2}\delta[((k-9))_{36}] + \tfrac{36}{2}\delta[((k+9))_{36}] \\ &= 18\delta[k-9] + 18\delta[k-27] \end{aligned}$$

Putting it all together we get

$$X_r[k] = \begin{cases} 18(\delta[k-3] + \delta[k-33]), & r = 0 \\ 18(\delta[k-9] + \delta[k-27]), & r = 1 \\ 0, & \text{otherwise} \end{cases}$$

10.10. The instantaneous frequency of the chirp signal is

$$\omega_i[n] = \omega_0 + \lambda n$$

This describes a line with slope λ and intercept ω_0. Thus,

$$\lambda = \frac{\Delta y}{\Delta x} = \frac{(0.5\pi - 0.25\pi)}{(19000 - 0)} = 41.34 \times 10^{-6} \text{ rad}$$

$$\omega_0 = 0.25\pi \text{ rad}$$

10.11. Using

$$\Delta f = \frac{1}{NT}$$

and assuming no aliasing occured when the continuous-time signal was sampled, we find that the frequency spacing between spectral samples is

$$\begin{aligned}\Delta f &= \frac{1}{(1024)(1/10,000)} \\ &= 9.77 \text{ Hz}\end{aligned}$$

or

$$\Delta\Omega = 2\pi\Delta f = 61.4 \text{ rad/s}$$

10.12. *No*, the peaks will not have the same height. The peaks in $V_2(e^{j\omega})$ will be larger than those in $V_1(e^{j\omega})$.

First, note that the Fourier transform of the rectangular window has a higher peak than that of the Hamming window. If this is not obvious, consider Figure 7.21, and recall that the Fourier transform of an L-point window $w[n]$, evaluated at DC ($\omega = 0$), is

$$W(e^{j0}) = \sum_{n=0}^{L-1} w[n]$$

Let the rectangular window be $w_R[n]$, and the Hamming window be $w_H[n]$. It is clear from the figure (where M = L+1) that

$$\sum_{n=0}^{L-1} w_R[n] > \sum_{n=0}^{L-1} w_H[n]$$

Therefore,

$$W_R(e^{j0}) > W_H(e^{j0})$$

Thus, the Fourier transform of the rectangular window has a higher peak than that of the Hamming window.

Now recall that the multiplication of two signals in the time domain corresponds to a periodic convolution in the frequency domain. So in the frequency domain, $V_1(e^{j\omega})$ is the convolution of two scaled impulses from the sinusoid, with the Fourier transform of the L-point Hamming window, $W_H(e^{j\omega})$. This results in two scaled copies of $W_H(e^{j\omega})$, centered at the frequencies of the sinusoid. Similarly, $V_2(e^{j\omega})$ consists of two scaled copies of $W_R(e^{j\omega})$, also centered at the frequencies of the sinusoid. The scale factor is the same in both cases, resulting from the Fourier transform of the sinusoid.

Since the peaks of the Fourier transform of the rectangular window are higher than those of the Hamming window, the peaks in $V_2(e^{j\omega})$ will be larger than those in $V_1(e^{j\omega})$.

10.13. (a) The best sidelobe attenuation expected under these constraints is

$$\begin{aligned} L &\simeq \frac{24\pi(A_{sl}+12)}{155\Delta_{ml}}+1 \\ 512 &\simeq \frac{24\pi(A_{sl}+12)}{155(\pi/100)}+1 \\ A_{sl} &\simeq 21 \text{ dB} \end{aligned}$$

(b) The two sinusoidal components are separated by at least $\pi/50$ radians. Since the largest allowable mainlobe width is $\pi/100$ radians, we know that the peak of the DFT magnitude of the weaker sinusoidal component will not be located in the mainlobe of the DFT magnitude of the stronger sinusoidal component. Thus, we only need to consider the sidelobe height of the stronger component.

Converting 21 dB attenuation back from dB gives

$$\begin{aligned} -21 \text{ dB} &= 20\log_{10} m \\ m &= 0.0891 \end{aligned}$$

Since the amplitude of the stronger sinusoidal component is 1, the amplitude of the weaker sinusoidal component must be greater than 0.0891 in order for the weaker sinusoidal component to be seen over the sidelobe of the stronger sinusoidal component.

10.14. (a) The length of a window is

$$\begin{aligned} L &= \left(16,000 \ \frac{\text{samples}}{\text{sec}}\right)(20 \times 10^{-3} \text{ sec}) \\ &= 320 \text{ samples} \end{aligned}$$

(b) The *frame rate* is the number of frames of data processed per second, or equivalently, the number of DFT computations done per second. Since the window is advanced 40 samples between computations of the DFT, the frame rate is

$$\begin{aligned} \text{frame rate} &= \left(16,000 \ \frac{\text{samples}}{\text{sec}}\right)\left(\frac{1 \text{ frame processed}}{40 \text{ samples}}\right) \\ &= 400 \ \frac{\text{frames}}{\text{sec}} \end{aligned}$$

(c) The most straightforward solution to this problem is to say that since the window length L is 320, we need $N \geq L$ in order to do the DFT. Therefore, a value of $N = 512$ meets the criteria of $N \geq L$, $N = 2^{\nu}$. However, since the windows overlap, we can find a smaller N.

Since the window advances 40 samples between computations, we really only need 40 *valid* samples for each DFT in order to reconstruct the original input signal. If we time alias the windowed data, we can use a smaller DFT length than the window length. With $N = 256$, 64 samples will be time aliased, and remaining 192 samples will be valid. However, with $N = 128$, all the samples will be aliased. Therefore, the minimum size of N is 256.

(d) Using the relation

$$\Delta f = \frac{1}{NT},$$

the frequency spacing for $N = 512$ is

$$\Delta f = \frac{16,000}{512} = 31.25 \text{ Hz}$$

and for $N = 256$ is

$$\Delta f = \frac{16,000}{256} = 62.5 \text{ Hz}$$

10.15. (a) Since $x[n]$ is real, $X[k]$ must be conjugate symmetric.

$$X[k] = X^*[((-k))_N]$$

We can use this conjugate symmetry property to find $X[k]$ for $k = 200$.

$$\begin{aligned} X[((-k))_N] &= X^*[k] \\ X[((-800))_{1000}] &= (1+j)^* \\ X[200] &= 1-j \end{aligned}$$

(b) Since an N-point DFT is periodic in k with period N, we know that

$$X[800] = 1+j$$

implies that

$$X[-200] = 1+j$$

Using the relation

$$\Omega_k = \frac{2\pi k}{NT},$$

we find

$$\begin{aligned} \Omega_{-200} &= \frac{-2\pi(200)}{(1000)(1/20,000)} \\ &= -2\pi(4000) \text{ rad/s} \\ \Omega_{200} &= \frac{2\pi(200)}{(1000)(1/20,000)} \\ &= 2\pi(4000) \text{ rad/s} \end{aligned}$$

Consequently,

$$\begin{aligned} X_c(j\Omega)\,|_{\Omega=-2\pi(4000)} &= \frac{1+j}{20,000} \\ X_c(j\Omega)\,|_{\Omega=2\pi(4000)} &= \frac{1-j}{20,000} \end{aligned}$$

Note that both expressions for $X_c(j\Omega)$ have been multiplied by the sampling period $T = 1/20,000$ because sampling the continuous-time signal $x_c(t)$ involves multiplication by $1/T$.

10.16. *All windows expect the Blackman satisfy the criteria.* Using the table, and noting that the window length $N = M + 1$, we find

Rectangular:

$$\begin{aligned}
\Delta w_{ml} &= \frac{4\pi}{M+1} \\
&= \frac{4\pi}{256} \\
&= \frac{\pi}{64} \leq \frac{\pi}{25} \text{ rad}
\end{aligned}$$

The resolution of the rectangular window satisfies the criteria.

Bartlett, Hanning, Hamming:

$$\begin{aligned}
\Delta w_{ml} &= \frac{8\pi}{M} \\
&= \frac{8\pi}{255} \\
&= \frac{\pi}{31.875} \leq \frac{\pi}{25} \text{ rad}
\end{aligned}$$

The resolution of the Bartlett, Hanning, and Hamming windows satisfies the criteria.

Blackman:

$$\begin{aligned}
\Delta w_{ml} &= \frac{12\pi}{M} \\
&= \frac{12\pi}{255} \\
&= \frac{\pi}{21.25} \not\leq \frac{\pi}{25} \text{ rad}
\end{aligned}$$

The Blackman window does not have a frequency resolution of at least $\pi/25$ radians. Therefore, this window does not satisfy the criteria.

10.17. The equivalent continuous-time frequency spacing is

$$\Delta f = \frac{1}{NT}$$

Thus, to satisfy the criterion that the frequency spacing between consecutive DFT samples is 1 Hz or less we must have

$$\begin{aligned}
\Delta f &\leq 1 \\
\frac{1}{NT} &\leq 1 \\
T &\geq \frac{1}{N} \\
T &\geq \frac{1}{1024} \text{ sec}
\end{aligned}$$

However, we must also satisfy the Sampling Theorem to avoid aliasing. We therefore have the addition restriction that,

$$\begin{aligned}
\frac{1}{T} &\geq 200 \text{ Hz} \\
T &\leq \frac{1}{200} \text{ sec}
\end{aligned}$$

Putting the two constraints together we find

$$\frac{1}{1024} \leq T \leq \frac{1}{200}$$

$$T_{\text{min}} = \frac{1}{1024} \text{ sec}$$

10.18. We have

$$\begin{aligned} v[n] &= \cos(2\pi n/5)w[n] \\ &= \left[\frac{e^{j2\pi n/5}+e^{-j2\pi n/5}}{2}\right]w[n] \\ V(e^{j\omega}) &= \frac{1}{2}W(e^{j(\omega-2\pi/5)})+\frac{1}{2}W(e^{j(\omega+2\pi/5)}) \end{aligned}$$

The rectangular window's transform is

$$W(e^{j\omega}) = \frac{\sin(16\omega)}{\sin(\omega/2)}e^{-j\omega 31/2}$$

In order to label $V(e^{j\omega})$ correctly, we must find the mainlobe height, strongest sidelobe height, and the first nulls of $W(e^{j\omega})$.

Mainlobe Height of $W(e^{j\omega})$: The peak height is at $\omega = 0$ for which we can use l'hôpital's rule to find

$$W(e^{j0}) = 32\left.\frac{\cos(16\omega)}{\cos(\omega/2)}\right|_{\omega=0} = 32$$

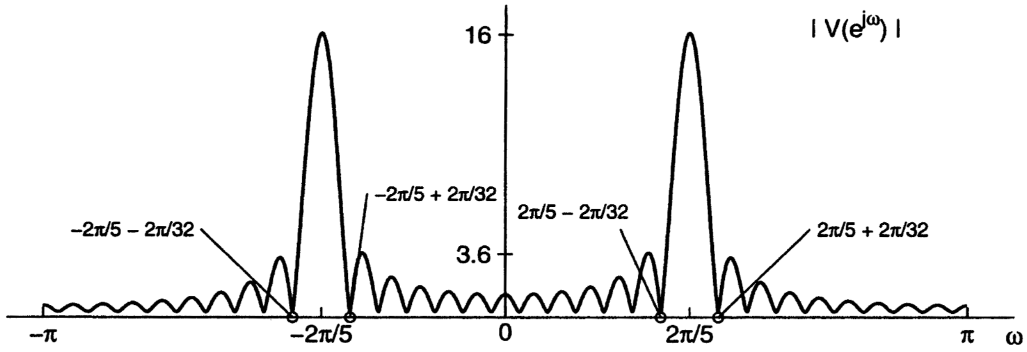
Strongest Sidelobe height of $W(e^{j\omega})$: The strongest sidelobe height for the rectangular window is 13 dB below the main peak height. Therefore, since 13 dB $= 0.2239$ we have

$$\text{Strongest Sidelobe height} = 0.2239(32) \approx 7.2$$

First Nulls of $W(e^{j\omega})$: The first nulls can be found be noting that $W(e^{j\omega}) = 0$ when $\sin(16\omega) = 0$ Thus, the first nulls occur at

$$\omega = \pm\frac{2\pi}{32}$$

Therefore, $|V(e^{j\omega})|$ looks like



Note that the numbers used above for the heights are not exact because we are adding two copies of $W(e^{j\omega})$ to get $V(e^{j\omega})$ and the exact values for the heights will depend on relative phase and location of the two copies. However, they are a very good approximation and the error is small.

10.19. *The signals $x_2[n]$, $x_3[n]$, and $x_6[n]$ could be $x[n]$*, as described below.

Looking at the figure, it is clear that there are two nonzero DFT coefficients at $k = 8$, and $k = 16$. These correspond to frequencies

$$\begin{aligned}
\omega_1 &= \frac{(2\pi)(8)}{128} \\
&= \frac{\pi}{8} \text{ rad} \\
\omega_2 &= \frac{(2\pi)(16)}{128} \\
&= \frac{\pi}{4} \text{ rad}
\end{aligned}$$

Also notice that the magnitude of the DFT coefficient at $k = 16$ is about 3 times that of the DFT coefficient at $k = 8$.

- $x_1[n]$: The second cosine term has a frequency of $.26\pi$ rad, which is neither $\pi/8$ rad or $\pi/4$ rad. Consequently, $x_1[n]$ is not consistent with the information shown in the figure.
- $x_2[n]$: This signal is consistent with the information shown in the figure. The peaks occur at the orrect locations, and are scaled properly.
- $x_3[n]$: This signal is consistent with the information shown in the figure. The peaks occur at the correct locations, and are scaled properly.
- $x_4[n]$: This signal has a cosine term with frequency $\pi/16$ rad, which is neither $\pi/8$ rad or $\pi/4$ rad. Consequently, $x_4[n]$ is not consistent with the information shown in the figure.
- $x_5[n]$: This signal has sinusoids with the correct frequencies, but the scale factors on the two terms are not consistent with the information shown in the figure.
- $x_6[n]$: This signal is consistent with the information shown in the figure. Note that phase information is not represented in the DFT magnitude plot.

10.20

For a 256-point window, the main lobe width is $\pi/64$ for the rectangular case and approximately $\pi/32$ for the Hamming case. This means that the two components of $x_1[n]$, which differ in frequency by $\pi/64$, will be resolvable using a rectangular window, but only marginally resolvable using a Hamming window. Since both components are of equal amplitude, the sidelobe level is not an issue in this case. Recommendation: Use the rectangular window for $x_1[n]$.

The two components of $x_2[n]$ are separated in frequency by $3\pi/32$. This is a sufficiently large separation compared to the main lobe width of either window to ensure that the two components can be resolved no matter which window is used. The higher-frequency component, however, is 35 dB smaller in amplitude than the lower frequency component. This component turns out to be only marginally above the level of the sidelobes of the lower-frequency component when the rectangular window is used. Recommendation: Use the Hamming window for $x_2[n]$.

The two components of $x_3[n]$ differ in frequency by only $\pi/1024$. This is so much less than the main lobe width of even the rectangular window that the two components will not be resolvable using either window. Recommendation: It does not matter which window is used, since neither will allow both frequency components to be identified.

10.21. We should choose *Method 2*.

Method 1: This doubles the number of samples we take of the frequency variable, but does not change the frequency resolution. The size of the main lobe from the window remains the same.

Method 2: *This improves the frequency resolution since the main lobe from the window gets smaller.*

Method 3: This increases the time resolution (the ability to distinguish events in time), but does not affect the frequency resolution.

Method 4: This will decrease the frequency resolution since the main lobe from the window increases. This is a strange thing to do since there are samples of x[n] that do not get used in the transform.

Method 5: This will only improve the resolution if we can ignore any problems due to sidelobe leakage. For example, changing to a rectangular window will improve our ability to resolve two equal amplitude sinusoids. In most cases, however, we need to worry about sidelobe levels. A large sidelobe might mask the presence of a low amplitude signal. Since we do not know ahead of time the nature of the signal we are trying to analyze, changing to a rectangular window may actually make things worse. Thus, in general, changing to a rectangular window will not necessarily increase the frequency resolution.

10.22. Using the approximation given in the chapter

$$L \simeq \frac{24\pi(A_{sl} + 12)}{155\Delta_{ml}} + 1$$

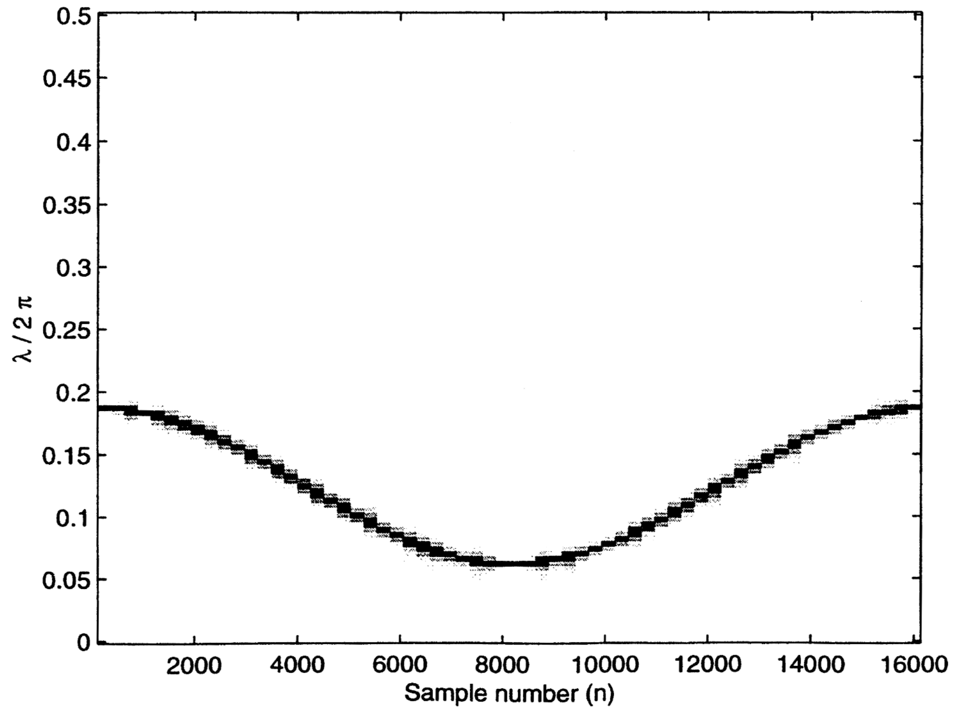
we find for $A_{sl} = 30$ dB and $\Delta_{ml} = \frac{\pi}{40}$ rad,

$$\begin{aligned} L &\simeq \frac{24\pi(30 + 12)}{155(\pi/40)} + 1 \\ &\simeq 261.1 \rightarrow 262 \end{aligned}$$

10.23. The 'instantaneous frequency' of $x[n]$, denoted as $\lambda[n]$, can be determined by taking the derivative with respect to n of the argument of the cosine term. This gives

$$\begin{aligned}
\lambda[n] &= \frac{d}{dn}\left[\frac{\pi n}{4} + 1000\sin\left(\frac{\pi n}{8000}\right)\right] \\
&= \frac{\pi}{4} + \frac{\pi}{8}\cos\left(\frac{\pi n}{8000}\right) \\
\frac{\lambda[n]}{2\pi} &= \frac{1}{8} + \frac{1}{16}\cos\left(\frac{\pi n}{8000}\right)
\end{aligned}$$

Once $\lambda[n]/2\pi$ is known, it is simple to sketch the spectrogram, shown below.



Here, we see a cosine plot shifted up the frequency $(\lambda/2\pi)$ axis by a constant. As is customary in a spectrogram, only the frequencies $0 \leq \lambda/2\pi \leq 0.5$ are plotted.

10.24. (a) Sampling the continuous-time input signal

$$x(t) = e^{j(3\pi/8)10^4 t}$$

with a sampling period $T = 10^{-4}$ yields a discrete-time signal

$$x[n] = x(nT) = e^{j3\pi n/8}$$

In order for $X_w[k]$ to be nonzero at exactly one value of k, it is necessary for the frequency of the complex exponential of $x[n]$ to correspond to that of a DFT coefficent, $w_k = 2\pi k/N$. Thus,

$$\begin{aligned} \frac{3\pi}{8} &= \frac{2\pi k}{N} \\ N &= \frac{16k}{3} \end{aligned}$$

The smallest value of k for which N is an integer is $k = 3$. Thus, the smallest value of N such that $X_w[k]$ is nonzero at exactly one value of k is

$$N = 16$$

(b) The rectangular windows, $w_1[n]$ and $w_2[n]$, differ only in their lengths. $w_1[n]$ has length 32, and $w_2[n]$ has length 8. Recall that compared to that of a longer window, the Fourier transform of a shorter window has a larger mainlobe width and higher sidelobes. Since the DFT is a sampled version of the Fourier transform, we might try to look for these features in the two plots. We notice that the second plot, Figure P10.31-3, appears to have a larger mainlobe width and higher sidelobes. As a result, we conclude that Figure P10.31-2 corresponds to $w_1[n]$, and P10.31-3 corresponds to $w_2[n]$.

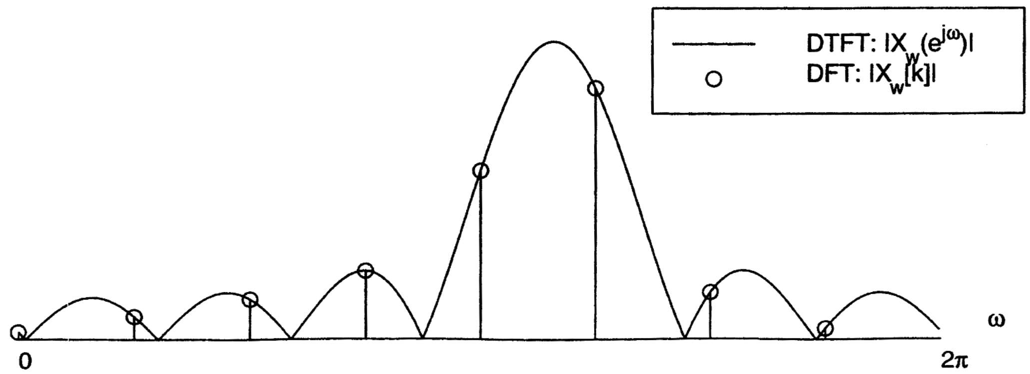
(c) A simple technique to estimate the value of ω_0 is to find the value of k at which the peak of $|X_w[k]|$ occurs. Then, the estimate, is

$$\hat{\omega}_0 = \frac{2\pi k}{N}$$

The corresponding value of $\hat{\Omega}_0$ is

$$\hat{\Omega}_0 = \frac{2\pi k}{NT}$$

This estimate is not exact, since the peak of the Fourier transform magnitude $|X_w(e^{j\omega})|$ might occur between two values of the DFT magnitude $|X_w[k]|$, as shown below.



The maximum possible error, $\Omega_{\text{max error}}$, of the frequency estimate is one half of the frequency resolution of the DFT.

$$\begin{aligned}\Omega_{\text{max error}} &= \frac{1}{2}\frac{2\pi}{NT} \\ &= \frac{\pi}{NT}\end{aligned}$$

For the system parameters of $N = 32$, and $T = 10^{-4}$, this is

$$\Omega_{\text{max error}} = 982 \text{ rad/s}$$

(d) To develop a procedure to get an exact estimate of Ω_0, it helps to derive $X_w[k]$. First, let's find the Fourier transform of $x_w[n] = x[n]w[n]$, where $w[n]$ is an N-point rectangular window.

$$\begin{aligned}X_w(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{j\omega_0 n}e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} e^{-j(\omega-\omega_0)n}\end{aligned}$$

Let $\omega' = \omega - \omega_0$. Then,

$$\begin{aligned}X_w(e^{j\omega}) &= \sum_{n=0}^{N-1} e^{-j\omega' n} \\ &= \frac{1-e^{-j\omega' N}}{1-e^{-j\omega'}} \\ &= \frac{(e^{j\omega' N/2} - e^{-j\omega' N/2})e^{-j\omega' N/2}}{(e^{-j\omega'/2} - e^{-j\omega'/2})e^{-j\omega'/2}} \\ &= \frac{\sin(\omega' N/2)}{\sin(\omega'/2)}e^{-j\omega'(N-1)/2} \\ &= \frac{\sin[(\omega-\omega_0)N/2]}{\sin[(\omega-\omega_0)/2]}e^{-j(\omega-\omega_0)(N-1)/2}\end{aligned}$$

Note that $X_w(e^{j\omega})$ has generalized linear phase. Having established this equation for $X_w(e^{j\omega})$, we now find $X_w[k]$. Recall that $X_w[k]$ is simply the Fourier transform $X_w(e^{j\omega})$ evaluated at the frequencies $\omega = 2\pi k/N$, for $k = 0, \ldots, N-1$. Thus,

$$X_w[k] = \frac{\sin[(2\pi k/N - \omega_0)N/2]}{\sin[(2\pi k/N - \omega_0)/2]}e^{-j(2\pi k/N-\omega_0)(N-1)/2}$$

Note that the phase of $X_w[k]$, using the above equation, is

$$\angle X_w[k] = -\frac{(2\pi k/N - \omega_0)(N-1)}{2} + m\pi$$

where the $m\pi$ term comes from the fact that the term

$$\frac{\sin[(2\pi k/N - \omega_0)N/2]}{\sin[(2\pi k/N - \omega_0)/2]}$$

can change sign (i.e. become negative or positive), and thereby offset the phase by π radians. In addition, this term accounts for wrapping the phase, so that the phase stays in the range $[-\pi, \pi]$.

Re-expressing the equation for $\angle X_w[k]$, we find

$$\omega_0 = \frac{2(\angle X_w[k] - m\pi)}{N-1} + \frac{2\pi k}{N}$$

Let $X_{1w}[k]$ be the DFT of the 32-point sequence $x_{1w}[n] = x[n]w_1[n]$, and let $X_{2w}[k]$ be the DFT of the 8-point sequence $x_{2w}[n] = x[n]w_2[n]$. Note that the kth DFT coefficient of $X_{2w}[k]$ corresponds to $X_{1w}[4k]$. Thus, we can relate the 8 DFT coeffients of $X_{2w}[k]$ to 8 of the DFT coefficients in $X_{1w}[k]$. Using the $k = 0$th DFT coefficient for simplicity, we find

$$\begin{aligned}\omega_0 &= \frac{2(\angle X_{w1}[0] - m\pi)}{32-1} = \frac{2(\angle X_{w2}[0] - p\pi)}{8-1} \\ &= \frac{\angle X_{w1}[0] - m\pi}{15.5} = \frac{\angle X_{w2}[0] - p\pi}{3.5}\end{aligned}$$

A solution that satisfies these equations, with m and p integers, will yield a precise estimate of ω_0. We can accelerate solving these equations by determining which values of m and p to check. This is done by looking at the peak of $|X_w[k]|$ in a procedure similar to Part (c). Suppose that the indices for two largest values of $|X_w[k]|$ are k_{min} and k_{max}. Then, we know that the peak of $|X(e^{j\omega})|$ will occur in the range

$$\frac{2\pi k_{min}}{N} \leq \omega_0 \leq \frac{2\pi k_{max}}{N}$$

By re-expressing the equation for $\angle X_w[k]$, we see that

$$\begin{aligned}m_{min} &= \frac{1}{\pi}\left[2\angle X_{w1}[k_{min}] + \left(\frac{2\pi k_{min}}{N} - \hat{\omega}_0\right)(N-1)\right] \\ m_{max} &= \frac{1}{\pi}\left[2\angle X_{w1}[k_{max}] + \left(\frac{2\pi k_{max}}{N} - \hat{\omega}_0\right)(N-1)\right]\end{aligned}$$

In these equations, $\hat{\omega}_0$ is the estimate found in Part (c). So we would look for values of m in the range $[\lfloor m_{min} \rfloor, \lceil m_{max} \rceil]$. Similar expressions hold for p.

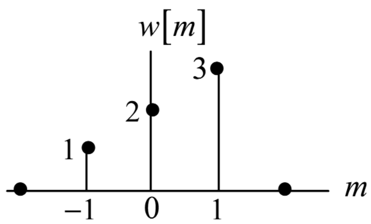
Once ω_0 is known, we can find Ω_0 using the relation $\Omega_0 = \omega_0/T$.

10.25

A.

$$\begin{aligned} y_q[n] &= \sum_{m=-\infty}^{\infty} h_q[m]x[n-m] \\ &= \sum_{m=-\infty}^{\infty} x[n+m]h_q[-m] \\ &= \sum_{m=-\infty}^{\infty} e^{-j2\pi qm/M} x[n+m]h_0[-m] \\ &= X[n,\lambda)\big|_{\lambda=2\pi q/M}. \end{aligned}$$

The associated window is $w[m]=\delta[m+1]+2\delta[m]+3\delta[m-1]$.



B. We have $M=N$. Note that $w[m]=0$ outside of $m=-1,0,1$.

$$\begin{aligned} y_q[n] &= \sum_{m=0}^{2} x[n+m-1]w[m-1]e^{-j2\pi q(m-1)/N} \\ &= e^{j2\pi q/N}\{\text{the } N\text{-point DFT of } x[n+m-1]w[m-1]\}. \end{aligned}$$

$$\begin{aligned} e^{-j2\pi q/N} y_q[Rn] &= e^{-j2\pi q/N} v[q,n] \\ &= \{\text{the } N\text{-point DFT of } x[Rn+m-1]w[m-1]\}, \end{aligned}$$

so

$$x[Rn+m-1]w[m-1] = \text{IDFT}\{e^{-j2\pi q/N} v[q,n]\}.$$

In general, from this last expression and knowing $w[m-1]$ for $m=0,1,2$ we can obtain $x[Rn-1], x[Rn], x[Rn+1]$. For example, if $R=2$ we can obtain $x[-1], x[0], x[1]$ from $n=0$, $x[1], x[2], x[3]$ from $n=1$, etc.

C. As long as R is less than or equal to the length of $w[m-1]$ we can recover all of the values of $x[n]$. For $R=5$ there are gaps in the recovery.

10.26. Problem 2 from Spring 2000 final exam
Appears in: Fall05 PS10.

Problem

We are interested in obtaining 256 equally-spaced samples of the z-transform of $x_w[n]$. $x_w[n]$ is a windowed version of an arbitrary sequence $x[n]$ where $x_w[n] = x[n]w[n]$ and $w[n] = 1$, $0 \leq n \leq 255$ and $w[n] = 0$ otherwise. The z-transform of $x_w[n]$ is defined as

$$X_w(z) = \sum_{n=0}^{255} x[n]z^{-n}.$$

The samples $X_w[k]$ that we would like to compute are

$$X_w[k] = X_w(z)\big|_{z=0.9e^{j\frac{2\pi}{256}k}} \qquad k = 0, 1, \ldots, 255.$$

We would like to process the signal $x[n]$ with a modulated filter bank, as indicated in Figure 10.2-1

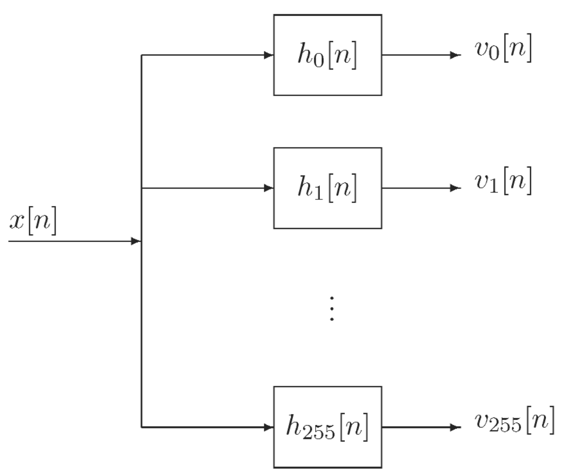


Figure 10.2-1

Each filter in the filter bank has an impulse response that is related to the prototype *causal* lowpass filter $h_0[n]$ as follows:

$$h_k[n] = h_0[n]e^{-j\omega_k n} \qquad k = 1, 2, \ldots, 255.$$

Each output of the filter bank is sampled once, at time $n = N_k$, to obtain $X_w[k]$, i.e.

$$X_w[k] = v_k[N_k].$$

Determine $h_0[n]$, ω_k and N_k so that

$$X_w[k] = v_k[N_k] = X_w(z)\big|_{z=0.9e^{j\frac{2\pi}{256}k}} \qquad k = 0, 1, \ldots, 255.$$

Solution from Fall05 PS10

$X_w[k]$ is defined as

$$X_w[k] = \sum_{n=0}^{255} x[n]0.9^{-n}e^{-j\frac{2\pi}{256}kn},$$

which is what we'd like our system to eventually implement. In terms of $v_k[n]$ this is

$$X_w[k] = v_k[N_k] = \sum_{n=-\infty}^{\infty} x[n]h_k[N_k - n] = \sum_{n=-\infty}^{\infty} x[n]h_0[N_k - n]e^{-j\omega_k(N_k-n)}.$$

We can allow the limits of the sum to go from $n = 0$ to $n = 255$ if we restrict $h_0[N_k - n]$ to be possibly nonzero only for $N_k - n \geq 0$ and $N_k - n \leq 255$, or equivalently, for $N_k - 255 \leq n \leq N_k$. Since the prototype filter must be causal, $N_k - 255$ (the lower limit on the filter's possibly nonzero region) must be greater than or equal to 0. N_k can then be judiciously chosen to be

$$N_k = 256 \quad \forall k.$$

We now have

$$v_k[256] = \sum_{n=0}^{255} x[n]h_0[256 - n]e^{-j\omega_k(256-n)}.$$

Putting issues with the exponential term aside for the moment, we know we'd like to have

$$h_0[256 - n] = \begin{cases} 0.9^{-n}, & 0 \leq n \leq 255 \\ 0, & \text{otherwise} \end{cases}.$$

With a change of variables this becomes

$$h_0[n] = \begin{cases} 0.9^{n-256}, & 1 \leq n \leq 256 \\ 0, & \text{otherwise} \end{cases},$$

and so we now have

$$v_k[256] = \sum_{n=0}^{255} x[n]0.9^{-n}e^{-j\omega_k(256-n)}.$$

We'd still like

$$e^{-j\omega_k(256-n)} = e^{-j\frac{2\pi}{256}kn},$$

which is satisfied for

$$\omega_k = -\frac{2\pi}{256}k.$$

We now have

$$v_k[256] = \sum_{n=0}^{255} x[n]0.9^{-n}e^{j\frac{2\pi}{256}k(256-n)} = \sum_{n=0}^{255} x[n]0.9^{-n}e^{j2\pi k}e^{-j\frac{2\pi}{256}kn} = \sum_{n=0}^{255} x[n]0.9^{-n}e^{-j\frac{2\pi}{256}kn}.$$

Solution from Spring2000 Final

N/A

10.27

A. The system shown in the figure computes an $N = 512$-point time-dependent DFT $G_k[n] = G[rL, k]$, where $n = rL, \quad -\infty < r < \infty$. The coefficients $a_l, \quad l = 0, \dots, N-1$ determine the window function.

To minimize the number of multiplies per second, windowing and DFT calculations should be made as infrequently as possible. This will occur if L is given its maximum value of $L = 256$.

If we choose $L = 256$, then $R = 1$.

B. For the system of part A with $L = 256$ and $R = 1$, we have

$$X_k[n] = G_k[n] = \sum_{l=0}^{511} a_l x[256n - l] e^{-j\frac{2\pi kl}{512}}, \quad k = 0, \dots 511.$$

For the system of part B convolution gives

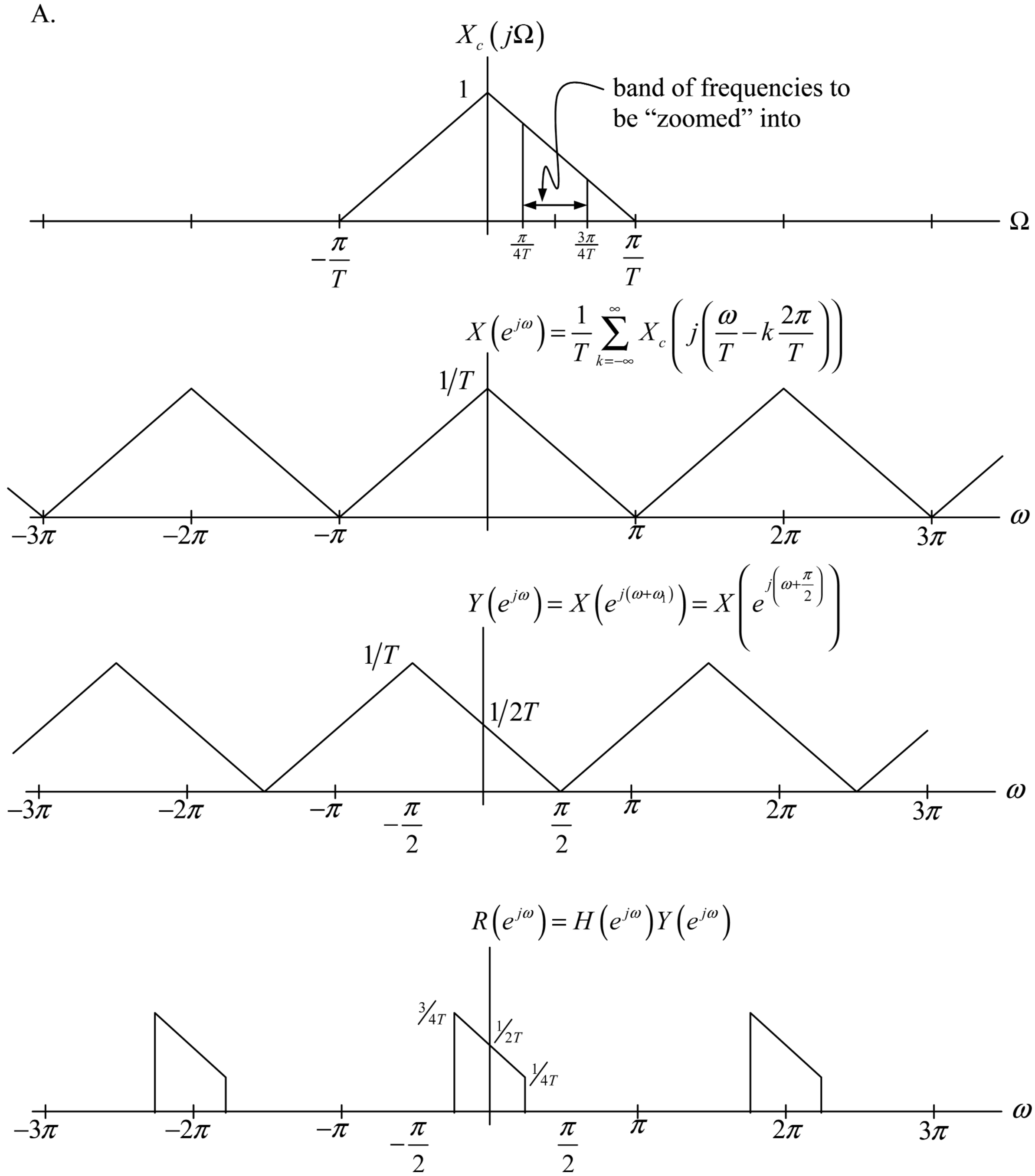
$$Y_k[n] = \sum_{l=0}^{255} x[nM - l](0.93)^l e^{-j\omega_k l}, \quad k = 0, \dots, 255.$$

If the systems are to provide identical outputs we must have $M = 256$ (choice (a)),

$\omega_k = \dfrac{2\pi k}{512}$ (choice (b)), and $a_l = \begin{cases} (0.93)^l, & l = 0, \dots, 255 \\ 0, & l = 265, \dots, 511, \end{cases}$ (choice (a)).

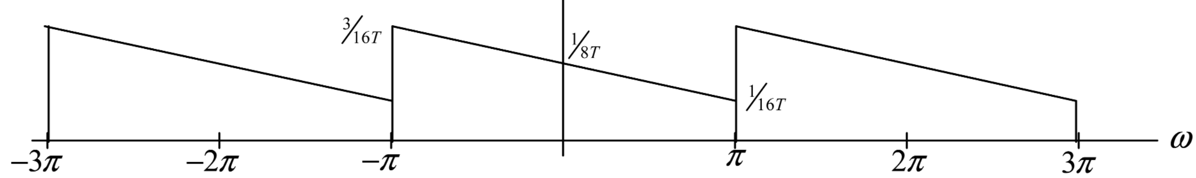
10.28

A.



$X_c(j\Omega)$
1
band of frequencies to be "zoomed" into
Ω
$-\frac{\pi}{T}$ $\frac{\pi}{4T}$ $\frac{3\pi}{4T}$ $\frac{\pi}{T}$
$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right)\right)$
$1/T$
-3π -2π $-\pi$ π 2π 3π ω
$Y(e^{j\omega}) = X(e^{j(\omega+\omega_1)}) = X\left(e^{j\left(\omega+\frac{\pi}{2}\right)}\right)$
$1/T$
$1/2T$
-3π -2π $-\pi$ $-\frac{\pi}{2}$ $\frac{\pi}{2}$ π 2π 3π ω
$R(e^{j\omega}) = H(e^{j\omega})Y(e^{j\omega})$
$3/4T$
$1/2T$
$1/4T$
-3π -2π $-\pi$ $-\frac{\pi}{2}$ $\frac{\pi}{2}$ π 2π 3π ω

$$R_d\left(e^{j\omega}\right)=\frac{1}{M}\sum_{r=0}^{M-1}R\left(e^{j(\omega-2\pi r)/M}\right)=\frac{1}{4}R\left(e^{j\frac{\omega}{4}}\right)$$



B. The spectrum of the discrete-time system first gets centered on ω_1. We then pick frequencies between $\omega_1-\dfrac{\pi}{M}$ and $\omega_1+\dfrac{\pi}{M}$ and "zoom" into those. Therefore the band of continuous frequencies is given by $\left(\omega_1-\dfrac{\pi}{M}\right)\dfrac{1}{T}$ to $\left(\omega_1+\dfrac{\pi}{M}\right)\dfrac{1}{T}$. When $\omega_1=\pi/2$ and $M=4$ we zoom into the band $\left(\dfrac{\pi}{2}-\dfrac{\pi}{4}\right)\dfrac{1}{T}$ to $\left(\dfrac{\pi}{2}+\dfrac{\pi}{4}\right)\dfrac{1}{T}$, or $\dfrac{\pi}{4T}$ to $\dfrac{3\pi}{4T}$.

C. As is evident from the figure in part A, $V[k]$ is going to be formed by sampling the DTFT of $x_d[n]$ from 0 to 2π. The windowing operation distorts the true $R_d\left(e^{j\omega}\right)$ but does not affect the end points in the frequency domain.

i. The frequencies in the range $0\le k\le N/2$ correspond to the frequencies between ω_1 and $\omega_1+\dfrac{\pi}{M}$ in $X\left(e^{j\omega}\right)$. Thus the continuous-time frequenies that were sampled are given by $\Omega_k=\left(\omega_1+\dfrac{2\pi k}{NM}\right)\dfrac{1}{T}$.

ii. Similarly, the frequencies in the range $N/2<k\le N-1$ correspond to the frequencies between $\omega_1-\dfrac{\pi}{M}$ and ω_1. Hence $\Omega_k=\left(\omega_1-\dfrac{\pi}{M}+\dfrac{2\pi\left(k-\dfrac{N}{2}\right)}{NM}\right)\dfrac{1}{T}$.

10.29. In this problem, we relate the DFT $X[k]$ of a discrete-time signal $x[n]$ to the continuous-time Fourier transform $X_c(j\Omega)$ of the continuous-time signal $x_c(t)$. Since $x[n]$ is obtained by sampling $x_c(t)$,

$$\begin{aligned} x[n] &= x_c(nT) \\ X(e^{j\omega}) &= \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} + j\frac{2\pi r}{T}\right) \end{aligned}$$

Over one period, assuming no aliasing, this is

$$X(e^{j\omega}) = \frac{1}{T}X_c\left(j\frac{\omega}{T}\right) \qquad \text{for } -\pi \le \omega \le \pi$$

which is equivalent to

$$X(e^{j\omega}) = \begin{cases} \frac{1}{T}X_c\left(j\frac{\omega}{T}\right), & \text{for } 0 \le \omega < \pi \\ \frac{1}{T}X_c\left(j\frac{\omega-2\pi}{T}\right), & \text{for } \pi \le \omega < 2\pi \end{cases}$$

Since the DFT is a sampled version of $X(e^{j\omega})$,

$$X[k] = X(e^{j\omega})\big|_{\omega=2\pi k/N} \qquad \text{for } 0 \le k \le N-1$$

we find

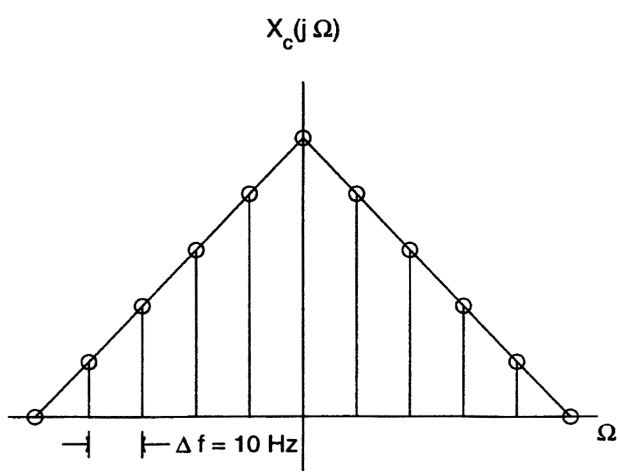
$$X[k] = \begin{cases} \frac{1}{T}X_c\left(j\frac{2\pi k}{NT}\right), & \text{for } 0 \le k < \frac{N}{2} \\ \frac{1}{T}X_c\left(j\frac{2\pi(k-N)}{NT}\right), & \text{for } \frac{N}{2} \le k \le N-1 \end{cases}$$

Breaking up the DFT into two terms like this is necessary to relate the negative frequencies of $X_c(j\Omega)$ to the proper indicies $\frac{N}{2} \le k \le N-1$ in $X[k]$.

Method 1: Using the above equation for $X[k]$, and plugging in values of $N = 4000$, and $T = 25\mu$s, we find

$$X_1[k] = \begin{cases} 40{,}000X_c\left(j2\pi \cdot 10 \cdot k\right), & \text{for } 0 \le k \le 1999 \\ 40{,}000X_c\left(j2\pi \cdot 10 \cdot (k-4000)\right), & \text{for } 2000 \le k \le 3999 \end{cases}$$

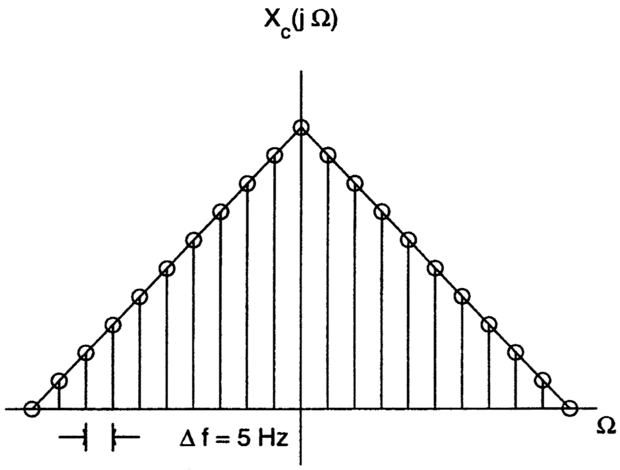
Therefore, we see this does not provide the desired samples. A sketch is provided below, for a triangular-shaped $X_c(j\Omega)$.



Method 2: This time we plug in values of $N = 4000$, and $T = 50\mu s$ to find

$$X_2[k] = \begin{cases} 20{,}000X_c\left(j2\pi \cdot 5 \cdot k\right), & \text{for } 0 \le k \le 1999 \\ 20{,}000X_c\left(j2\pi \cdot 5 \cdot (k - 4000)\right), & \text{for } 2000 \le k \le 3999 \end{cases}$$

Therefore, we see this *does* provide the desired samples. A sketch is provided below.



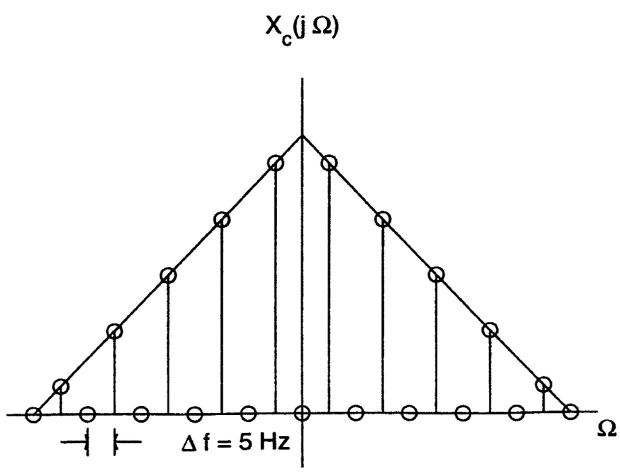
Method 3: Noting that $x_3[n] = x_2[n] + x_2\left[n - \frac{N}{2}\right]$, we get

$$X_3[k] = X_2[k] + (-1)^k X_2[k]$$

$$X_3[k] = \begin{cases} 2X_2[k], & \text{for } k \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

$$X_3[k] = \begin{cases} 40{,}000X_c\left(j2\pi \cdot 5 \cdot k\right), & \text{for } k \text{ even, and } 0 \le k \le 1999 \\ 40{,}000X_c\left(j2\pi \cdot 5 \cdot (k - 4000)\right), & \text{for } k \text{ even, and } 2000 \le k \le 3999 \\ 0, & \text{otherwise} \end{cases}$$

This system provides the desired samples only for k an even integer. A sketch is provided below.



10.30. (a) In this problem, we relate the DFT $X[k]$ of a discrete-time signal $x[n]$ to the continuous-time Fourier tranform $X_c(j\Omega)$ of the continuous-time signal $x_c(t)$. Since $x[n]$ is obtained by sampling $x_c(t)$,

$$\begin{aligned} x[n] &= x_c(nT) \\ X(e^{j\omega}) &= \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} + j\frac{2\pi r}{T}\right) \end{aligned}$$

Over one period, assuming no aliasing, this is

$$X(e^{j\omega}) = \frac{1}{T}X_c\left(j\frac{\omega}{T}\right) \qquad \text{for } -\pi \le \omega \le \pi$$

which is equivalent to

$$X(e^{j\omega}) = \begin{cases} \frac{1}{T}X_c\left(j\frac{\omega}{T}\right), & \text{for } 0 \le \omega < \pi \\ \frac{1}{T}X_c\left(j\frac{\omega-2\pi}{T}\right), & \text{for } \pi \le \omega < 2\pi \end{cases}$$

Since the DFT is a sampled version of $X(e^{j\omega})$,

$$X[k] = X(e^{j\omega})\big|_{\omega=2\pi k/N} \qquad \text{for } 0 \le k \le N-1$$

we find

$$X[k] = \begin{cases} \frac{1}{T}X_c\left(j\frac{2\pi k}{NT}\right), & \text{for } 0 \le k < \frac{N}{2} \\ \frac{1}{T}X_c\left(j\frac{2\pi(k-N)}{NT}\right), & \text{for } \frac{N}{2} \le k \le N-1 \end{cases}$$

Breaking up the DFT into two terms like this is necessary to relate the negative frequencies of $X_c(j\Omega)$ to the proper indicies $\frac{N}{2} \le k \le N-1$ in $X[k]$.

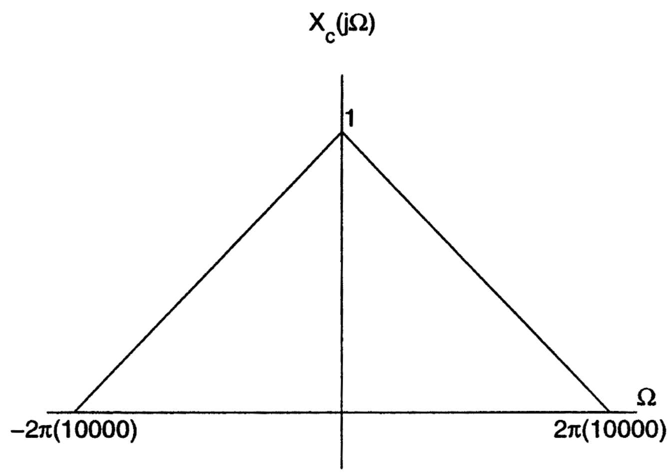
The effective frequency spacing is

$$\begin{aligned} \Delta\Omega &= \frac{2\pi}{NT} \\ &= \frac{2\pi}{(1000)(1/20,000)} \\ &= 2\pi(20) \text{ rad/s} \end{aligned}$$

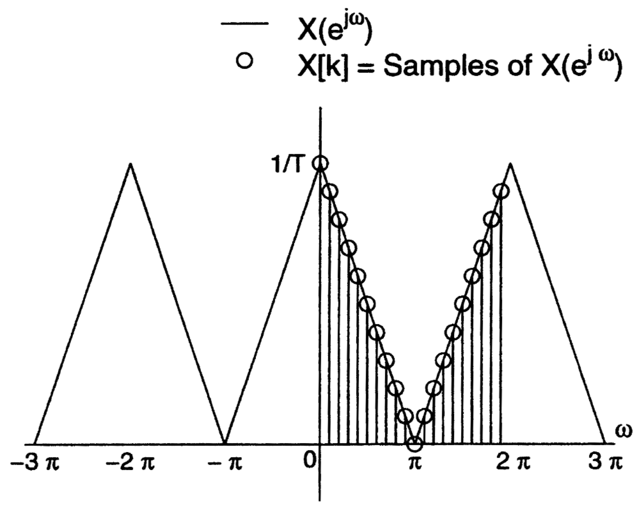
(b) Next, we determine if the designer's assertion that

$$Y[k] = \alpha X_c(j2\pi \cdot 10 \cdot k)$$

is correct. To understand the effect of each step in the procedure, it helps to draw some frequency domain plots. Assume the spectrum of the original signal $x_c(t)$ looks like



Sampling this continuous-time signal will produce the discrete-time signal $x[n]$, with a spectrum

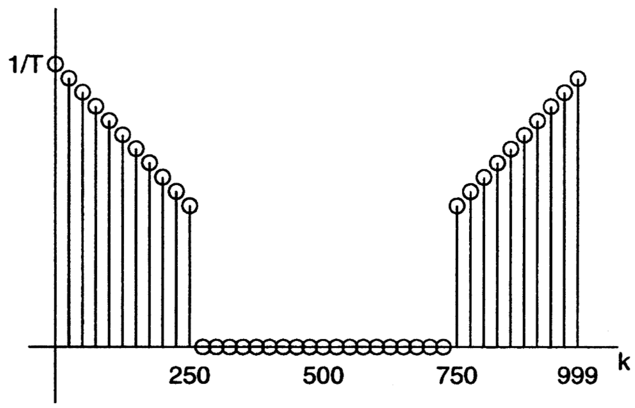


Next, we form

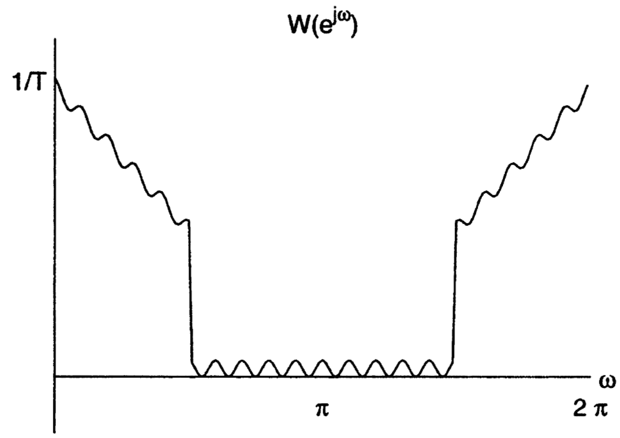
$$W[k] = \begin{cases} X[k], & 0 \le k \le 250 \\ 0, & 251 \le k \le 749 \\ X[k], & 750 \le k \le 999 \end{cases}$$

and find $w[n]$ as the inverse DFT of $W[k]$.

W[k]



Before going on, we should plot the Fourier transform, $W(e^{j\omega})$, of $w[n]$. It will look like



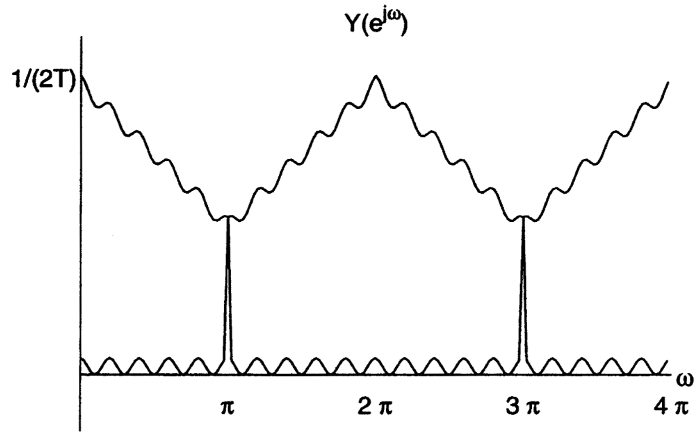
$W(e^{j\omega})$ goes through the DFT points and therefore is equal to samples of $X_c(j\Omega)$ at these points for $0 \le k \le 250$ and $750 \le k \le 999$, but it is *not equal* to $X_c(j\Omega)$ between those frequencies. Furthermore, $W(e^{j\omega}) = 0$ at the DFT frequencies for $251 \le k \le 749$, but it is not zero between those frequencies; i.e. we can not do ideal lowpass filtering using the DFT.

Now we define

$$y[n] = \begin{cases} w[2n], & 0 \le n \le 499 \\ 0, & 500 \le n \le 999 \end{cases}$$

and let $Y[k]$ be the DFT of $y[n]$. First note that $Y(e^{j\omega})$ is

$$Y(e^{j\omega}) = \frac{1}{2}W(e^{j\omega/2}) + \frac{1}{2}W(e^{j(\omega-2\pi)/2})$$



$Y[k]$ is equal to samples of the $Y(e^{j\omega})$

$$\begin{aligned} Y[k] &= Y(e^{j\omega})|_{\omega=2\pi k/N} \\ &= \frac{1}{2}W\left(e^{j\frac{2\pi}{N}\frac{k}{2}}\right) + \frac{1}{2}W\left(e^{j\frac{2\pi}{N}\left(\frac{k-N}{2}\right)}\right) \end{aligned}$$

Now putting all that we know together, we see that for $k = 0, 1, \ldots, 500$, $Y[k]$ is related to $X_c(j\Omega)$ as follows.

$$Y[k] = \begin{cases} \frac{1}{2T}X_c(j2\pi \cdot 10 \cdot k), & k \text{ even}, k \neq 500 \\ \frac{1}{T}X_c(j2\pi \cdot 10 \cdot k), & k = 500 \\ \frac{1}{2T}W(e^{j\pi k/N}) + \frac{1}{2T}W(e^{j\pi(k-N)/N}) & k \text{ odd} \end{cases}$$

In other words, the even indexed DFT samples are not aliased, but the odd indexed values (and $k = 500$) are aliased. The designer's assertion is not correct.

10.31. (a) Starting with definition of the time-dependent Fourier transform,

$$Y[n,\lambda) = \sum_{m=-\infty}^{\infty} y[n+m]w[m]e^{-j\lambda m}$$

we plug in

$$y[n+m] = \sum_{k=0}^{M} h[k]x[n+m-k]$$

to get

$$\begin{aligned} Y[n,\lambda) &= \sum_{m=-\infty}^{\infty}\sum_{k=0}^{M} h[k]x[n+m-k]w[m]e^{-j\lambda m} \\ &= \sum_{k=0}^{M} h[k] \sum_{m=-\infty}^{\infty} x[n+m-k]w[m]e^{-j\lambda m} \\ &= \sum_{k=0}^{M} h[k]X[n-k,\lambda) \\ &= h[n] * X[n,\lambda) \end{aligned}$$

where the convolution is for the variable n.

(b) Starting with

$$\check{Y}[n,\lambda) = e^{-j\lambda n}Y[n,\lambda)$$

we find

$$\begin{aligned} \check{Y}[n,\lambda) &= e^{-j\lambda n}\left[\sum_{k=0}^{M} h[k]X[n-k,\lambda)\right] \\ &= e^{-j\lambda n}\left[\sum_{k=0}^{M} h[k]e^{j(n-k)\lambda}\check{X}[n-k,\lambda)\right] \\ &= \sum_{k=0}^{M} h[k]e^{-j\lambda k}\check{X}[n-k,\lambda) \end{aligned}$$

If the window is long compared to M, then a small time shift in $\check{X}[n,\lambda)$ won't radically alter the spectrum, and

$$\check{X}[n-k,\lambda) \simeq \check{X}[n,\lambda)$$

Consequently,

$$\begin{aligned} \check{Y}[n,\lambda) &\simeq \sum_{k=0}^{M} h[k]e^{-j\lambda k}\check{X}[n,\lambda) \\ &\simeq H(e^{j\lambda})\check{X}[n,\lambda) \end{aligned}$$

10.32

A. Spectrograms (a) and (c) were computed with a rectangular window; note the high sidelobes.

B. Spectrograms (a) & (b) have approximately the same frequency resolution, as do spectrograms (c) & (d). Note that the horizontal bars have approximately the same width in pairs having the same resolution.

C. Spectrogram (c) has the shortest time window. Note that this spectrogram has the broadest mainlobe in frequency and the shortest transient region at the ends of the bars.

D. The length of the window in spectrogram (b) is $L = 400$ samples. The length of the "fuzzy" region is essentially this window length. It takes that much time to engage and disengage.

E. Spectrograms do not give absolute amplitudes and they give no phase information. We have

$$x_c(t) = \begin{cases} A_1 \cos\left(0.4\pi\times 10^4 t + \phi_1\right) + A_2 \cos\left(0.7\pi\times 10^4 t + \phi_2\right), & 0 \le t \le 1000\times 10^{-4} \\ A_1 \cos\left(0.4\pi\times 10^4 t + \phi_1\right) + 0, & 0.1 \le t < 0.2 \\ A_1 \cos\left(0.4\pi\times 10^4 t + \phi_1\right) + A_3 \cos\left(0.5\pi\times 10^4 t + \phi_3\right), & 0.2 \le t. \end{cases}$$

10.33. Plugging in the relation for $c_{vv}[m]$ into the equation for $I(\omega)$ gives

$$\begin{aligned}
I(\omega) &= \frac{1}{LU} \sum_{m=-(L-1)}^{L-1} \left[\sum_{n=0}^{L-1} v[n]v[n+m] \right] e^{-j\omega m} \\
&= \frac{1}{LU} \sum_{n=0}^{L-1} v[n] \sum_{m=-(L-1)}^{L-1} v[n+m]e^{-j\omega m}
\end{aligned}$$

Let $\ell = n+m$ in the second summation. This gives

$$\begin{aligned}
I(\omega) &= \frac{1}{LU} \sum_{n=0}^{L-1} v[n] \sum_{\ell=n-(L-1)}^{n+(L-1)} v[\ell]e^{-j\omega(\ell-n)} \\
&= \frac{1}{LU} \sum_{n=0}^{L-1} v[n]e^{j\omega n} \sum_{\ell=n-(L-1)}^{n+(L-1)} v[\ell]e^{-j\omega \ell}
\end{aligned}$$

Note that for all values of $0 \le n \le L-1$, the second summation will be over *all* non-zero values of $v[\ell]$ in the range $0 \le \ell \le L-1$. As a result,

$$\begin{aligned}
I(\omega) &= \frac{1}{LU} \sum_{n=0}^{L-1} v[n]e^{j\omega n} \sum_{\ell=0}^{L-1} v[\ell]e^{-j\omega \ell} \\
&= \frac{1}{LU} V^*(e^{j\omega})V(e^{j\omega}) \\
&= \frac{1}{LU} |V(e^{j\omega})|^2
\end{aligned}$$

Note that in this analysis, we have assumed that $v[n]$ is a real sequence.

10.34. (a) Since $x[n]$ has length L, the aperiodic function, $c_{xx}[m]$, will be $2L-1$ points long. Therefore, in order for the aperiodic correlation function to equal the periodic correlation fuction, $\tilde{c}_{xx}[m]$, for $0 \le m \le L-1$, we require that the inverse DFT is not time aliased. So, the minimum inverse DFT length N_{min} is

$$N_{\text{min}} = 2L - 1$$

(b) If we require M points to be unaliased, we can have $L-M$ aliased points. Therefore, for $\tilde{c}_{xx}[m] = c_{xx}[m]$ for $0 \le m \le M-1$, the minimum inverse DFT length N_{min} is

$$\begin{aligned} N_{\text{min}} &= 2L - 1 - (L - M) \\ &= L + M - 1 \end{aligned}$$

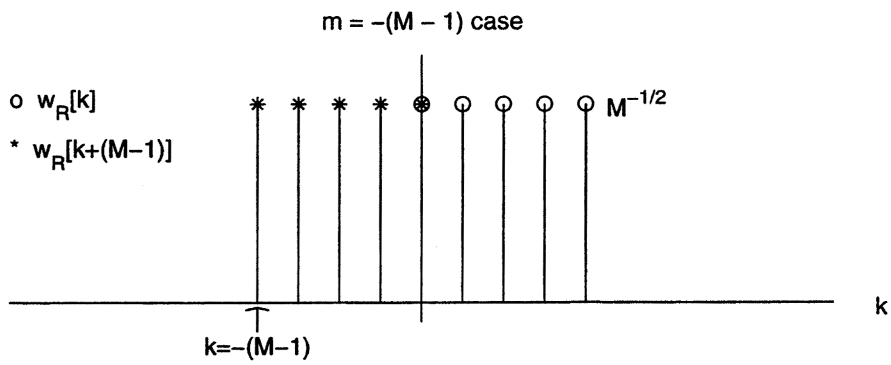
10.35. (a) Let

$$w_R[m] = \frac{1}{\sqrt{M}}(u[n] - u[n-M])$$

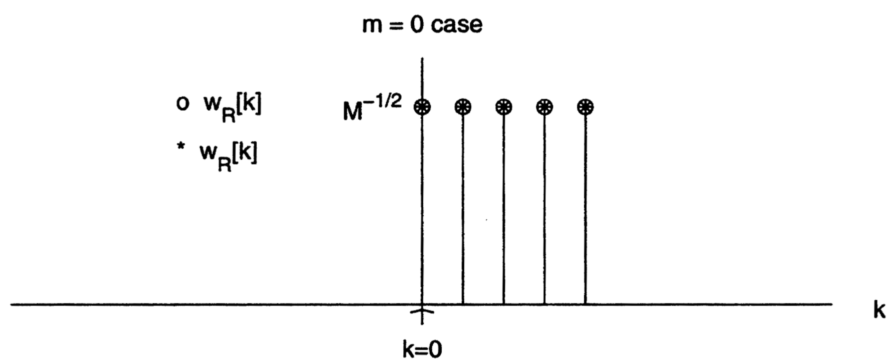
be a scaled rectangular pulse. Then we can write the aperiodic autocorrelation as,

$$\begin{aligned} w_B[m] &= \sum_{n=-\infty}^{\infty} w_R[n]w_R[n+m] \\ &= \sum_{k=-\infty}^{\infty} w_R[k-m]w_R[k] \\ &= \sum_{k=-\infty}^{\infty} w_R[k]w_R[-(m-k)] \\ &= w_R[m] * w_R[-m] \end{aligned}$$

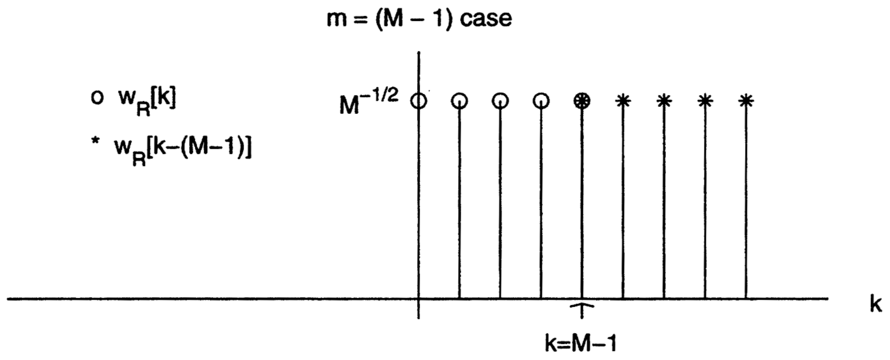
The convolution above is the triangular signal described by the symmetric Bartlett window formula. This is shown graphically below for a few critical cases of m.
Consider $m = -(M-1)$. This is first value of m for which the two signals overlap.



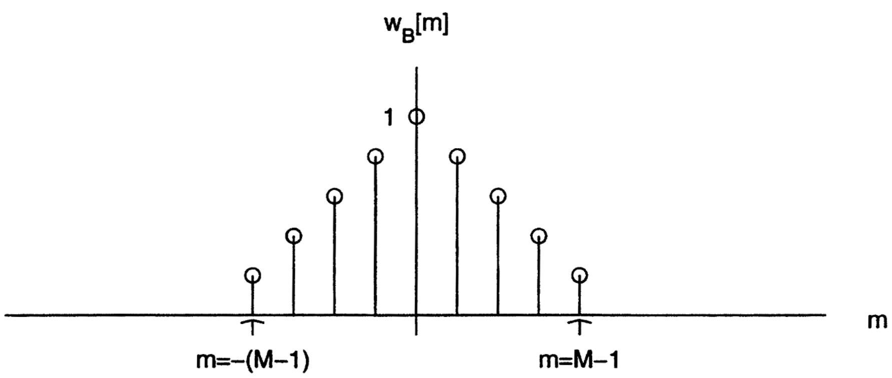
At $m = 0$, all non-zero samples overlap.



Consider $m = (M-1)$. This is last value of m for which the two signals overlap.



The final result of the aperiodic autocorrelation is



Stated mathematically, this is

$$w_B[m] = \begin{cases} 1 - |m|/M, & |m| \leq M - 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) The transform of the causal scaled rectangular pulse $w_R[n]$ is

$$W_R(e^{j\omega}) = \frac{1}{\sqrt{M}}\frac{\sin(\omega M/2)}{\sin(\omega/2)}e^{-j\omega(M-1)/2}$$

From part (a), we know that the Bartlett window can be found by convolving $w_R[m]$ with $w_R[-m]$. In the frequency domain, we therefore have,

$$\begin{aligned} W_B(e^{j\omega}) &= W_R(e^{j\omega})W_R(e^{-j\omega}) \\ &= \left[\frac{1}{\sqrt{M}}\frac{\sin(\omega M/2)}{\sin(\omega/2)}e^{-j\omega(M-1)/2}\right]\left[\frac{1}{\sqrt{M}}\frac{\sin(-\omega M/2)}{\sin(-\omega/2)}e^{j\omega(M-1)/2}\right] \\ &= \frac{1}{M}\left[\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right]^2 \end{aligned}$$

(c) The power spectrum, defined as the Fourier transform of the aperiodic autocorrelation sequence, is always nonnegative. Thus, any window that can be represented as an aperiodic autocorrelation sequence will have a nonnegative Fourier transform. So to generate other finite-length window sequences, $w[n]$, that have nonnegative Fourier transforms, simply take the aperiodic autocorrelation of an input sequence, $x[n]$.

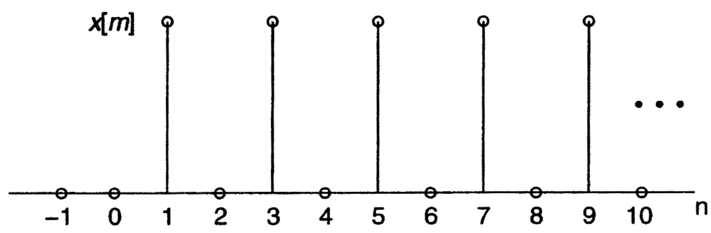
$$w[n] = \sum_{m=-\infty}^{\infty} x[m]x[n+m]$$

The signal $w[n]$ will have a nonnegative Fourier transform.

10.36. (a) Using the definition of the time-dependent Fourier transform we find

$$\begin{aligned}
X[0,k] &= \sum_{m=0}^{13} x[m]e^{-j(2\pi/7)km} \\
&= \sum_{m=0}^{6} x[m]e^{-j(2\pi/7)km} + \sum_{l=7}^{13} x[l]e^{-j(2\pi/7)kl} \\
&= \sum_{m=0}^{6} x[m]e^{-j(2\pi/7)km} + \sum_{m=0}^{6} x[m+7]e^{-j(2\pi/7)km}e^{-j2\pi k} \\
&= \sum_{m=0}^{6} (x[m]+x[m+7])e^{-j(2\pi/7)km}
\end{aligned}$$

By plotting $x[m]$



we see that $x[m] + x[m+7] = 1$ for $0 \le m \le 6$. Thus,

$$\begin{aligned}
X[0,k] &= \sum_{m=0}^{6} (1)e^{-j(2\pi/7)km} \\
&= \mathcal{DFT}\{1\} \\
&= 7\delta[k]
\end{aligned}$$

(b) If we follow the same procedure we used in part (a) we find

$$\begin{aligned}
X[n,k] &= \sum_{m=0}^{13} x[n+m]e^{-j(2\pi/7)km} \\
&= \sum_{m=0}^{6} x[n+m]e^{-j(2\pi/7)km} + \sum_{l=7}^{13} x[n+l]e^{-j(2\pi/7)kl} \\
&= \sum_{m=0}^{6} (x[n+m]+x[n+m+7])e^{-j(2\pi/7)km}
\end{aligned}$$

With $n \ge 0$ we have $x[n+m] + x[n+m+7] = 1$ for $0 \le m \le 6$, and so

$$\begin{aligned}
X[n,k] &= \mathcal{DFT}\{1\} \\
&= 7\delta[k]
\end{aligned}$$

Therefore, for $0 \le n \le \infty$ we have

$$\begin{aligned}
\sum_{k=0}^{6} X[n,k] &= \sum_{k=0}^{6} 7\delta[k] \\
&= 7
\end{aligned}$$

10.37. (a) **Rectangular:** The Fourier transform of the rectangular window is given by

$$W_R(e^{j\omega}) = \sum_{m=-(M-1)}^{M-1} (1)e^{-j\omega m}$$

Let $n = m + (M-1)$. Then, $m = n - (M-1)$, and

$$\begin{aligned} W_R(e^{j\omega}) &= \sum_{n=0}^{2(M-1)} e^{-j\omega[n-(M-1)]} \\ &= e^{j\omega(M-1)} \sum_{n=0}^{2(M-1)} e^{-j\omega n} \end{aligned}$$

Using the relation

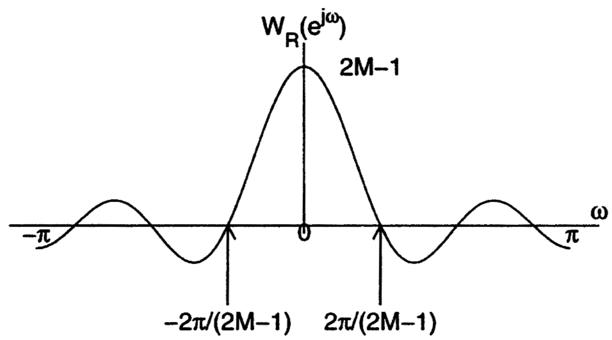
$$\sum_{n=0}^{M-1} a^n = \frac{1-a^M}{1-a}$$

we find

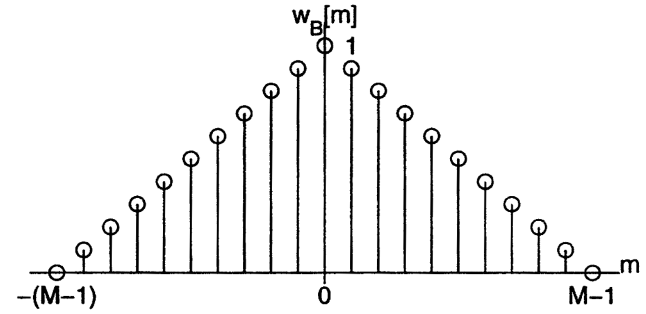
$$\begin{aligned} W_R(e^{j\omega}) &= e^{j\omega(M-1)} \frac{1-e^{-j\omega[2(M-1)+1]}}{1-e^{-j\omega}} \\ &= e^{j\omega(M-1)} \frac{1-e^{-j\omega(2M-1)}}{1-e^{-j\omega}} \\ &= \frac{e^{j\omega(M-1)} - e^{-j\omega M}}{1-e^{-j\omega}} \\ &= \frac{e^{-j\omega/2}[e^{j\omega(M-1/2)} - e^{-j\omega(M-1/2)}]}{e^{-j\omega/2}[e^{j\omega/2} - e^{-j\omega/2}]} \\ &= \frac{e^{j\omega(M-1/2)} - e^{-j\omega(M-1/2)}}{e^{j\omega/2} - e^{-j\omega/2}} \\ &= \frac{2j\sin[\omega(M-\frac{1}{2})]}{2j\sin(\omega/2)} \\ &= \frac{\sin[\omega(M-\frac{1}{2})]}{\sin(\omega/2)} \end{aligned}$$

$$W_R(e^{j\omega}) = \frac{\sin[\omega\frac{2M-1}{2}]}{\sin(\omega/2)}$$

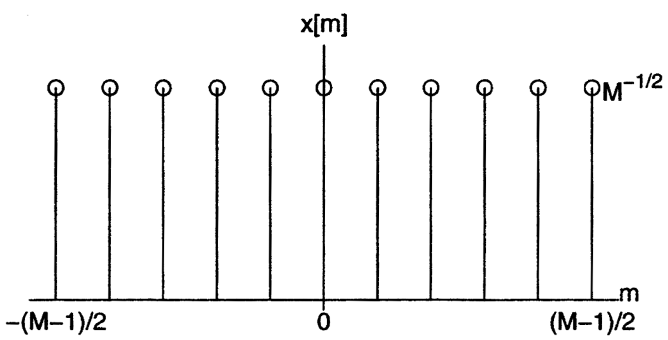
where $2M-1$ is the window length. A sketch of $W_R(e^{j\omega})$ appears below.



Bartlett (triangular): $W_B(e^{j\omega})$ is the Fourier transform of a triangular signal,



which is the convolution of a rectangular signal,



with itself. That is, $w_B[m] = x[m] * x[m]$.
Above, we found the Fourier tranform of a rectangular window, as

$$W_R(e^{j\omega}) = \frac{\sin[\omega \frac{2M-1}{2}]}{\sin(\omega/2)}$$

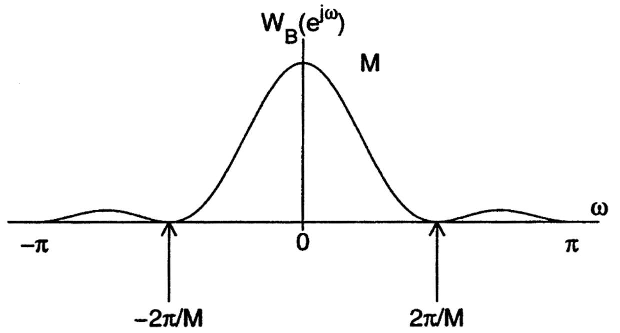
where $2M-1$ was the length of the window. We can use this result to find the Fourier transform of $x[m]$. The signal $x[m]$ is similar to the rectangular window, the difference being that it is scaled by $\frac{1}{\sqrt{M}}$ and has a length $2\frac{M-1}{2}+1=M$. Therefore,

$$X(e^{j\omega}) = \frac{1}{\sqrt{M}}\frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

The time domain convolution, $w_B[m] = x[m]*x[m]$ corresponds to a multiplication, $W_B(e^{j\omega}) = [X(e^{j\omega})]^2$ in the frequency domain. As a result,

$$\begin{aligned} W_B(e^{j\omega}) &= [X(e^{j\omega})]^2 \\ &= \frac{1}{M}\left[\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right]^2 \end{aligned}$$

A sketch of $W_B(e^{j\omega})$ appears below.



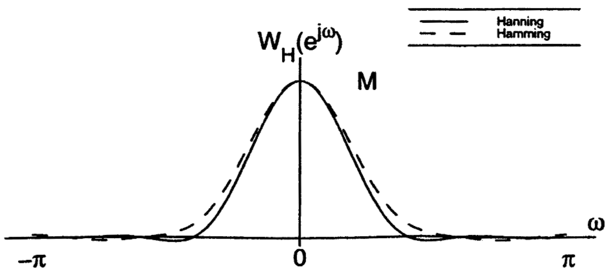
Hanning/Hamming: Starting with

$$\begin{aligned} w_H[m] &= (\alpha + \beta\cos[\pi m/(M-1)])\, w_R[m] \\ w_H[m] &= \left(\alpha + \frac{\beta}{2}e^{j\pi m/(M-1)} + \frac{\beta}{2}e^{-j\omega m/(M-1)}\right) w_R[m] \end{aligned}$$

We take the Fourier transform to find

$$\begin{aligned} W_H(e^{j\omega}) &= \alpha W_R(e^{j\omega}) + \frac{\beta}{2}\left(W_R(e^{j[\omega-\pi/(M-1)]}) + W_R(e^{j[\omega+\pi/(M-1)]})\right) \\ &= \alpha\frac{\sin[\omega\,(M-\frac{1}{2})]}{\sin(\omega/2)} + \frac{\beta}{2}\left[\frac{\sin[(\omega-\frac{\pi}{M-1})(M-\frac{1}{2})]}{\sin[(\omega-\frac{\pi}{M-1})/2]}\right] \\ &+ \frac{\beta}{2}\left[\frac{\sin[(\omega+\frac{\pi}{M-1})(M-\frac{1}{2})]}{\sin[(\omega+\frac{\pi}{M-1})/2]}\right] \end{aligned}$$

A sketch of $W_H(e^{j\omega})$ appears below.



(b) **Rectangular:** The approximate mainlobe width, and the approximate variance ratio, F, for the rectangular window are found below for large M.
In part (a), we found the Fourier transform of the rectangular window as

$$W_R(e^{j\omega}) = \frac{\sin[\omega(M-\frac{1}{2})]}{\sin(\omega/2)}$$

The numerator becomes zero when the argument of its sine term equals πn.

$$\begin{aligned}\frac{(2M-1)\omega}{2} &= \pi n\\ \omega &= \frac{2\pi n}{2M-1}\end{aligned}$$

Plugging in $n = 1$ gives us half the mainlobe bandwidth.

$$\begin{aligned}\frac{1}{2}\text{Mainlobe bandwidth} &= \frac{2\pi}{2M-1}\\ \text{Mainlobe bandwidth} &= \frac{4\pi}{2M-1}\\ \text{Mainlobe bandwidth} &\simeq \frac{2\pi}{M}\end{aligned}$$

$$\begin{aligned}\mathrm{F} &= \frac{1}{Q}\sum_{m=-(M-1)}^{(M-1)} w^2[m]\\ &= \frac{1}{Q}(2M-1)\\ &\simeq \frac{2M}{Q}\end{aligned}$$

Bartlett (triangular): The approximate mainlobe width, and the approximate variance ratio, F, for the Bartlett window are found below for large M.

In part (a), we found the Fourier transform of the Bartlett window as

$$W_B(e^{j\omega}) = \frac{1}{M}\left[\frac{\sin(\omega M/2)}{\sin(\omega/2)}\right]^2$$

The numerator becomes zero when the argument of its sine term equals πn.

$$\begin{aligned}\frac{\omega M}{2} &= \pi n\\ \omega &= \frac{2\pi n}{M}\end{aligned}$$

Plugging in $n = 1$ gives us half the mainlobe bandwidth.

$$\begin{aligned}
\frac{1}{2}\text{Mainlobe bandwidth} &= \frac{2\pi}{M} \\
\text{Mainlobe bandwidth} &= \frac{4\pi}{M}
\end{aligned}$$

To compute F, we use the relations

$$\begin{aligned}
\sum_{m=0}^{M-1} m &= \frac{M(M-1)}{2} \\
\sum_{m=0}^{M-1} m^2 &= \frac{M(M-1)(2M-1)}{6}
\end{aligned}$$

$$\begin{aligned}
F &= \frac{1}{Q}\sum_{m=-(M-1)}^{(M-1)}\left(1-\frac{|m|}{M}\right)^2 \\
&= \frac{1}{Q}\left[2\sum_{m=0}^{M-1}\left(1-\frac{m}{M}\right)^2-1\right] \\
&= \frac{1}{Q}\left[2\sum_{m=0}^{M-1}1-\frac{4}{M}\sum_{m=0}^{M-1}m+\frac{2}{M^2}\sum_{m=0}^{M-1}m^2-1\right] \\
&= \frac{1}{Q}\left[2M-\frac{4(M-1)M}{2M}+\frac{2(M-1)M(2M-1)}{6M^2}-1\right] \\
&\simeq \frac{1}{Q}\left[2M-2M+\frac{2M}{3}\right] \\
&\simeq \frac{2M}{3Q}
\end{aligned}$$

Hanning/Hamming: We can approximate the mainlobe bandwidth by analyzing the Fourier transform derived in Part (a). Looking at one of the terms from this expression,

$$\frac{\beta}{2}\left[\frac{\sin[(\omega-\frac{\pi}{M-1})(M-\frac{1}{2})]}{\sin[(\omega-\frac{\pi}{M-1})/2]}\right]$$

we note that the numerator is zero whenever the its argument equals πn, or

$$\begin{aligned}
\left(\omega-\frac{\pi}{M-1}\right)\left(M-\frac{1}{2}\right) &= \pi n \\
\omega &= \frac{n\pi}{M-(1/2)}+\frac{\pi}{M-1} \\
&\simeq \frac{n\pi}{M}+\frac{\pi}{M} \\
&\simeq \frac{\pi(n+1)}{M}
\end{aligned}$$

So the mainlobe bandwidth for this term is

$$\begin{aligned}
\frac{1}{2}\text{Mainlobe bandwidth} &\simeq \frac{\pi}{M} \\
\text{Mainlobe bandwidth} &\simeq \frac{2\pi}{M}
\end{aligned}$$

Note that the peak value for this term occurs at a frequency $\omega \simeq \pi/M$.

A similar analysis can be applied to the other terms in Fourier transform derived in Part (a). The mainlobe bandwidth for the term

$$\frac{\beta}{2}\left[\frac{\sin[(\omega+\frac{\pi}{M-1})(M-\frac{1}{2})]}{\sin[(\omega+\frac{\pi}{M-1})/2]}\right]$$

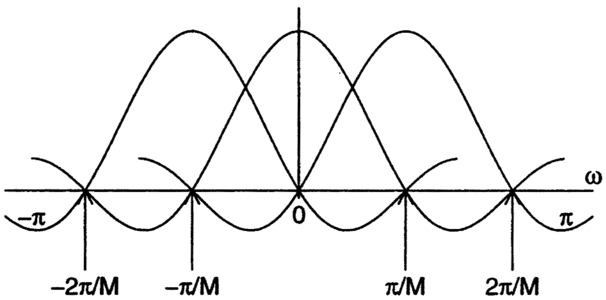
is also $2\pi/M$. Note that the peak value for this term occurs at a frequency $\omega \simeq -\pi/M$.

Finally, the mainlobe bandwidth for the term

$$\alpha\frac{\sin[\omega\left(M-\frac{1}{2}\right)]}{\sin(\omega/2)}$$

is also $2\pi/M$. Note that the peak value for this term occurs at a frequency $\omega = 0$.

A sample plot of these three terms, for $\beta = 2\alpha$ and large M is shown below.



Thus, for large M, the mainlobe bandwidth is bounded by

$$\frac{2\pi}{M} < \text{Mainlobe bandwidth} < \frac{4\pi}{M}$$

Therefore, a reasonable approximation for the mainlobe bandwidth is

$$\text{Mainlobe bandwidth} \simeq \frac{3\pi}{M}$$

$$\begin{aligned} \mathrm{F} &= \frac{1}{Q} \sum_{m=-(M-1)}^{M-1} \left(\alpha + \beta \cos\left(\frac{\pi m}{M-1} \right) \right)^2 \\ &= \frac{1}{Q} \left[\sum_{m=-(M-1)}^{M-1} \alpha^2 + 2\alpha\beta \sum_{m=-(M-1)}^{M-1} \cos\left(\frac{\pi m}{M-1} \right) + \beta^2 \sum_{m=-(M-1)}^{M-1} \cos^2\left(\frac{\pi m}{M-1} \right) \right] \end{aligned}$$

Using the relation

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

we get

$$\begin{aligned} \mathrm{F} &= \frac{1}{Q} \left[\sum_{m=-(M-1)}^{M-1} \alpha^2 + 2\alpha\beta \sum_{m=-(M-1)}^{M-1} \cos\left(\frac{\pi m}{M-1} \right) \right. \\ &\quad \left. + \frac{\beta^2}{2} \sum_{m=-(M-1)}^{M-1} (1) + \frac{\beta^2}{2} \sum_{m=-(M-1)}^{M-1} \cos\left(\frac{2\pi m}{M-1} \right) \right] \end{aligned}$$

Noting that

$$\begin{aligned} \sum_{m=-(M-1)}^{M-1} \cos\left(\frac{\pi m}{M-1} \right) &= -1 \\ \sum_{m=-(M-1)}^{M-1} \cos\left(\frac{2\pi m}{M-1} \right) &= 1 \end{aligned}$$

we conclude

$$\begin{aligned} \mathrm{F} &= \frac{1}{Q} \left[(2M-1)\alpha^2 - 2\alpha\beta + \frac{\beta^2}{2}(2M-1) + \frac{\beta^2}{2} \right] \\ &\simeq \frac{2M}{Q} \left(\alpha^2 + \frac{\beta^2}{2} \right) \end{aligned}$$

10.38. For each part, we use the definition of the time-dependent Fourier transform,

$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}.$$

(a) *Linearity*: using $x[n] = ax_1[n] + bx_2[n]$,

$$\begin{aligned}
X[n,\lambda) &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m} \\
&= \sum_{m=-\infty}^{\infty} (ax_1[n+m] + bx_2[n+m])\, w[m]e^{-j\lambda m} \\
&= a\sum_{m=-\infty}^{\infty} x_1[n+m]w[m]e^{-j\lambda m} + b\sum_{m=-\infty}^{\infty} x_2[n+m]w[m]e^{-j\lambda m} \\
&= aX_1[n,\lambda) + bX_2[n,\lambda)
\end{aligned}$$

(b) *Shifting*: using $y[n] = x[n-n_0]$,

$$\begin{aligned}
Y[n,\lambda) &= \sum_{m=-\infty}^{\infty} y[n+m]w[m]e^{-j\lambda m} \\
&= \sum_{m=-\infty}^{\infty} x[n-n_0+m]w[m]e^{-j\lambda m} \\
&= X[n-n_0,\lambda)
\end{aligned}$$

(c) *Modulation:* using $y[n] = e^{j\omega_0 n}x[n]$,

$$\begin{aligned}
y[n+m] &= e^{j\omega_0(n+m)}x[n+m] \\
Y[n,\lambda) &= \sum_{m=-\infty}^{\infty} y[n+m]w[m]e^{-j\lambda m} \\
&= \sum_{m=-\infty}^{\infty} e^{j\omega_0(n+m)}x[n+m]w[m]e^{-j\lambda m} \\
&= \sum_{m=-\infty}^{\infty} e^{j\omega_0 n}x[n+m]w[m]e^{-j(\lambda-\omega_0)m} \\
&= e^{j\omega_0 n}X[n,\lambda-\omega_0)
\end{aligned}$$

(d) *Conjugate Symmetry:* for $x[n]$ and $w[n]$ real,

$$\begin{aligned}
X[n,\lambda) &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m} \\
&= \left[\sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{j\lambda m}\right]^* \\
&= [X[n,-\lambda)]^* \\
&= X^*[n,-\lambda)
\end{aligned}$$

10.39. (a) We are given that $\phi_c(\tau) = \mathcal{E}\{x_c(t)x_c(t+\tau)\}$. Since $x[n] = x_c(nT)$,

$$\begin{aligned}\phi[m] &= \mathcal{E}\{x[n]x[n+m]\} \\ &= \mathcal{E}\{x_c(nT)x_c(nT+mT)\} \\ &= \phi_c(mT)\end{aligned}$$

(b) $P(\omega)$ and $P_c(\Omega)$ are the transforms of $\phi[m]$ and $\phi_c(\tau)$ respectively. Since $\phi[m]$ is a sampled version of $\phi_c(\tau)$, $P(\omega)$ and $P_c(\Omega)$ are related by

$$P(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} P_c\left(\frac{\omega - 2\pi k}{T}\right)$$

(c) The condition is that no aliasing occurs when sampling. Thus, we require that $P_c(\Omega) = 0$ for $|\Omega| \geq \frac{\pi}{T}$ so that

$$P(\omega) = \frac{1}{T}P_c\left(\frac{\omega}{T}\right), \qquad |\omega| < \pi$$

10.40. In this problem, we are given

- $x[n] = A\cos(\omega_0 n + \theta) + e[n]$
- θ is a uniform random variable on 0 to 2π
- $e[n]$ is an independent, zero mean random variable

(a) Computing the autocorrelation function,

$$\begin{aligned}
\phi_{xx}[m] &= \mathcal{E}\{x[n]x[n+m]\} \\
&= \mathcal{E}\{(A\cos(\omega_0 n + \theta) + e[n])(A\cos(\omega_0(n+m)+\theta) + e[n+m])\} \\
&= \mathcal{E}\{A^2\cos(\omega_0 n + \theta)\cos(\omega_0(n+m)+\theta)\} \\
&\quad + \mathcal{E}\{Ae[n]\cos(\omega_0(n+m)+\theta)\} + \mathcal{E}\{Ae[n+m]\cos(\omega_0 n+\theta)\} \\
&\quad + \mathcal{E}\{e[n]e[n+m]\} \\
&= A^2\mathcal{E}\{\cos(\omega_0 n+\theta)\cos(\omega_0(n+m)+\theta)\} \\
&\quad + A\mathcal{E}\{e[n]\}\mathcal{E}\{\cos(\omega_0(n+m)+\theta)\} + A\mathcal{E}\{e[n+m]\}\mathcal{E}\{\cos(\omega_0 n+\theta)\} \\
&\quad + \mathcal{E}\{e[n]e[n+m]\}
\end{aligned}$$

First, note that

$$\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

Therefore, the first term can be re-expressed as

$$A^2\mathcal{E}\left\{\frac{1}{2}\cos(2\omega_0 n + \omega_0 m + 2\theta) + \frac{1}{2}\cos(\omega_0 m)\right\}$$

Next, note that

$$\mathcal{E}\{e[n]\} = 0$$

As a result, the two middle terms drop out. Finally, note that since $e[n]$ is a sequence of zero-mean variables that are uncorrelated with each other,

$$\mathcal{E}\{e[n]e[n+m]\} = \sigma_e^2\delta[m], \qquad \text{where } \sigma_e^2 = \mathcal{E}\{e^2[n]\}$$

Putting this together, we get

$$\phi_{xx}[m] = A^2\mathcal{E}\left\{\frac{1}{2}\cos(2\omega_0 n + \omega_0 m + 2\theta) + \frac{1}{2}\cos(\omega_0 m)\right\} + \sigma_e^2\delta[m]$$

Since $\frac{1}{2\pi}\int_0^{2\pi}\cos(2\omega_0 n + \omega_0 m + 2\theta)d\theta = 0$, we have

$$\phi_{xx}[m] = \frac{A^2}{2}\cos(\omega_0 m) + \sigma_e^2\delta[m]$$

(b) Since the Fourier transform of $\cos(\omega_0 m)$ is $\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$ for $|\omega| \le \pi$,

$$\Phi_{xx}(e^{j\omega}) = P_{xx}(\omega) = \frac{A^2\pi}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] + \sigma_e^2$$

10.41. (a) Plugging in the equation

$$I[k] = I(\omega_k) = \frac{1}{L}|V[k]|^2$$

into the relation

$$\text{var}[I(\omega)] \simeq P_{xx}^2(\omega)$$

we find that

$$\begin{aligned} \text{var}\left[\frac{1}{L}|V[k]|^2\right] &\simeq P_{xx}^2(\omega) \\ \text{var}\left[|V[k]|^2\right] &\simeq L^2 P_{xx}^2(\omega) \end{aligned}$$

This equation can be used to find the approximate variance of $|X[k]|^2$. We substitute the signal $X[k]$ for $V[k]$, the DFT length N for L, and use the power spectrum

$$P_{xx}(w) = \sigma_x^2$$

This gives

$$\text{var}\left[|X[k]|^2\right] = N^2\sigma_x^4$$

(b) The cross-correlation is found below.

$$\begin{aligned} \mathcal{E}\{X[k]X^*[r]\} &= \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} \mathcal{E}\{x[n_1]x[n_2]\} W_N^{kn_1} W_N^{-rn_2} \\ &= \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} \sigma_x^2\delta[n_1 - n_2] W_N^{kn_1} W_N^{-rn_2} \\ &= \sum_{n=0}^{N-1} \sigma_x^2 W_N^{(k-r)n} \\ &= \sigma_x^2\left[\frac{1 - W_N^{N(k-r)}}{1 - W_N^{(k-r)}}\right] \\ &= N\sigma_x^2\delta[k-r] \end{aligned}$$

Note that the cross-correlation is zero everywhere except when $k = r$. This is what one would expect for white noise, since samples for which $k \neq r$ are completely uncorrelated.

10.42. (a) The length of the data record is

$$\begin{aligned} Q &= 10 \text{ seconds} \cdot \frac{20,000 \text{ samples}}{\text{second}} \\ Q &= 200,000 \text{ samples} \end{aligned}$$

(b) To achieve a 10 Hz or less spacing between samples of the power spectrum, we require

$$\begin{aligned} \frac{1}{NT} &\leq 10 \text{ Hz} \\ N &\geq \frac{1}{10T} \\ &\geq \frac{20,000}{10} \\ &\geq 2,000 \text{ samples} \end{aligned}$$

Since N must also be a power of 2, we choose $N = 2048$.

(c)

$$\begin{aligned} K &= \frac{Q}{L} \\ &= \frac{200,000}{2048} \\ &= 97.66 \text{ segments} \end{aligned}$$

If we zero-pad the last segment so that it contains 2048 samples, we will have $K = 98$ segments.

(d) The key to reducing the variance is to use more segments. Two methods are discussed below. Note that in both methods, we want the segments to be length $L = 2048$ so that we maintain the frequency spacing.

(i) Decreasing the length of the segments to $\frac{1}{10}$th their length, and then zero-padding them to $L = 2048$ samples will increase K by a factor of 10. Accordingly, the variance will decrease by a factor of 10. However, the frequency resolution will be reduced.

(ii) If we increase the data record to 2,000,000 samples, we can keep the window length the same and increase K by a factor of 10.

10.43. (a) Taking the expected value of

$$\overline{\phi}[m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{I}(\omega)e^{j\omega m}d\omega$$

gives

$$\begin{aligned}\mathcal{E}\left\{\overline{\phi}[m]\right\} &= \mathcal{E}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{I}(\omega)e^{j\omega m}d\omega\right\} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{E}\left\{\overline{I}(\omega)\right\}e^{j\omega m}d\omega\end{aligned}$$

Using the relation

$$\mathcal{E}\left\{\overline{I}(w)\right\} = \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)C_{ww}(e^{j(\omega-\theta)})d\theta$$

we find

$$\begin{aligned}\mathcal{E}\left\{\overline{\phi}[m]\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)C_{ww}(e^{j(\omega-\theta)})d\theta\right]e^{j\omega m}d\omega \\ &= \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{ww}(e^{j(\omega-\theta)})e^{j\omega m}d\omega\right]d\theta\end{aligned}$$

Substituting $\omega' = \omega - \theta$ in the inner integral yields

$$\begin{aligned}\mathcal{E}\left\{\overline{\phi}[m]\right\} &= \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)\left[\frac{1}{2\pi}\int_{-\pi-\theta}^{\pi-\theta} C_{ww}(e^{j\omega'})e^{j(\omega'+\theta)m}d\omega'\right]d\theta \\ &= \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)e^{j\theta m}\left[\frac{1}{2\pi}\int_{-\pi-\theta}^{\pi-\theta} C_{ww}(e^{j\omega'})e^{j\omega' m}d\omega'\right]d\theta\end{aligned}$$

Note we can change the limits of integration of the inner integral to be $[-\pi, \pi]$ because we are integrating over the whole period. Doing this gives

$$\begin{aligned}\mathcal{E}\left\{\overline{\phi}[m]\right\} &= \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)e^{j\theta m}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{ww}(e^{j\omega'})e^{j\omega' m}d\omega'\right]d\theta \\ &= \frac{1}{2\pi LU}\int_{-\pi}^{\pi} P_{xx}(\theta)e^{j\theta m}\left\{c_{ww}[m]\right\}d\theta \\ &= \frac{1}{LU}c_{ww}[m]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(\theta)e^{j\theta m}d\theta\right] \\ &= \frac{1}{LU}c_{ww}[m]\phi_{xx}[m]\end{aligned}$$

(b)

$$\overline{\phi}_p[m] = \frac{1}{N}\sum_{k=0}^{N-1} \overline{I}[k]e^{j2\pi km/N}$$

By applying the sampling theorem to Fourier transforms, we see that

$$\begin{aligned}\overline{\phi}_p[m] &= \sum_{r=-\infty}^{\infty} \overline{\phi}_{xx}[m+rN] \\ \mathcal{E}\left\{\overline{\phi}_p[m]\right\} &= \sum_{r=-\infty}^{\infty} \mathcal{E}\left\{\overline{\phi}_{xx}[m+rN]\right\} \\ &= \frac{1}{LU}\sum_{r=-\infty}^{\infty} c_{ww}[m+rN]\phi_{xx}[m+rN]\end{aligned}$$

which is a time aliased version of $\mathcal{E}\left\{\overline{\phi}_{xx}[m]\right\}$.

(c) N should be chosen so that no time aliasing occurs. Since $\overline{\phi}_{xx}[m]$ is $2L-1$ points long, we should choose $N \geq 2L$.

10.44. (a) For $0 \leq m \leq M$,

$$\begin{aligned}
\hat{\phi}_{xx}[m] &= \frac{1}{Q}\sum_{n=0}^{Q-m-1} x[n]x[n+m] \\
&= \frac{1}{Q}\left[\sum_{n=0}^{M-1} x[n]x[n+m] + \sum_{n=M}^{2M-1} x[n]x[n+m] + \ldots + \sum_{n=(K-1)M}^{KM-1} x[n]x[n+m]\right] \\
&= \frac{1}{Q}\left[\sum_{n=0}^{M-1} x[n]x[n+m]\right. \\
&\quad \left. + \sum_{n=0}^{M-1} x[n+M]x[n+M+m] + \ldots + \sum_{n=0}^{M-1} x[n+(K-1)M]x[n+(K-1)M+m]\right] \\
&= \frac{1}{Q}\sum_{i=0}^{K-1}\sum_{n=0}^{M-1} x[n+iM]x[n+iM+m] \\
&= \frac{1}{Q}\sum_{i=0}^{K-1} c_i[m]
\end{aligned}$$

where

$$c_i[m] = \sum_{n=0}^{M-1} x[n+iM]x[n+iM+m] \qquad \text{for } 0 \leq m \leq M-1$$

(b) We can rewrite the expression for $c_i[m]$ from part (a) as

$$\begin{aligned}
c_i[m] &= \sum_{n=0}^{M-1} x[n+iM]x[n+iM+m] \\
&= \sum_{n=0}^{M-1} x[n+iM]x[n+iM+m] + \sum_{n=M}^{N-1} 0 \cdot x[n+iM+m] \\
&= \sum_{n=0}^{N-1} x_i[n]y_i[n+m]
\end{aligned}$$

where

$$x_i[n] = \begin{cases} x[n+iM], & 0 \leq n \leq M-1 \\ 0, & M \leq n \leq N-1 \end{cases}$$

and

$$y_i[n] = x[n+iM] \qquad \text{for } 0 \leq n \leq N-1$$

Thus, the correlations $c_i[m]$ can be obtained by computing N-point *linear* correlations. Next, we show that for $N \geq 2M-1$, circular correlation is equivalent to linear correlation.

Note that the circular correlation of $x_i[n]$ with $y_i[n]$,

$$\tilde{c}_{yx}[m] = \sum_{n=0}^{N-1} x_i[n]y_i[((n+m))_N]$$

can be expressed as

$$\begin{aligned}
\tilde{c}_{yx}[m] &= \tilde{c}_{xy}[-m] \\
&= \sum_{n=0}^{N-1} x_i[((n-m))_N]y_i[n] \\
&= \sum_{n=0}^{N-1} x_i'[((m-n))_N]y_i[n]
\end{aligned}$$

where $x'[n] = x[-n]$. Note that this is a circular *convolution* of $x_i[-n]$ with $y_i[n]$. Thus, we have expressed the circular correlation of $x_i[n]$ with $y_i[n]$ as a circular convolution of $x_i[-n]$ with $y_i[n]$. Now recall from chapter 8 that the circular convolution of two M point signals is equivalent to their linear convolution when $N \geq 2M - 1$. Since we can express the circular correlation in terms of a circular convolution, this result applies to circular correlation as well. Therefore, we see that if $N \geq 2M - 1$,

$$c_i[m] = \tilde{c}_i[m] \qquad \text{for } 0 \leq m \leq M-1$$

Thus, the minimum value of N is $2M - 1$.

(c) A procedure for computing $\hat{\phi}_{xx}[m]$ is described below.

step 1: Compute $X_i[k]$ and $Y_i[k]$, which are the $N \geq 2M - 1$ point DFTs of $x_i[n]$ and $y_i[n]$.
step 2: Multiply $X_i[k]$ and $Y_i^*[k]$ point by point, yielding $C_i[k] = \tilde{C}_i[k] = X_i[k]Y_i^*[k]$.
step 3: Repeat the above two steps for all data (K times), then compute

$$\hat{\Phi}_{xx}[k] = \frac{1}{Q}\sum_{i=0}^{K-1} C_i[k] \qquad \text{for } 0 \leq k \leq N-1$$

step 4: Take the N point inverse DFT of $\hat{\Phi}_{xx}[k]$ to get $\hat{\phi}_{xx}[m]$.

Assuming that a radix-2 FFT, requiring $\frac{N}{2}\log_2 N$ complex multiplications is used to compute the forward and inverse DFTS, the number of complex multiplications is

$$\begin{array}{ll} 2 \cdot \frac{N}{2}\log_2 N \cdot K = KN\log_2 N, & \text{for step 1} \\ KN, & \text{for step 2} \\ N, & \text{for divide by } Q \text{ operation in step 3} \\ \frac{N}{2}\log_2 N & \text{for step 4} \end{array}$$

So the total number of complex multiplications is $(K + \frac{1}{2})N\log_2 N + (K+1)N$.

(d) The procedure developed in part (c) would compute the cross-correlation estimate $\hat{\phi}_{xy}$ without any major modifications. All we need to do is redefine $y_i[n]$ as

$$y_i[n] = y[n + iM], \qquad 0 \leq n \leq N-1$$

and $x_i[n]$ is the same as it was before, namely

$$x_i[n] = \begin{cases} x[n+iM], & 0 \leq n \leq M-1 \\ 0, & M \leq n \leq N-1 \end{cases}$$

Note that for $m < 0$, $\hat{\phi}_{xy}[m] = \hat{\phi}_{yx}[-m]$.

(e) For $N = 2M$,

$$\begin{aligned} y_i[n] &= x[n+iM], \qquad \text{for } 0 \leq n \leq 2M-1 \\ &= x[n+iM](u[n] - u[n-M]) + x[n+iM](u[n-M] - u[n-2M]) \\ &= x[n+iM](u[n] - u[n-M]) + x[n-M+(i+1)M](u[n-M] - u[n-2M]) \\ &= x_i[n] + x_{i+1}[n-M] \end{aligned}$$

Taking the DFT of this expression yields

$$Y_i[k] = X_i[k] + (-1)^k X_{i+1}[k]$$

A procedure for computing $\hat{\phi}_{xx}$ for $0 \leq m \leq M-1$ is described below.

step 1: Compute the N point DFT $X_i[k]$ for $i = 0, 1, \ldots, K$.
step 2: Compute $Y_i[k] = X_i[k] + (-1)^k X_{i+1}[k]$ for $i = 0, 1, \ldots, K-1$.
step 3: Let $A_0[k] = 0$ and compute

$$A_i[k] = A_{i-1}[k] + X_i[k]Y_i^*[k], \quad i = 1, \ldots, K-1$$

step 4: Define $V[k] = A_{K-1}[k]$. Compute $v[m]$, the N point inverse DFT of $V[k]$.
step 5: Compute

$$\hat{\phi}_{xx}[m] = \frac{1}{Q}v[m]$$

Assuming that a radix-2 FFT, requiring $\frac{N}{2}\log_2 N$ complex multiplications is used to compute the forward and inverse DFTs, the number of complex multiplications is

$(K+1)\frac{N}{2}\log_2 N$,	for step 1
0,	for step 2
$(K-1)N$,	for step 3
$\frac{N}{2}\log_2 N$,	for step 4
N,	for the divide by Q in step 5

So the total number of complex multiplications is $\frac{K+2}{2}N\log_2 N + KN$. Note that for large N and K, this procedure requires roughly half the number of complex multiplications as the procedure described in part (c).

10.45. (a) Using the relations,

$$\begin{aligned} c[n,m] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X[n,\lambda)|^2 e^{j\lambda m} d\lambda \\ X[n,\lambda) &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m} d\lambda \end{aligned}$$

we find

$$\begin{aligned} c[n,m] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X[n,\lambda)|^2 e^{j\lambda m} d\lambda \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X[n,\lambda)X[n,-\lambda)e^{j\lambda m} d\lambda \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{l=-\infty}^{\infty} x[n+l]w[l]e^{-j\lambda l}\right)\left(\sum_{r=-\infty}^{\infty} x[n+r]w[r]e^{j\lambda r}\right) e^{j\lambda m} d\lambda \\ &= \sum_{l=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} x[n+l]w[l]x[n+r]w[r]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\lambda l}e^{j\lambda r}e^{j\lambda m} d\lambda\right) \\ &= \sum_{l=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} x[n+l]w[l]x[n+r]w[r]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\lambda(-l+r)}e^{j\lambda m} d\lambda\right) \end{aligned}$$

Using the Fourier transform relation,

$$\delta[n-n_0] \longleftrightarrow e^{-j\omega n_0}$$

we find

$$c[n,m] = \sum_{l=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} x[n+l]w[l]x[n+r]w[r]\delta[m-l+r]$$

The $\delta[m-l+r]$ term is zero everwhere except when $m-l+r=0$. Therefore, we can replace the two sums of l and r with one sum over r, by substituting $l=m+r$.

$$\begin{aligned} c[n,m] &= \sum_{r=-\infty}^{\infty} x[n+m+r]w[m+r]x[n+r]w[r] \\ &= \sum_{r=-\infty}^{\infty} x[n+r]w[r]x[n+m+r]w[m+r] \end{aligned}$$

(b) First, note that

$$\begin{aligned} |X[n,\lambda)|^2 &= X[n,-\lambda)X[n,\lambda) \\ &= |X[n,-\lambda)|^2 \end{aligned}$$

Starting with the definition of $c[n,m]$,

$$c[n,m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X[n,\lambda)|^2 e^{j\lambda m} d\lambda$$
$$c[n,-m] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X[n,\lambda)|^2 e^{-j\lambda m} d\lambda$$

Thus, the time-dependent autocorrelation function is an even function of m for n fixed. Next, we use this fact to obtain the equivalent expression for $c[n,m]$.

$$\begin{aligned} c[n,m] &= \sum_{r=-\infty}^{\infty} x[n+r]w[r]x[m+n+r]w[m+r] \\ &= \sum_{r=-\infty}^{\infty} x[n+r]w[r]x[-m+n+r]w[-m+r] \end{aligned}$$

Substituting $r' = n+r$ gives

$$\begin{aligned} &= \sum_{r'=-\infty}^{\infty} x[r']w[r'-n]x[r'-m]w[(r'-m)-n] \\ &= \sum_{r'=-\infty}^{\infty} x[r']x[r'-m]w[r'-n]w[-(m+n-r')] \\ &= \sum_{r'=-\infty}^{\infty} x[r']x[r'-m]h_m[n-r'] \end{aligned}$$

where

$$h_m[r] = w[-r]w[-(m+r)]$$

(c) To compute $c[n,m]$ by causal operations, we see that

$$h_m[r] = w[-r]w[-(m+r)]$$

requires that $w[r]$ must be zero for

$$\begin{aligned} -r &< 0 \\ r &> 0 \end{aligned}$$

and $w[r]$ must be zero for

$$\begin{aligned} -(m+r) &< 0 \\ m+r &> 0 \\ r &> -m \end{aligned}$$

Thus, $w[r]$ must be zero for $r > \min(0,-m)$. If m is positive, then $w[r]$ must be zero for $r > 0$. This is equivalent to the requirement that $w[-r]$ must be zero for $r < 0$.

(d) Plugging in

$$w[-r] = \begin{cases} a^r, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

into $h_m[r] = w[-r]w[-(m+r)]$, we find

$$h_m[r] = \begin{cases} a^{2r+m}, & r \geq 0, r \geq -m \\ 0, & \text{otherwise} \end{cases}$$

Taking the z-transform of this expression gives

$$\begin{aligned} H_m(z) &= \sum_{r=-\infty}^{\infty} h_m[r]z^{-r} \\ &= \sum_{r=0}^{\infty} a^{2r+m}z^{-r} \\ &= a^m \sum_{r=0}^{\infty} (a^2z^{-1})^r \end{aligned}$$

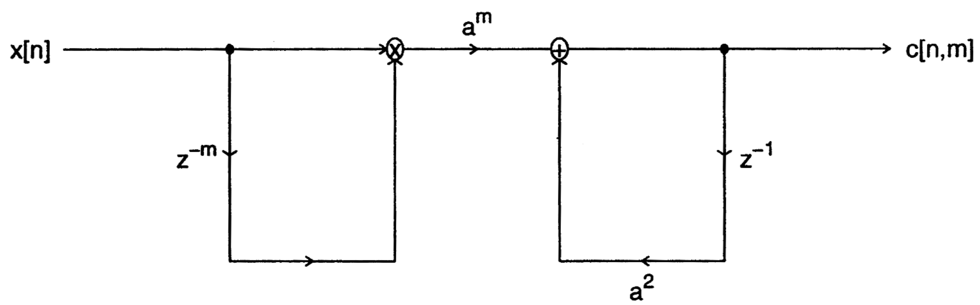
Again we have assumed that m is positive. If $|z| > a^2$, then

$$\begin{aligned} H_m(z) &= \frac{a^m}{1 - a^2 z^{-1}} \\ h_m[r] &= a^m \delta[r] + a^2 h_m[r-1] \end{aligned}$$

Using this in the equation for $c[n,m]$ gives

$$\begin{aligned} c[n,m] &= \sum_{r=-\infty}^{\infty} x[r]x[r-m]h_m[n-r] \\ &= \sum_{r=-\infty}^{\infty} x[r]x[r-m]\left(a^m \delta[n-r] + a^2 h_m[n-r-1]\right) \\ &= a^m x[n]x[n-m] + a^2 \sum_{r=-\infty}^{\infty} x[r]x[r-m]h_m[n-r-1] \\ &= a^m x[n]x[n-m] + a^2 c[n-1,m] \end{aligned}$$

A block diagram of this system appears below.



(e) Next, consider the system

$$w[-r] = \begin{cases} ra^r, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

$$\begin{aligned} h_m[r] &= \{ra^r u[r]\}\left\{(r+m)a^{r+m}u[r+m]\right\} \\ &= a^m r^2 a^{2r} + a^m m r a^{2r} \qquad\qquad r \geq 0; r \geq -m \end{aligned}$$

To get the z-transform $H_m(z)$, recall the z-transform property: $rx[r] \leftrightarrow -z\frac{dX(z)}{dz}$. Using this property, we find

$$\begin{aligned} ra^{2r}u[r] &\Longleftrightarrow \frac{a^2 z^{-1}}{(1-a^2 z^{-1})^2} \\ r^2 a^{2r} u[r] &\Longleftrightarrow \frac{a^2 z^{-1}(1 + a^2 z^{-1})}{(1-a^2 z^{-1})^3} \end{aligned}$$

Again we have assumed that m is positive. Thus,

$$\begin{aligned} H_m(z) &= a^m \left[\frac{a^2 z^{-1}(1+a^2 z^{-1})}{(1-a^2 z^{-1})^3}\right] + ma^m \left[\frac{a^2 z^{-1}}{(1-a^2 z^{-1})^2}\right] \\ &= \frac{a^{m+2} z^{-1}(1+a^2 z^{-1}) + ma^m (a^2 z^{-1})(1-a^2 z^{-1})}{(1-a^2 z^{-1})^3} \end{aligned}$$

$$
\begin{aligned}
&= \frac{a^{m+2}z^{-1}(1 + a^2z^{-1} + m - ma^2z^{-1})}{(1 - a^2z^{-1})^3} \\
&= \frac{a^{m+2}(1+m)z^{-1} + a^{m+4}(1-m)z^{-2}}{1 - 3a^2z^{-1} + 3a^4z^{-2} - a^6z^{-3}}
\end{aligned}
$$

Cross-multiplying and taking the inverse z-transform gives

$$h_m[r] - 3a^2h_m[r-1] + 3a^4h_m[r-2] - a^6h_m[r-3] = a^{m+2}(1+m)\delta[r-1] + a^{m+4}(1-m)\delta[r-2]$$

$$h_m[r] = 3a^2h_m[r-1] - 3a^4h_m[r-2] + a^6h_m[r-3] + a^{m+2}(1+m)\delta[r-1] + a^{m+4}(1-m)\delta[r-2]$$

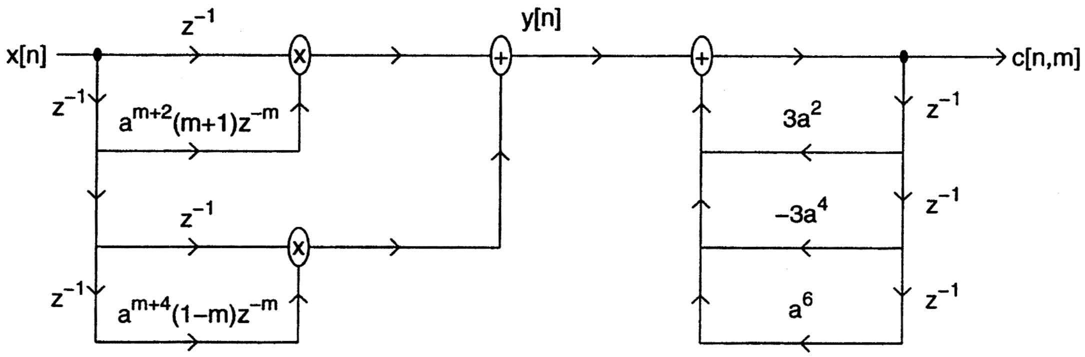
Using this relation for $h_m[r]$ in

$$c[n,m] = \sum_{r=-\infty}^{\infty} x[r]x[r-m]h_m[n-r]$$

we get

$$
\begin{aligned}
c[n,m] &= \sum_{r=-\infty}^{\infty} x[r]x[r-m]\left(3a^2h_m[n-r-1] - 3a^4h_m[n-r-2] + a^6h_m[n-r-3]\right) \\
&\quad + \sum_{r=-\infty}^{\infty} x[r]x[r-m]\left(a^{m+2}(1+m)\delta[n-r-1] + a^{m+4}(1-m)\delta[n-r-2]\right) \\
&= 3a^2c[n-1,m] - 3a^4c[n-2,m] + a^6c[n-3,m] \\
&\quad + a^{m+2}(1+m)x[n-1]x[n-1-m] + a^{m+4}(1-m)x[n-2]x[n-2-m]
\end{aligned}
$$

A block diagram of this system appears below.



10.46. (a) Looking at the figure, we see that

$$\begin{aligned} X[n,\lambda) &= \left\{\left(x[n]e^{-j\lambda n}\right) * h_0[n]\right\} e^{j\lambda n} \\ &= \left[\sum_{m=-\infty}^{\infty} x[n-m]e^{-j\lambda(n-m)}h_0[m]\right] e^{j\lambda n} \\ &= \sum_{m=-\infty}^{\infty} x[n-m]h_o[m]e^{j\lambda m} \end{aligned}$$

Let $m' = -m$. Then,

$$\begin{aligned} X[n,\lambda) &= \sum_{m'=\infty}^{-\infty} x[n+m']h_0[-m']e^{-j\lambda m'} \\ &= \sum_{m'=-\infty}^{\infty} x[n+m']h_0[-m']e^{-j\lambda m'} \\ &= X[n,\lambda) \end{aligned}$$

if $h_0[-m] = w[m]$. Next, we show that for λ fixed, $X[n,\lambda)$ behaves as a linear, time-invariant system.

Linear: Inputting the signal $ax_1[n] + bx_2[n]$ into the system yields

$$\sum_{m=-\infty}^{\infty} \left(ax_1[n+m] + bx_2[n+m]\right) h_0[-m]e^{-j\lambda m} =$$

$$\sum_{m=-\infty}^{\infty} ax_1[n+m]h_0[-m]e^{-j\lambda m} + \sum_{m=-\infty}^{\infty} bx_2[n+m]h_0[-m]e^{-j\lambda m} = aX_1[n,\lambda) + bX_2[n,\lambda)$$

The system is linear.

Time invariant: Shifting the input $x[n]$ by an amount l yields

$$\sum_{m=-\infty}^{\infty} x[n+m+l]h_0[-m]e^{-j\lambda m} = X[n+l,\lambda)$$

which is the output shifted by l samples. The system is time-invariant.

Next, we find the impulse response and frequency response of the system. To find the impulse response, denoted as $h[n]$, we let $x[n] = \delta[n]$.

$$\begin{aligned} h[n] &= \sum_{m=-\infty}^{\infty} \delta[n+m]w[m]e^{-j\lambda m} \\ &= w[-n]e^{j\lambda n} \\ &= h_0[n]e^{j\lambda n} \end{aligned}$$

Taking the DTFT gives the frequency response, denoted as $H(e^{j\omega})$.

$$H(e^{j\omega}) = H_0(e^{j(\omega-\lambda)})$$

(b) We find $S(e^{j\omega})$ to be

$$\begin{aligned} s[n] &= \left(x[n]e^{-j\lambda n}\right) * w[-n] \\ S(e^{j\omega}) &= X\left(e^{j(\omega+\lambda)}\right) W(e^{-j\omega}) \\ S(e^{j\omega}) &= X\left(e^{j(\omega+\lambda)}\right) H_0(e^{j\omega}) \end{aligned}$$

Note that most typical window sequences are lowpass in nature, and are centered around a frequency of $\omega = 0$. Since $H_0(e^{j\omega}) = W(e^{-j\omega})$ is the Fourier transform of a window which is lowpass in nature, the signal $S(e^{j\omega})$ is also lowpass.

The signal $s[n] = \check{X}[n, \lambda)$ is multiplied by a complex exponential $e^{j\lambda n}$. This modulation shifts the frequency response of $S(e^{j\omega})$ so that it is centered at $\omega = \lambda$.

$$\begin{aligned} h[n] &= s[n]e^{j\lambda n} \\ H(e^{j\omega}) &= S\left(e^{j(\omega-\lambda)}\right) \end{aligned}$$

Since $S(e^{j\omega})$ is lowpass filter centered at $\omega = 0$, the overall system is a bandpass filter centered at $\omega = \lambda$.

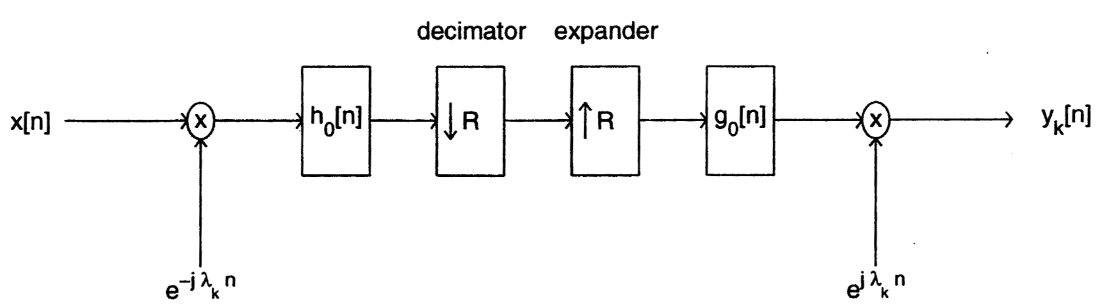
(c) First, it is shown that the individual outputs $y_k[n]$ are samples (in the λ dimension) of the time-dependent Fourier transform.

$$\begin{aligned} y_k[n] &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda_k m} \\ &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j2\pi km/N} \\ &= X[n,\lambda)|_{\lambda=2\pi k/N} \end{aligned}$$

Next, it is shown that the overall output is $y[n] = Nw[0]x[n]$.

$$\begin{aligned} y[n] &= \sum_{k=0}^{N-1} y_k[n] \\ &= \sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j2\pi km/N} \\ &= \sum_{m=-\infty}^{\infty}\sum_{k=0}^{N-1} x[n+m]w[m]e^{-j2\pi km/N} \\ &= \sum_{m=-\infty}^{\infty} x[n+m]w[m]\underbrace{\sum_{k=0}^{N-1} e^{-j2\pi km/N}}_{N\delta[m]} \\ &= Nw[0]x[n] \end{aligned}$$

(d) Consider a single channel,



In the frequency domain, the input to the decimator is

$$X\left(e^{j(\omega+\lambda_k)}\right)H_0(e^{j\omega})$$

so the output of the decimator is

$$\frac{1}{R}\sum_{l=0}^{R-1} X\left(e^{j((\omega-2\pi l)/R+\lambda_k)}\right)H_0\left(e^{j(\omega-2\pi l)/R}\right)$$

The output of the expander is

$$\frac{1}{R}\sum_{l=0}^{R-1} X\left(e^{j(\omega+\lambda_k-2\pi l/R)}\right)H_0\left(e^{j(\omega-2\pi l/R)}\right)$$

The output $Y_k(e^{j\omega})$ is then

$$Y_k(e^{j\omega}) = \frac{1}{R}\sum_{l=0}^{R-1} G_0\left(e^{j(\omega-\lambda_k)}\right) X\left(e^{j(\omega-2\pi l/R)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right)$$

The overall system output is formed by summing these terms over k.

$$\begin{aligned} Y(e^{j\omega}) &= \sum_{k=0}^{N-1} Y_k(e^{j\omega}) \\ &= \frac{1}{R}\sum_{l=0}^{R-1}\sum_{k=0}^{N-1} G_0\left(e^{j(\omega-\lambda_k)}\right) X\left(e^{j(\omega-2\pi l/R)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right) \end{aligned}$$

To cancel the aliasing, we rewrite the equation as follows:

$$\begin{aligned} Y(e^{j\omega}) &= X(e^{j\omega})\frac{1}{R}\sum_{k=0}^{N-1} H_0\left(e^{j(\omega-\lambda_k)}\right) G_0\left(e^{j(\omega-\lambda_k)}\right) \\ &+ \underbrace{\sum_{l=1}^{R-1} X\left(e^{j(\omega-2\pi l/R)}\right)\frac{1}{R}\sum_{k=0}^{N-1} G_0\left(e^{j(\omega-\lambda_k)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right)}_{\text{Aliasing Component}} \end{aligned}$$

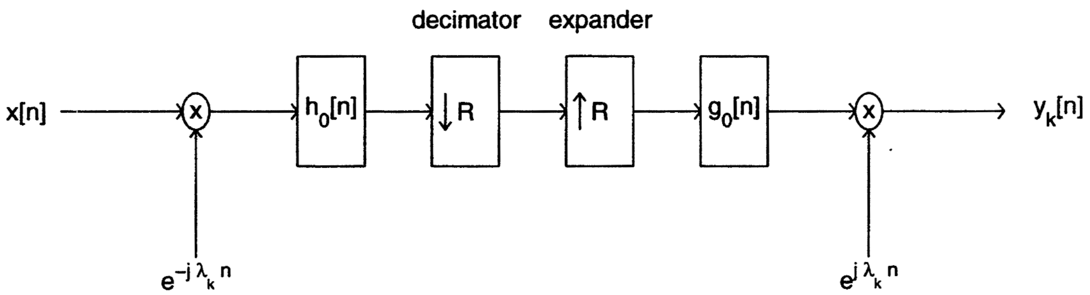
Therefore, we require the following relations to be satisfied so that $y[n] = x[n]$:

$$\begin{aligned} \sum_{k=0}^{N-1} G_0\left(e^{j(\omega-\lambda_k)}\right) H_0\left(e^{j(\omega-\lambda_k-2\pi l/R)}\right) &= 0, \qquad \forall\,\omega, \text{ and } l = 1,\ldots,R-1 \\ \sum_{k=0}^{N-1} H_0\left(e^{j(\omega-\lambda_k)}\right) G_0\left(e^{j(\omega-\lambda_k)}\right) &= R, \qquad \forall\,\omega \end{aligned}$$

(e) *Yes*, it is possible. $G_o(e^{j\omega}) = NH_0(e^{j\omega})$ will yield exact reconstruction.

(f) See chapter 7 in "Multirate Digital Signal Processing" by Crochiere and Rabiner, 1983.

(g) Once again, we consider a single channel,



From Part (a), we know that the output of the filter $h_0[n]$ is

$$\check{X}[n,\lambda_k] = \sum_{m=-\infty}^{\infty} x[m]h_0[n-m]e^{-j\lambda_k m}$$

or, using $\lambda_k = 2\pi k/N$,

$$\check{X}[n,k] = \sum_{m=-\infty}^{\infty} x[m]h_0[n-m]e^{-j2\pi km/N}$$

Therefore, the output of the decimator is

$$\check{X}[Rn,k] = \sum_{m=-\infty}^{\infty} x[m]h_0[Rn-m]e^{-j2\pi km/N}$$

Recall that in general, the output of an expander with expansion factor R is

$$x_e[n] = \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell R]$$

This relation is given in chapter 3. Therefore, the output of the expander is

$$\sum_{\ell=-\infty}^{\infty} \check{X}[R\ell,k]\delta[n-\ell R]$$

This signal is then convolved with $g_0[n]$, giving

$$\sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} \check{X}[R\ell,k]\delta[m-\ell R]g_0[n-m] = \sum_{\ell=-\infty}^{\infty} \check{X}[R\ell,k]g_0[n-\ell R]$$

Therefore,

$$\begin{aligned}
y_k[n] &= \sum_{\ell=-\infty}^{\infty} g_0[n-\ell R]\left(\sum_{m=-\infty}^{\infty} x[m]h_0[Rl-m]e^{-j2\pi km/N}\right)e^{j2\pi kn/N} \\
y[n] &= \sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} g_0[n-\ell R]\left(\sum_{m=-\infty}^{\infty} x[m]h_0[Rl-m]e^{-j2\pi km/N}\right)e^{j2\pi kn/N} \\
&= \sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} g_0[n-\ell R]\sum_{m=-\infty}^{\infty} x[m]h_0[Rl-m]e^{-j2\pi k(m-n)/N} \\
&= \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g_0[n-\ell R]h_0[Rl-m]x[m]\sum_{k=0}^{N-1} e^{j2\pi k(n-m)/N}
\end{aligned}$$

Now recall that $\sum_{k=0}^{N-1} e^{j2\pi k(n-m)/N} = N\delta[((n-m))_N]$, by considering it as a Fourier series expansion, or as an inverse DFT of $Ne^{-j2\pi mk/N}$. Thus,

$$\sum_{k=0}^{N-1} e^{j2\pi k(n-m)/N} = N\sum_{r=-\infty}^{\infty}\delta[n-m-rN]$$

where r is an integer. Therefore,

$$\begin{aligned}
y[n] &= \sum_{\ell=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} g_0[n-\ell R]h_0[\ell R-m]x[m]N\sum_{r=-\infty}^{\infty}\delta[n-m-rN] \\
&= N\sum_{\ell=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} g_0[n-\ell R]h_0[\ell R-n+rN]x[n-rN] \\
&= N\sum_{r=-\infty}^{\infty} x[n-rN]\sum_{\ell=-\infty}^{\infty} g_0[n-\ell R]h_0[\ell R+rN-n]
\end{aligned}$$

Therefore, if we want $y[n] = x[n]$, we require

$$\sum_{\ell=-\infty}^{\infty} g_0[n - \ell R] h_0[\ell R + rN - n] = \delta[r]$$

for all values n.

(h) Intuitively, we see that it is possible since we are keeping the necessary number of samples. If $g_0[n] = \delta[n]$ find that

$$\begin{aligned} \sum_{\ell=-\infty}^{\infty} \delta[n - \ell R] h_0[\ell R + rN - n] &= h_0[rN] \\ &= \delta[r] \end{aligned}$$

since $h_0[rN]$ is zero for all values of r, except $r = 0$, where it is equal to 1. Thus, the condition derived in Part (g) is satisfied.

(i) See Rabiner and Crochiere or Portnoff. (Hint: consider an overlap and add FFT algorithm.)

10.47. Note that $h[n]$ is real in this problem.

(a) First, we express $y[n]$ as the convolution of $h[n]$ and $x[n]$.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The autocorrelation of $y[n]$ is then

$$\begin{aligned}
\phi_{yy}[m] &= \mathcal{E}\{y[n+m]y[n]\} \\
&= \mathcal{E}\left\{\sum_{k=-\infty}^{\infty} h[k]x[n+m-k] \sum_{l=-\infty}^{\infty} h[l]x[n-l]\right\} \\
&= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h[k]h[l]\mathcal{E}\{x[n+m-k]x[n-l]\} \\
&= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h[k]h[l]\phi_{xx}[l+m-k]
\end{aligned}$$

Since $x[n]$ is white noise, it has the autocorrelation function

$$\phi_{xx}[l+m-k] = \sigma_x^2\delta[l+m-k]$$

Substituting this into the expression for $\phi_{yy}[m]$ gives

$$\begin{aligned}
\phi_{yy}[m] &= \sigma_x^2 \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h[k]h[l]\delta[l+m-k] \\
&= \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l+m]h[l]
\end{aligned}$$

Note that

$$\phi_{yy}[m] = \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l-m]h[l]$$

is also a correct answer, since $\phi_{yy}[m] = \phi_{yy}[-m]$.

(b) Taking the DTFT of $\phi_{yy}[m]$ will give the power density spectrum $\Phi_{yy}(\omega)$.

$$\begin{aligned}
\Phi_{yy}(\omega) &= \sum_{m=-\infty}^{\infty}\left\{\sigma_x^2 \sum_{l=-\infty}^{\infty} h[l+m]h[l]\right\} e^{-j\omega m} \\
&= \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l] \sum_{m=-\infty}^{\infty} h[l+m]e^{-j\omega m}
\end{aligned}$$

Substituting $k = l+m$ into the second summation gives

$$\begin{aligned}
\Phi_{yy}(\omega) &= \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l] \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega(k-l)} \\
&= \sigma_x^2 \sum_{l=-\infty}^{\infty} h[l]e^{j\omega l} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\
&= \sigma_x^2 \sum_{l=-\infty}^{\infty} h[-l]e^{-j\omega l} \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \\
&= \sigma_x^2 H^*(e^{j\omega})H(e^{j\omega}) \\
&= \sigma_x^2 \left|H(e^{j\omega})\right|^2
\end{aligned}$$

(c) This problem can be approached either in the time domain or the z-transform domain.

Time domain: Since all the a_k's are zero for a MA process,

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

so $y[n]$ is nonzero for $0 \leq n \leq M$. Note that the autocorrelation sequence,

$$\phi_{yy}[m] = \sum_{n=-\infty}^{\infty} y[n+m]y[n]$$

can be re-written as a convolution

$$\phi_{yy}[m] = \sum_{n=-\infty}^{\infty} g[m-n]y[n]$$

where $g[n] = y[-n]$. Therefore,

$$\phi_{yy}[n] = y[-n] * y[n]$$

Since $y[-n]$ is nonzero for $-M \leq n \leq 0$, and $y[n]$ is nonzero for $0 \leq n \leq M$, we see that their convolution $\phi_{yy}[m]$ is nonzero only in the interval $|m| \leq M$.

Z-transform domain: Note that

$$\Phi_{yy}(z) = \sigma_x^2 H(z)H^*(z)$$

If all the a_k's $= 0$, then

$$\begin{aligned} H(z) &= \sum_{k=0}^{M} b_k z^{-k} \\ \Phi_{yy}(z) &= \sum_{k=0}^{M} b_k z^{-k} \sum_{\ell=0}^{M} b_\ell^* z^{\ell} \end{aligned}$$

The relation for $\Phi_{yy}(z)$ above is found by multiplying two polynomials in z. The highest power of z in $\Phi_{yy}(z)$ is z^M which arises from the multiplication of the $k=0$ and $l=M$ coefficients. The smallest power of z in $\Phi_{yy}(z)$ is z^{-M} which arises from the multiplication of the $k=M$ and $l=0$ coefficents. Thus, $\phi_{yy}[m]$ is nonzero only in the interval $|m| \leq M$.

(d) For an AR process,

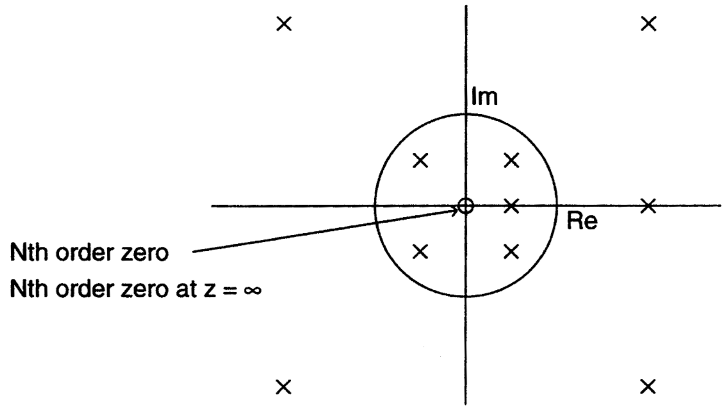
$$\begin{aligned} H(z) &= \frac{b_0}{1 - \sum_{k=1}^{N} a_k z^{-k}} \\ &= \frac{b_0}{\prod_{k=1}^{N}(1-\alpha_k z^{-1})} \end{aligned}$$

Since

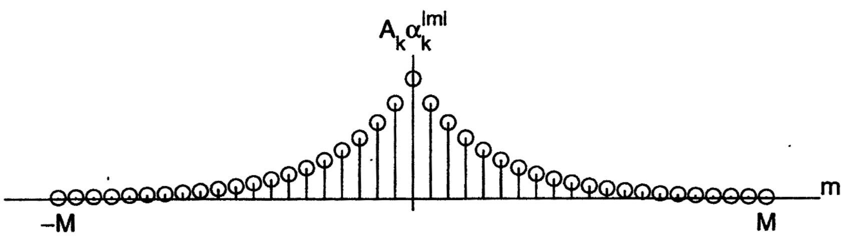
$$\Phi_{yy}(z) = \sigma_x^2 H(z)H^*(z)$$

$$\Phi_{yy}(z) = \frac{b_0^2}{\prod_{k=1}^{N}(1-\alpha_k z^{-1})(1-\alpha_k^* z)}$$

Thus, the poles for $\Phi_{yy}(z)$ come in conjugate reciprocal pairs. A sample pole-zero diagram appears below.



By performing a partial fraction expansion on $\Phi_{yy}(z)$ we find that each pole pair contributes a sequence of the form $A_k\alpha_k^{|m|}$



and therefore

$$\phi_{yy}[m] = \sum_{k=1}^{N} A_k\alpha_k^{|m|}$$

(e) For an AR process, with $b_0 = 1$,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\sum_{k=1}^{N} a_k z^{-k}}$$

which means that

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + x[n]$$

The autocorrelation function is then

$$\begin{aligned}
\phi_{yy}[m] &= \phi_{yy}[-m] \\
&= \mathcal{E}\left\{y[n-m]y[n]\right\} \\
&= \mathcal{E}\left\{y[n-m]\left(\sum_{k=1}^{N} a_k y[n-k] + x[n]\right)\right\} \\
&= \sum_{k=1}^{N} a_k \mathcal{E}\left\{y[n-m]y[n-k]\right\} + \mathcal{E}\left\{y[n-m]x[n]\right\} \\
&= \sum_{k=1}^{N} a_k \phi_{yy}[m-k] + \phi_{yx}[-m] \\
&= \sum_{k=1}^{N} a_k \phi_{yy}[m-k] + \phi_{xy}[m]
\end{aligned}$$

For $m = 0$,

$$\phi_{yy}[0] = \sum_{k=1}^{N} a_k \phi_{yy}[-k] + \phi_{xy}[0]$$

The $\phi_{xy}[0]$ term is

$$\begin{aligned}
\phi_{xy}[0] &= \mathcal{E}\left\{x[n]y[n]\right\} \\
&= \mathcal{E}\left\{x[n]\left(\sum_{k=1}^{N} a_k y[n-k] + x[n]\right)\right\} \\
&= \sum_{k=1}^{N} a_k \mathcal{E}\left\{x[n]y[n-k]\right\} + \mathcal{E}\left\{x[n]x[n]\right\} \\
&= \sum_{k=1}^{N} a_k \mathcal{E}\left\{x[n]y[n-k]\right\} + \sigma_x^2
\end{aligned}$$

Note that $x[n]$ is uncorrelated with the $y[n-k]$, for $k = 1, \ldots, N$. Therefore,

$$\phi_{xy}[0] = \sigma_x^2$$

Thus,

$$\begin{aligned}
\phi_{yy}[0] &= \sum_{k=1}^{N} a_k \phi_{yy}[-k] + \sigma_x^2 \\
&= \sum_{k=1}^{N} a_k \phi_{yy}[k] + \sigma_x^2
\end{aligned}$$

since $\phi_{yy}[k] = \phi_{yy}[-k]$. For $m \geq 1$,

$$\begin{aligned}
\phi_{yy}[m] &= \sum_{k=1}^{N} a_k \phi_{yy}[m-k] + \phi_{xy}[m] \\
&= \sum_{k=1}^{N} a_k \phi_{yy}[m-k]
\end{aligned}$$

since $\phi_{xy}[m]$ is zero for all $m \geq 1$.

(f) By symmetry of the autocorrelation sequence, we know that

$$\begin{aligned}
\phi_{yy}[m-k] &= \phi_{yy}[k-m] \\
&= \phi_{yy}[|m-k|]
\end{aligned}$$

Thus,

$$\sum_{k=1}^{N} a_k \phi_{yy}[|m-k|] = \sum_{k=1}^{N} a_k \phi_{yy}[m-k]$$

Using the result from part (e), we get

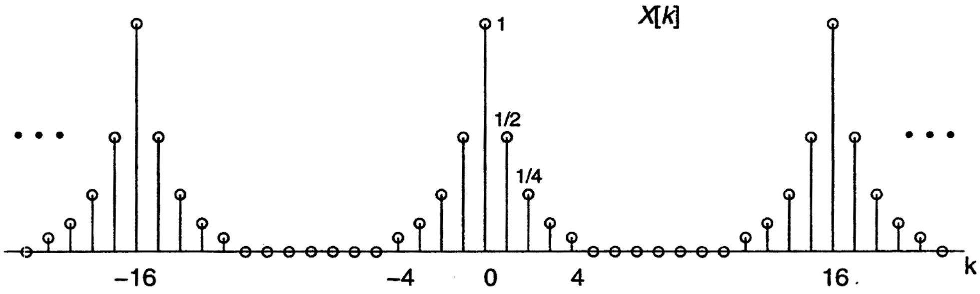
$$\sum_{k=1}^{N} a_k \phi_{yy}[|m-k|] = \phi_{yy}[m]$$

for $m = 1, 2, \ldots, N$.

10.48. (a) Sampling $x_c(t)$ we get

$$\begin{aligned} x[n] &= x_c(nT) \\ &= \frac{1}{16}\sum_{k=-4}^{4}\left(\frac{1}{2}\right)^{|k|} e^{j(2\pi/16)kn} \end{aligned}$$

Define the periodic sequence $X[k]$ to be



Then we see that we can write $x[n]$ in terms of $X[k]$:

$$\begin{aligned} x[n] &= \frac{1}{16}\sum_{k=-4}^{4} X[k]e^{j(2\pi/16)kn} \\ &= \frac{1}{16}\sum_{k=-8}^{7} X[k]e^{j(2\pi/16)kn} \\ &= \text{IDFS}\{X[k]\} \end{aligned}$$

However, since the period we use in the sum of the IDFS is unimportant we can also write

$$\begin{aligned} x[n] &= \frac{1}{16}\sum_{k=0}^{15} X[k]e^{j(2\pi/16)kn} \\ &= \text{IDFS}\{X[k]\} \\ &= \text{IDFT}\{X_0[k]\} \end{aligned}$$

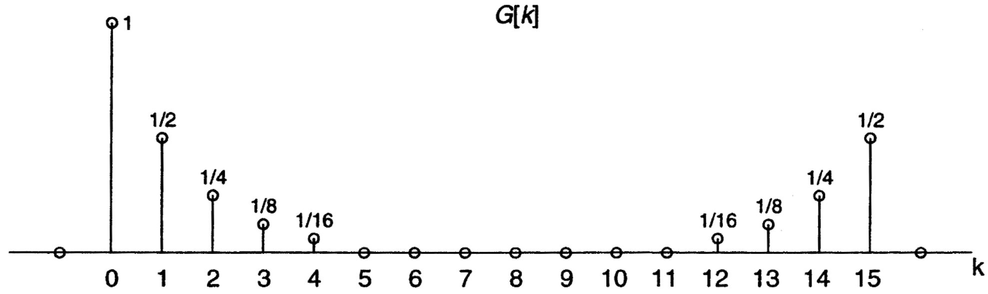
where $X_0[k]$ is the period of $X[k]$ starting at zero, i.e.,

$$X_0[k] = \begin{cases} X[k], & k = 0, \ldots, 15 \\ 0, & \text{otherwise} \end{cases}$$

Using this information we can now find $G[k]$

$$\begin{aligned} G[k] &= \text{DFT}\{g[n]\} \\ &= \text{DFT}\{x[n](u[n] - u[n-16])\} \\ &= \text{DFT}\{x[n]\} \\ &= \text{DFT}\{\text{IDFT}\{X_0[k]\}\} \end{aligned}$$

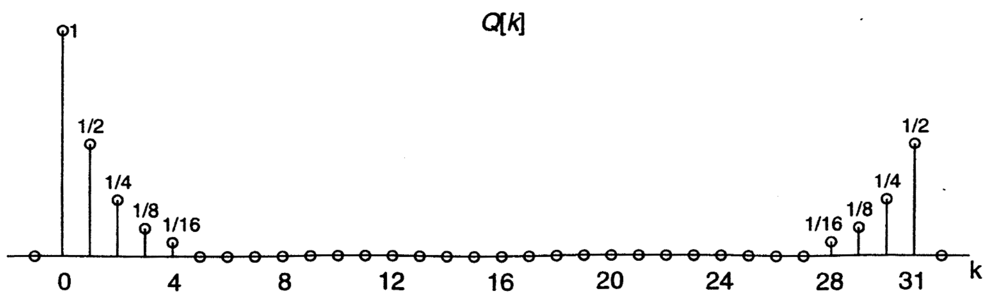
Thus, $G[k]$ looks like



(b) We want to find a sequence $Q[k]$ such that

$$\begin{aligned} q[n] &= \alpha x_c\left(\frac{n2\pi}{32}\right) \\ &= \frac{\alpha}{16}\sum_{k=-4}^{4}\left(\frac{1}{2}\right)^{|k|} e^{j(2\pi/32)kn} \end{aligned}$$

We can apply the same idea as we did in part (a), except now the DFS and DFT size should be 32 instead of 16. Going through the same steps will lead us to the sequence $Q[k]$ that looks like:



(Here we have assumed $\alpha = 1$). We see that we can interpolate in the time domain by zero padding in the *middle* of the DFT samples.

10.49. (a) Using the relation,

$$f_k = \begin{cases} \frac{k}{NT}, & 0 \le k \le N/2 \\ \frac{k-N}{NT}, & N/2 \le k \le N \end{cases}$$

where N is the DFT length and T is the sampling period, the continuous-time frequencies corresponding to the DFT indices $k = 32$ and $k = 231$ are

$$\begin{aligned} f_{32} &= \frac{32}{(256)(1/20,000)} \\ &= 2500 \text{ Hz} \\ f_{231} &= \frac{231-256}{(256)(1/20,000)} \\ &= -1953 \text{ Hz} \end{aligned}$$

(b) Since

$$\hat{x}[n] = x[n]w_R[n]$$

the DTFT of $\hat{x}[n]$ is simply the periodic convolution of $X(e^{j\omega})$ with $W_R(e^{j\omega})$.

$$\hat{X}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W_R(e^{j(\omega-\theta)})d\theta$$

(c) Multiplication in the time domain corresponds to periodic convolution in the frequency domain, as shown in part (b). To evaluate this periodic convolution at the frequency $\omega_{32} = 2\pi(32)/L$, (where $L = N = 256$) corresponding to the $k = 32$ DFT coefficient, we first shift the window $W_{avg}(e^{j\omega})$ to ω_{32}. Then, we multiply the shifted window with $X(e^{j\omega})$, and integrate the result. In order for

$$X_{avg}[32] = \alpha\hat{X}[31] + \hat{X}[32] + \alpha\hat{X}[33]$$

we must therefore have

$$W_{avg}(e^{j\omega}) = \begin{cases} 1, & \omega = 0 \\ \alpha, & \omega = \pm 2\pi/L \\ 0, & 2\pi k/L, \qquad \text{for } k = 2, 3, \ldots, L-2 \end{cases}$$

Note that we are only specifying $W_{avg}(e^{j\omega})$ at the DFT frequencies $\omega = 2\pi k/L$, for $k = 0, \ldots, L-1$.

(d) Note that the L point DFT of a rectangular window of length L is

$$\begin{aligned} W_R[k] &= \sum_{n=0}^{L-1}(1)e^{-j2\pi k/L} \\ &= \frac{1-e^{-j2\pi k}}{1-e^{-j2\pi kL}} \\ &= L\delta[k] \end{aligned}$$

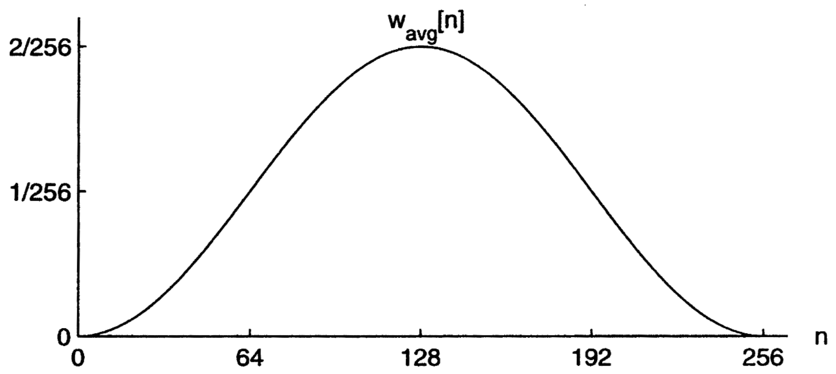
$W_{avg}(e^{j\omega})$ is only specified at DFT frequencies $\omega = 2\pi k/L$, and it can take on other values between these frequencies. Therefore, the DTFT of $W_{avg}(e^{j\omega})$ can be written in terms of $W_R(e^{j\omega})$ and two shifted versions of $W_R(e^{j\omega})$.

$$W_{avg}(e^{j\omega}) = \frac{\alpha}{L}W_R\left(e^{j(\omega+2\pi/L)}\right) + \frac{1}{L}W_R(e^{j\omega}) + \frac{\alpha}{L}W_R\left(e^{j(\omega-2\pi/L)}\right)$$

(e) Taking the inverse DTFT of $W_{avg}(e^{j\omega})$ gives $w_{avg}[n]$.

$$\begin{aligned} w_{avg}[n] &= \frac{\alpha}{L}w_R[n]e^{-j2\pi n/L} + \frac{1}{L}w_R[n] + \frac{\alpha}{L}w_R[n]e^{j2\pi n/L} \\ &= \left[\frac{1}{L} + \frac{2\alpha}{L}\cos\left(\frac{2\pi n}{L}\right)\right] w_R[n] \end{aligned}$$

A sketch of $w_{avg}[n]$ is provided below.

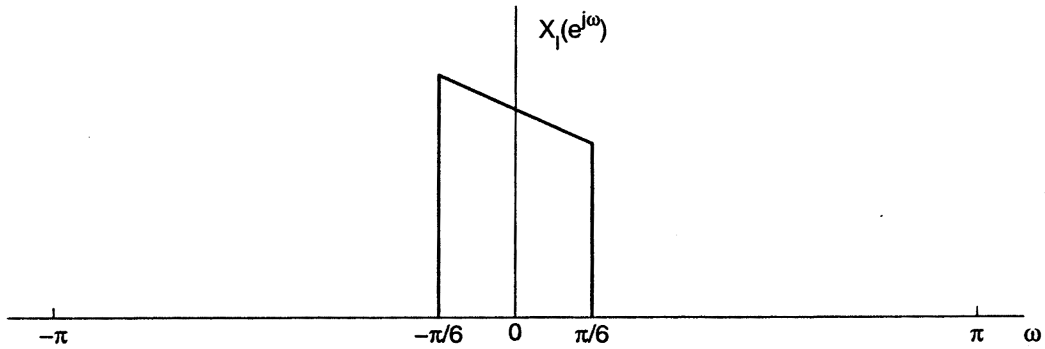


10.50. (a) After the lowpass filter, the highest frequency in the signal is $\Delta\omega$. To avoid aliasing in the downsampler we must have

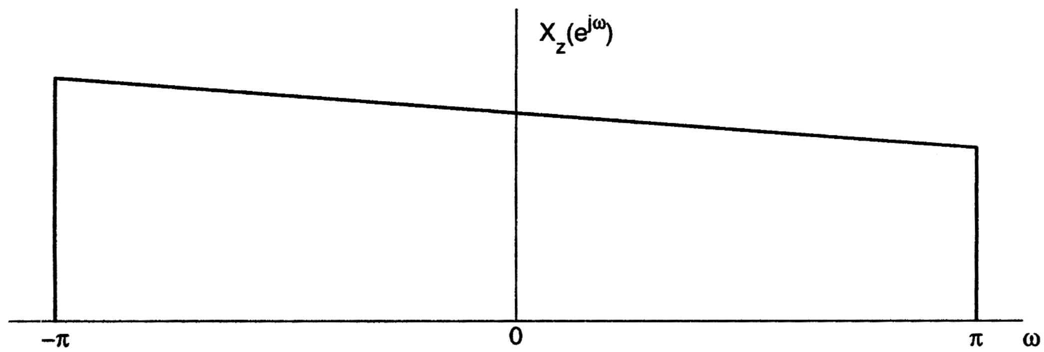
$$\begin{aligned} \Delta\omega M &\le \pi \\ M &\le \frac{\pi}{\Delta\omega} \\ &\le \frac{N}{2k_\Delta} \end{aligned}$$

$$M_{\text{max}} = \frac{N}{2k_\Delta}$$

(b) The fourier transform of $x_l[n]$ looks like



so $M = 6$ is the largest M we can use that avoids aliasing. With this choice of M the fourier transform of $x_z[n]$ looks like



Taking the DFT of $x_z[n]$ gives us N samples of $X_z(e^{j\omega})$ spaced $2\pi/N$ apart in frequency. By examining the figures above we see that these samples correspond to the desired samples of $X(e^{j\omega})$ which will be spaced $2\Delta\omega/N$ apart inside the region $-\Delta\omega < \omega < \Delta\omega$.

Note that after downsampling the endpoints of the region alias. Therefore, we cannot trust the values our new DFT provides at those points. However, the way the problem is set up we already know the values at the endpoints from the original DFT.

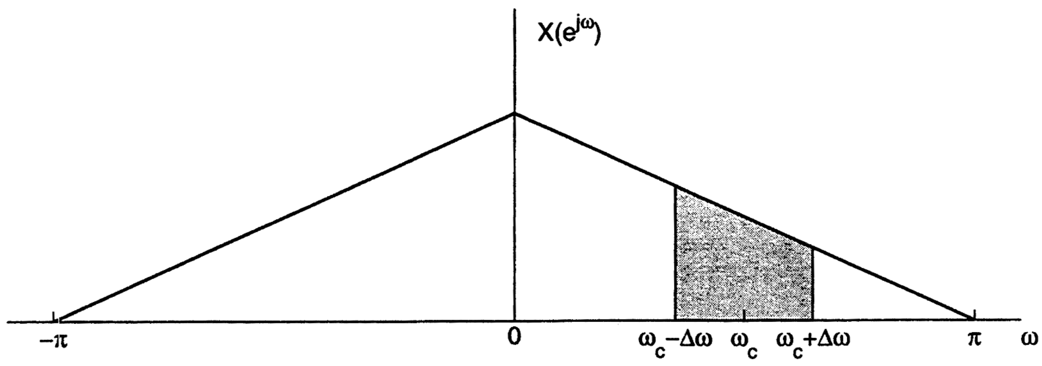
(c) The system $p[n]$ periodically replicates $X_N[n]$ to create $\tilde{X}_N[n]$. Then, the upsampler inserts $M-1$ zeros in betweeen each sample of $\tilde{X}_N[n]$. Thus, the samples $k_c - k_\Delta$ and $k_c + k_\Delta$ which border the zoom region in the original DFT map to $M(k_c - k_\Delta)$ and $M(k_c + k_\Delta)$. The system $h[n]$ then interpolates between the nonzero points filling in the "missing" samples. Since the linear phase filter is length 513 it adds a delay of $M/2 = 512/2 = 256$ samples so the desired samples of $\tilde{X}_{NM}[n]$ now lie in the region

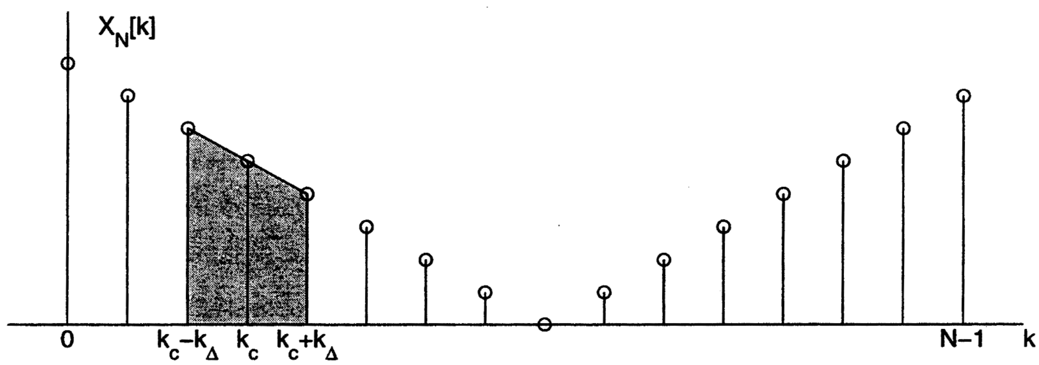
$$\begin{aligned} M(k_c - k_\Delta) + 256 &\le n \le M(k_c + k_\Delta) + 256 \\ k_c' - k_\Delta' &\le n \le k_c' + k_\Delta' \end{aligned}$$

where

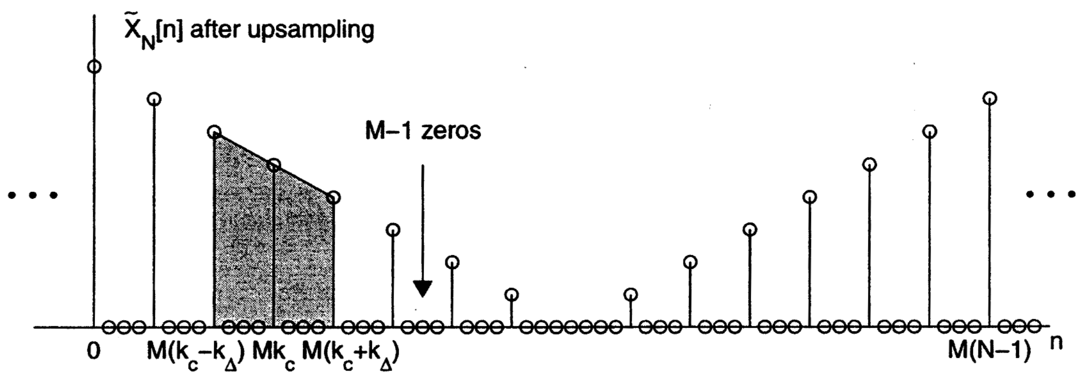
$$k'_c = Mk_c + 256$$
$$k'_\Delta = Mk_\Delta$$

(d) A typical sketch of $X(e^{j\omega})$ and $X_N[k]$ look like.

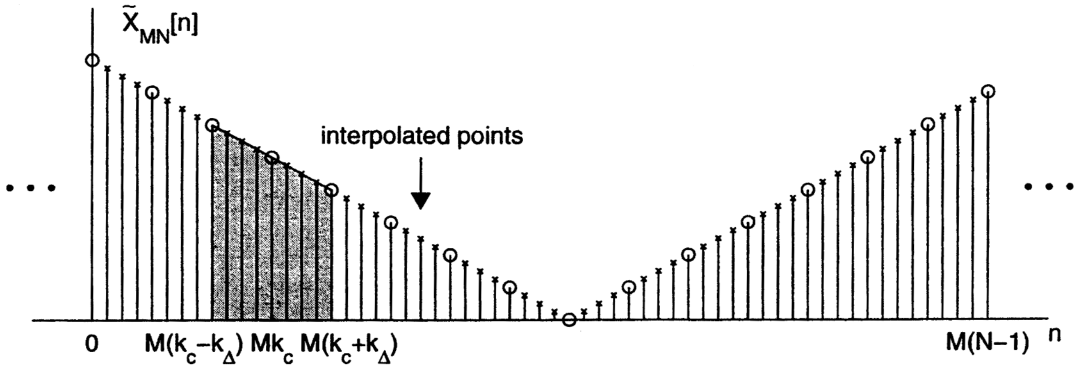




After periodically replicating and upsampling by M we have a signal that looks like



Filtering by $h[n]$ then interpolates between the samples. $\tilde{X}_{NM}[n]$ is shown below if we assume that $h[n]$ is the ideal zero phase filter. The points with an x correspond to the interpolated points.



Thus, we need to extract the points

$$\begin{aligned} M(k_c - k_\Delta) \quad &\le n \le \quad M(k_c + k_\Delta) \\ k'_c - k'_\Delta \quad &\le n \le \quad k'_c + k'_\Delta \end{aligned}$$

where

$$\begin{aligned} k'_c &= Mk_c \\ k'_\Delta &= Mk_\Delta \end{aligned}$$

11.1

Given

$$\phi_{ss}[i,k] = \sum_{n=-\infty}^{\infty} s[n-i]s[n-k],$$

let $m = n - i$. Then

$$\phi_{ss}[i,k] = \sum_{m=-\infty}^{\infty} s[m]s[m+i-k]$$
$$= r_{ss}[i-k].$$

Now $r_{ss}[k-i]$ is given by

$$r_{ss}[k-i] = \sum_{m=-\infty}^{\infty} s[m]s[m+k-i]$$
$$= \sum_{m=-\infty}^{\infty} s[m+k-i]s[m].$$

Substituting $n = m + k - i$ gives

$$r_{ss}[k-i] = \sum_{n=-\infty}^{\infty} s[n]s[n+i-k]$$
$$= r_{ss}[i-k].$$

Thus we can write

$$r_{ss}[|i-k|] = \sum_{m=-\infty}^{\infty} s[m]s[m+i-k].$$

Therefore

$$\phi_{ss}[i,k] = r_{ss}[|i-k|],$$

which is a function of $|i-k|$.

11.2

Given

$$E = \left\langle \left(s[n] - \sum_{k=1}^{p} a_k s[n-k] \right)^2 \right\rangle.$$

A. Then

$$\begin{aligned}
E &= \left\langle s^2[n] - s[n]\sum_{i=1}^{p} a_i s[n-i] - s[n]\sum_{k=1}^{p} a_k s[n-k] + \sum_{i=1}^{p} a_i s[n-i] \sum_{i=k}^{p} a_k s[n-k] \right\rangle \\
&= \left\langle s^2[n] \right\rangle - 2\left\langle s[n]\sum_{k=1}^{p} a_k s[n-k] \right\rangle + \left\langle \sum_{i=1}^{p} a_i s[n-i] \sum_{k=1}^{p} a_k s[n-k] \right\rangle \\
&= \phi_{ss}[0,0] - 2\sum_{k=1}^{p} a_k \left\langle s[n]s[n-k] \right\rangle + \sum_{i=1}^{p}\sum_{k=1}^{p} a_i a_k \left\langle s[n-i]s[n-k] \right\rangle \\
&= \phi_{ss}[0,0] - 2\sum_{k=1}^{p} a_k \phi_{ss}[0,k] + \sum_{i=1}^{p} a_i \sum_{k=1}^{p} a_k \phi_{ss}[i,k],
\end{aligned}$$

as required.

B. Now suppose

$$\sum_{k=1}^{p} a_k \phi_{ss}[i,k] = \phi_{ss}[i,0], \quad i = 1, \ldots, p.$$

Substituting gives

$$\begin{aligned}
E &= \phi_{ss}[0,0] - 2\sum_{k=1}^{p} a_k \phi_{ss}[0,k] + \sum_{i=1}^{p} a_i \phi_{ss}[i,0] \\
&= \phi_{ss}[0,0] - \sum_{k=1}^{p} a_k \phi_{ss}[0,k],
\end{aligned}$$

since $\phi_{ss}[k,0] = \phi_{ss}[0,k]$.

11.3

Given

$$h[n] = \sum_{k=1}^{p} a_k h[n-k] + G\delta[n].$$

A. We have

$$\begin{aligned}
r_{hh}[-m] &= \sum_{n=-\infty}^{\infty} h[n]h[n-m] \\
&= \sum_{n=-\infty}^{\infty}\left(\sum_{k=1}^{p} a_k h[n-k] + G\delta[n]\right)h[n-m] \\
&= \sum_{n=-\infty}^{\infty}\sum_{k=1}^{p} a_k h[n-k]h[n-m] + \sum_{n=-\infty}^{\infty} G\delta[n]h[n-m] \\
&= \sum_{k=1}^{p} a_k \sum_{n=-\infty}^{\infty} h[n-k]h[n-m] + Gh[-m] \\
&= \sum_{k=1}^{p} a_k \sum_{n=-\infty}^{\infty} h[n-k]h[n-m], \quad m = 1,\ldots,p,
\end{aligned}$$

since the all-pole system with impulse response $h[k]$ is causal. Now substitute $j = n-k$ to obtain

$$\begin{aligned}
r_{hh}[-m] &= \sum_{k=1}^{p} a_k \sum_{j=-\infty}^{\infty} h[j]h[j+k-m] \\
&= \sum_{k=1}^{p} a_k r_{hh}[k-m] \\
&= \sum_{k=1}^{p} a_k r_{hh}[|m-k|], \quad m = 1,\ldots,p,
\end{aligned}$$

where the last line reflects the fact that $r_{hh}[m]$ is an even function. Further, since $r_{hh}[-m] = r_{hh}[m]$ we have

$$r_{hh}[m] = \sum_{k=1}^{p} a_k r_{hh}[|m-k|], \quad m = 1,\ldots,p.$$

B. As above, we have

$$r_{hh}[-m] = \sum_{k=1}^{p} a_k \sum_{n=-\infty}^{\infty} h[n-k]h[n-m] + Gh[-m].$$

Now let $m = 0$. We have

$$r_{hh}[0] = \sum_{k=1}^{p} a_k \sum_{n=-\infty}^{\infty} h[n-k]h[n] + Gh[0]$$

$$= \sum_{k=1}^{p} a_k r_{hh}[k] + Gh[0].$$

But $h[0] = G\delta[0] = G$. Then

$$r_{hh}[0] = \sum_{k=1}^{p} a_k r_{hh}[k] + G^2,$$

as was to have been shown.

11.4. Appears in: Spring04 PS6, Fall03 PS6.

Problem

Consider a signal $x[n] = s[n] + w[n]$, where $s[n]$ is a first order autoregressive process that satisfies the difference equation

$$s[n] = 0.8s[n-1] + v[n]$$

where $v[n]$ is a white noise sequence with variance $\sigma_v^2 = 0.49$ and $w[n]$ is a white noise sequence with variance $\sigma_w^2 = 1$. The processes $v[n]$ and $w[n]$ are uncorrelated.

Determine the autocorrelation sequences $\phi_{ss}[m]$ and $\phi_{xx}[m]$.

Solution from Spring04 PS6

In addition to the information stated in the problem, students were also told that $v[n]$ and $w[n]$ were 0 mean sequences.

The signal $s[n]$ is generated by passing a 0 mean WSS sequence $v[n]$ through the LTI system

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$

Assuming that $h[n]$ is causal, $h[n] = (0.8)^n\, u[n]$, and

$$\phi_{ss}[m] = \phi_{vv}[n] * h[n] * h[-n]|_{n=m}$$

We can evaluate $h[n] * h[-n]$ using the same method as in problem 6.2 part (c), and we obtain

$$h[n] * h[-n] = \frac{1}{0.36}\,(0.8)^{|m|}$$

Therefore,

$$\phi_{ss}[m] = \frac{0.49}{0.36}\,(0.8)^{|m|}$$

Since $w[n]$ is uncorrelated with $v[n]$, $w[n]$ is also uncorrelated with $s[n]$, and

$$\phi_{xx}[m] = \phi_{ss}[m] + \phi_{ww}[m] = \phi_{ss}[m] + \delta[m]$$

Solution from Fall03 PS6

In addition to the information stated in the problem, students were also told that $v[n]$ and $w[n]$ were 0 mean sequences.

The signal $s[n]$ is generated by passing a 0 mean WSS sequence $v[n]$ through the LTI system

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$

Assuming that $h[n]$ is causal, $h[n] = (0.8)^n \, u[n]$, and

$$\phi_{ss}[m] = \phi_{vv}[n] * h[n] * h[-n]|_{n=m}$$

We can evaluate $h[n] * h[-n]$ using the same method as in problem 1 part (c), and we obtain

$$h[n] * h[-n] = \frac{1}{0.36} \, (0.8)^{|m|}$$

Therefore,

$$\phi_{ss}[m] = \frac{0.49}{0.36} \, (0.8)^{|m|}$$

Since $w[n]$ is uncorrelated with $v[n]$, $w[n]$ is also uncorrelated with $s[n]$, and

$$\phi_{xx}[m] = \phi_{ss}[m] + \phi_{ww}[m] = \phi_{ss}[m] + \delta[m]$$

11.5. Problem 5 in Spring2005 Midterm exam.

Problem

Recall: In the inverse filter approach to all-pole modeling of a deterministic signal $s[n]$, we consider

$$s[n] \rightarrow \boxed{\frac{1}{A}\left[1 - \sum_{k=1}^{p} a_k z^{-k}\right]} \rightarrow g[n]$$

and choose the coefficients a_1, a_2, $\ldots$, a_p to minimize

$$\mathcal{E} = \sum_{n=0}^{\infty} (g[n] - \delta[n])^2.$$

We then consider

$$\frac{A}{1 - \sum_{k=1}^{p} a_k z^{-k}}$$

to be the best available approximation to $S(z)$.

(a) Find the coefficients a_1 and a_2 of the best all-pole model for $s[n] = \delta[n] + \delta[n-2]$ with $p = 2$.

(b) Find the coefficients a_1, a_2 and a_3 of the best all-pole model for $s[n] = \delta[n] + \delta[n-2]$ with $p = 3$.

Solution from Spring05 midterm

Every term and notation in this problem matches the Parametric Signal Modeling handout. The relevant samples of the deterministic correlation sequence of $s[n]$ are:

$$\begin{aligned}\phi_{ss}[0] &= 2\\ \phi_{ss}[1] &= 0\\ \phi_{ss}[2] &= 1\end{aligned}$$

In matrix form, the autocorrelation normal equations are

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From this we have (uniquely) that $a_1 = 0$ and $a_2 = \frac{1}{2}$.

We now need one more sample of the deterministic autocorrelation:

$$\phi_{ss}[3] = 0.$$

The autocorrelation normal equations are now

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

which have the (unique) solution $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = 0$.

11.6

The all-pole spectrum estimate is given by

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{G}{1-\sum_{i=1}^{p} a_i e^{-j\omega i}}\right|.$$

For simplicity let us assume that $N > p$. If $\omega_k = 2\pi k/N, \quad k = 0,\ldots,N-1$, then

$$\left|H\left(e^{j\omega_k}\right)\right| = \left|\frac{G}{1-\sum_{i=1}^{p} a_i e^{-j2\pi ki/N}}\right|, \quad k = 0,\ldots,N-1.$$

Now let $a[0]=1$, $a[i]=-a_i, \quad i=1,\ldots,p$, and $a[i]=0, \quad i=p+1,\ldots,N-1$. Then

$$A[k] = 1-\sum_{i=1}^{p} a_i e^{-j2\pi ki/N}$$

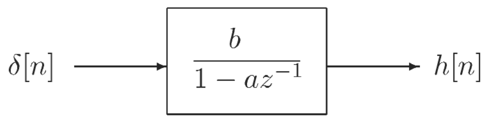
is the DFT of the finite sequence $a[i], \quad i=0,\ldots,N-1$. The algorithm is

1. Given the coefficients $a_i, \quad i=1,\ldots,p$, form $a[i], \quad i=0,\ldots,N-1$.
2. Use the FFT program to find $A[k]$, the DFT of $a[i]$.
3. The spectrum estimate is $\left|\frac{G}{A[k]}\right|, \quad k=0,\ldots,N-1$.

11.7. Appears in: Spring05 PS7, Fall04 PS6, Fall02 PS6, Fall01 PS9. Note: The Fall01 version has a few minor differences.

Problem

Consider a desired, causal impulse response $h_d[n]$ that we wish to approximate by the following system:



Our optimality criterion is to minimize the error function given by:

$$\varepsilon = \sum_{n=0}^{\infty} (h_d[n] - h[n])^2$$

(a) Supose a is given and we wish to determine the unknown parameter b which minimizes ε. Assume that $|a| < 1$. Is this a nonlinear problem? If so, show why. If not, determine b.

(b) Suppose b is given and we wish to determine the unknown parameter a which minimizes ε. Is this a nonlinear problem? If so, show why. If not, determine a.

Solution from Spring05 PS7

$H(z)$ is causal, thus:

$$H(z) = \frac{b}{1 - az^{-1}} \longrightarrow h[n] = ba^n u[n]$$

$$\varepsilon = \sum_{n=0}^{\infty} (h_d[n] - ba^n)^2$$

(a)

$$\frac{\partial \varepsilon}{\partial b} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(-a^n) = 0$$

$$\sum_{n=0}^{\infty} a^n h_d[n] = b \sum_{n=0}^{\infty} (a^2)^n = b \frac{1}{1 - a^2}$$

$$\Rightarrow b = (1 - a^2) \sum_{n=0}^{\infty} a^n h_d[n]$$

(b)

$$\frac{\partial \varepsilon}{\partial a} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(bna^{n-1}) = 0 \longrightarrow \textbf{Nonlinear}$$

This problem illustrates that even for this simple first order all-pole system, it's intractable to obtain coefficients by directly matching $h_d[n]$ and $h[n]$, thus justifying the inverse approach, which yields the normal equations.

Solution from Fall04 PS6

$H(z)$ is causal, thus:

$$H(z) = \frac{b}{1 - az^{-1}} \longrightarrow h[n] = ba^n u[n]$$

$$\varepsilon = \sum_{n=0}^{\infty} (h_d[n] - ba^n)^2$$

(a)

$$\frac{\partial \varepsilon}{\partial b} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(-a^n) = 0$$

$$\sum_{n=0}^{\infty} a^n h_d[n] = b \sum_{n=0}^{\infty} (a^2)^n = b \frac{1}{1 - a^2}$$

$$\Rightarrow b = (1 - a^2) \sum_{n=0}^{\infty} a^n h_d[n]$$

(b)

$$\frac{\partial \varepsilon}{\partial a} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(bna^{n-1}) = 0 \longrightarrow \textbf{Nonlinear}$$

This problem illustrates that even for this simple first order all-pole system, it's intractable to obtain coefficients by directly matching $h_d[n]$ and $h[n]$, thus justifying the inverse approach, which yields the normal equations.

Solution from Fall02 PS6

$H(z)$ is causal, thus:

$$H(z) = \frac{b}{1 - az^{-1}} \longrightarrow h[n] = ba^n u[n]$$

$$\varepsilon = \sum_{n=0}^{\infty} (h_d[n] - ba^n)^2$$

(a)

$$\frac{\partial \varepsilon}{\partial b} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(-a^n) = 0$$

$$\sum_{n=0}^{\infty} a^n h_d[n] = b \sum_{n=0}^{\infty} (a^2)^n = b \frac{1}{1 - a^2}$$

$$\Rightarrow b = (1 - a^2) \sum_{n=0}^{\infty} a^n h_d[n]$$

(b)

$$\frac{\partial \varepsilon}{\partial a} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(bna^{n-1}) = 0 \longrightarrow \textbf{Nonlinear}$$

This problem illustrates that even for this simple first order all-pole system, it's intractable to obtain coefficients by directly matching $h_d[n]$ and $h[n]$, thus justifying the inverse approach, which yields the normal equations.

Solution from Fall01 PS9

N/A

11.8

A. The prediction error sequence $\tilde{e}[n] = s[n] - \sum_{k=1}^{p} \beta_k s[n+k]$ is the convolution of the sequence $s[n]$ with the impulse response $h_B[n] = \delta[n] - \sum_{k=1}^{p} \beta_k \delta[n+k]$ of the prediction-error filter. Now $s[n]$ takes non-zero values only in the interval $0 \le n \le M-1$ and $h_B[n]$ takes non-zero values only in the interval $-p \le n \le 0$. Thus the convolution is non-zero only in the interval $N_1 = -p \le n \le N_2 = M-1$.

B. The mean-squared backward prediction error is defined as

$$\tilde{E} = \sum_{m=-\infty}^{\infty} \left(s[m] - \sum_{k=1}^{p} \beta_k s[m+k] \right)^2 .$$

Setting the derivatives of $\tilde{E}$ with respect to β_i, $i = 1, \ldots, p$, equal to zero gives

$$\frac{\partial \tilde{E}}{\partial \beta_i} = -\sum_{m=-\infty}^{\infty} \left(s[m] - \sum_{k=1}^{p} \beta_k s[m+k] \right) s[m+i] = 0, \quad i = 1, \ldots, p.$$

That is,

$$\sum_{m=-\infty}^{\infty} s[m]s[m+i] = \sum_{k=1}^{p} \beta_k \sum_{m=-\infty}^{\infty} s[m+k]s[m+i], \quad i = 1, \ldots, p.$$

Now define the autocorrelation function $r_{ss}[i-k]$ by

$$r_{ss}[i-k] = \sum_{m=-\infty}^{\infty} s[m+k]s[m+i], \quad i = 1, \ldots, p, \quad k = 1, \ldots, p.$$

Then we have the required normal equations

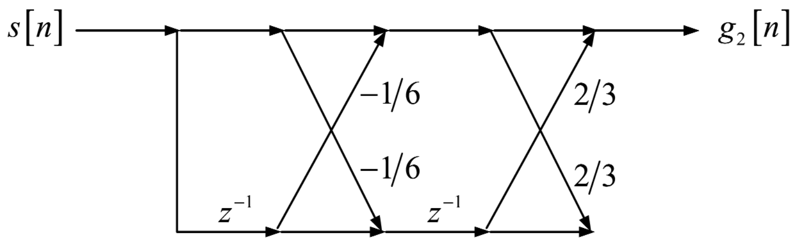
$$r_{ss}[i] = \sum_{k=1}^{p} \beta_k r_{ss}[i-k], \quad i = 1, \ldots, p.$$

C. The normal equations derived in part B for backward prediction are the same as those derived in the text for forward prediction. Consequently, the backward predictor coefficients $\{\beta_k\}$ are identical to the forward predictor coefficients $\{\alpha_k\}$.

11.9

A. By tracing paths through the flow graph it is easy to see that the coefficient of z^{-1} is $h_{inv}^{(4)}[1] = -1/18$.

B.



C. First we convert the reflection coefficients into system function coefficients for the inverse filter. We have

$$a_1^{(1)} = k_1 = 1/6$$

$$a_2^{(2)} = k_2 = -2/3$$

$$a_1^{(2)} = a_1^{(1)} - k_2 a_1^{(1)} = 5/18.$$

Then $H_{inv}^{(2)}(z) = 1 - \frac{5}{18}z^{-1} + \frac{2}{3}z^{-2}$, and

$$H^{(2)}(z) = \frac{1}{1 - \frac{5}{18}z^{-1} + \frac{2}{3}z^{-2}}.$$

11.10

We have

$$A^{(i)}(z) = 1 - a_1^{(i)} z^{-1} - a_2^{(i)} z^{-2} - \cdots - a_i^{(i)} z^{-i}$$
$$= \left(1 - z_1^{(i)} z^{-1}\right)\left(1 - z_2^{(i)} z^{-1}\right)\cdots\left(1 - z_i^{(i)} z^{-1}\right).$$

Multiplying out, we see that the coefficient $a_i^{(i)}$ of z^{-i} is

$$k_i = a_i^{(i)} = -(-1)^i z_1^{(i)} z_2^{(i)} \cdots z_i^{(i)}.$$

Then

$$|k_i| = \left|z_1^{(i)} z_2^{(i)} \cdots z_i^{(i)}\right| = \left|z_1^{(i)}\right|\left|z_2^{(i)}\right|\cdots\left|z_i^{(i)}\right|.$$

Now if $\left|z_j^{(i)}\right| < 1, \quad j = 1, \ldots, i$, it follows that $|k_i| < 1$. Consequently, if $|k_i| \geq 1$, then we must have $\left|z_j^{(i)}\right| \geq 1$ for some j.

11.11

A. We have a white noise input to the system with zero mean, unit variance, and $r_{xx}[m] = \sigma_x^2 \delta[m] = \delta[m]$. Taking the inverse z-transform of the given system function $H(z)$, we have the following impulse response:

$$h[n] = h_0 \delta[n] + h_1 \delta[n-1].$$

The autocorrelation of the output of the system can be determined as follows:

$$\begin{aligned} r_{yy}[m] &= h[m] * h[-m] * r_{xx}[m] \\ &= h[m] * h[-m] * \delta[m] \\ &= h[m] * h[-m] \\ &= h_0 h_1 \delta[m+1] + \left(h_0^2 + h_1^2\right) \delta[m] + h_0 h_1 \delta[m-1]. \end{aligned}$$

B. The impulse response of the forward prediction error filter is

$$h_A[n] = \delta[n] - a_1 \delta[n-1] - a_2 \delta[n-2],$$

and we have the following relation for the autocorrelation function of the error:

$$r_{ee}[m] = h_A[m] * h_A[-m] * r_{yy}[m].$$

Now

$$\begin{aligned} h_A[m] * h_A[-m] = &-a_2 \delta[m+2] + \left(-a_1 + a_1 a_2\right) \delta[m+1] + \left(1 + a_1^2 + a_2^2\right) \delta[m] \\ &+ \left(-a_1 + a_1 a_2\right) \delta[m-1] - a_2 \delta[m-2] \end{aligned}$$

and we have $r_{yy}[m]$ from part A. Convolving the two will give $r_{ee}[m]$. However, we want to minimize the variance of $e[n]$, and so we just need to compute $r_{ee}[0]$. We obtain

$$\begin{aligned} r_{ee}[0] &= \left(-a_1 + a_1 b_1\right)\left(h_0 h_1\right) + \left(1 + a_1^2 + a_2^2\right)\left(h_0^2 + h_1^2\right) + \left(-a_1 + a_1 b_1\right)\left(h_0 h_1\right) \\ &= \left(1 + a_1^2 + a_2^2\right)\left(h_0^2 + h_1^2\right) + 2\left(-a_1 + a_1 b_1\right)\left(h_0 h_1\right). \end{aligned}$$

Now we take partial derivatives with respect to a_1 and a_2, set them equal to zero, and solve for a_1 and a_2 respectively.

$$\frac{\partial r_{ee}[0]}{\partial a_1} = 2a_1\left(h_0^2 + h_1^2\right) + 2h_0h_1\left(-1 + a_2\right) = 0$$

$$\frac{\partial r_{ee}[0]}{\partial a_2} = 2a_2\left(h_0^2 + h_1^2\right) + 2h_0h_1a_1 = 0.$$

$$a_1 = \frac{h_0h_1\left(h_0^2 + h_1^2\right)}{\left(h_0^2 + h_1^2\right)^2 - h_0^2h_1^2}$$

$$a_2 = \frac{-h_0^2h_1^2}{\left(h_0^2 + h_1^2\right)^2 - h_0^2h_1^2}.$$

C. The impulse response of the backward prediction error filter is

$$h_B[n] = \delta[n] - b_1\delta[n+1] - b_2\delta[n+2],$$

and we have the following relation for the autocorrelation function of the error:

$$r_{\tilde{e}\tilde{e}}[m] = h_B[m] * h_B[-m] * r_{yy}[m].$$

Now

$$h_B[m] * h_B[-m] = -b_2\delta[m+2] + \left(-b_1 + b_1b_2\right)\delta[m+1] + \left(1 + b_1^2 + b_2^2\right)\delta[m] + \left(-b_1 + b_1b_2\right)\delta[m-1] - b_2\delta[m-2]$$

and we have $r_{yy}[m]$ from part A. Convolving the two will give $r_{\tilde{e}\tilde{e}}[m]$. We note, however, that $r_{\tilde{e}\tilde{e}}[m]$ is identical to $r_{ee}[m]$ with b_1 and b_2 substituted for a_1 and a_2, respectively. The solution proceeds exactly as in part B, yielding

$$b_1 = \frac{h_0h_1\left(h_0^2 + h_1^2\right)}{\left(h_0^2 + h_1^2\right)^2 - h_0^2h_1^2}$$

$$b_2 = \frac{-h_0^2h_1^2}{\left(h_0^2 + h_1^2\right)^2 - h_0^2h_1^2}.$$

We see that $b_1 = a_1$ and $b_2 = a_2$.

11.12

A. For the given all-pole model, $a_1 = a, a_2 = 0, a_3 = b$. Then

$$ar_{ss}[i-1] + br_{ss}[i-3] = r_{ss}[i], \quad i = 1,2,3.$$

B. First observe that

$$v[n] = x[n-1] + \frac{1}{2}x[n-2] + z[n].$$

Now

$$\phi_{vv}[n] = \overline{v[n]v[n+m]}$$

$$= \overline{\left(x[n-1] + \frac{1}{2}x[n-2] + z[n]\right)\left(x[n-1+m] + \frac{1}{2}x[n-2+m] + z[n]\right)}.$$

Multiplying out and averaging term-by-term gives

$$\phi_{vv}[m] = \frac{1}{2}\delta[m+1] + \frac{9}{4}\delta[m] + \frac{1}{2}\delta[m-1],$$

where we have used the facts that $\phi_{xx}[m] = \delta[m]$, $\phi_{zz}[m] = \delta[m]$, and $x[n]$ and $z[n]$ are independent and each of zero mean.

C. Let the output of the system $H_1(z)$ with input $x[n]$ be designated $x_1[n]$. Then $y_1[n] = x_1[n] + z[n]$. Let $H_2(z) = 1 - az^{-1} - bz^{-3}$.

Now the response of an LTI system to a zero mean input also has zero mean. Thus $\overline{x_1[n]} = 0$. Then $\overline{y_1[n]} = \overline{x_1[n] + z[n]} = \overline{x_1[n]} + \overline{z[n]} = 0$, and consequently $\overline{w_1[n]} = 0$.

By linearity, $w_1[n] = w_x[n] + w_z[n]$, where $w_x[n]$ is the response of $H_2(z)$ to $x_1[n]$ and $w_z[n]$ is the response of $H_2(z)$ to $z[n]$. It can be shown by direct calculation that $w_x[n]$ and $w_z[n]$ are uncorrelated owing to the independence of $x[n]$ and $z[n]$, and consequently $\phi_{w_1w_1}[m] = \phi_{w_xw_x}[m] + \phi_{w_zw_z}[m]$. Now $w_x[n]$ results from passing $x[n]$ through $H_1(z)$ and then through $H_2(z)$. Since $H_2(z)$ is the inverse of $H_1(z)$, $w_x[n]$ will be white because $x[n]$ is white. On the other hand, $w_z[n]$ is not white; its spectrum is shaped by the frequency response of $H_2(z)$. We have $\phi_{w_1w_1}[m] = \phi_{w_xw_x}[m] + \phi_{w_zw_z}[m] = \delta[m] + \phi_{w_zw_z}[m] \neq c\delta[m]$ for any constant c.

D. Since $w_1[n]$ has zero mean, the variance of $w_1[n]$ is $\phi_{w_1w_1}[0]$. Now

$$\begin{aligned}\phi_{w_1w_1}[0] &= \phi_{w_xw_x}[0] + \phi_{w_zw_z}[0] \\ &= 1 + \phi_{w_zw_z}[0] \\ &= 1 + \overline{w_z^2[n]}.\end{aligned}$$

We have

$$\begin{aligned}\overline{w_z^2[n]} &= \overline{\left(z[n] - az[n-1] - bz[n-3]\right)^2} \\ &= 1 + a^2 + b^2.\end{aligned}$$

Therefore $\phi_{w_1w_1}[0] = 2 + a^2 + b^2$.

11.13

A. Using $g[n]=s[n]-a_1 s[n-1]-a_2 s[n-2]$,

$$g[0]-G\delta[0]=4-G$$
$$g[1]-G\delta[1]=8-4a_1$$
$$g[2]-G\delta[2]=4-8a_1-4a_2$$
$$g[3]-G\delta[3]=2-4a_1-8a_2$$
$$g[4]-G\delta[4]=1-2a_1-4a_2$$
$$g[5]-G\delta[5]=\tfrac{1}{2}-a_1-2a_2$$

B. The linear equations are $4-G=0$ and

$$\begin{bmatrix} 4 & 8 & 4 & 2 & 1 \\ 0 & 4 & 8 & 4 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 8 & 4 \\ 4 & 8 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}=\begin{bmatrix} 4 & 8 & 4 & 2 & 1 \\ 0 & 4 & 8 & 4 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix}.$$

C. $G=4$.

D. Evaluating $g[1]-G\delta[1]$ yields $a_1 = 2$.
Using this value and evaluating $g[2]-G\delta[2]$ yields $a_2 = -3$.
Using these values, we get a nonzero value for $g[3]-G\delta[3]$. Therefore we do not expect $\mathcal{E}$ to be zero.

E. Using $g[n]=s[n]-as[n-1]$ and $r[n]=b_0\delta[n]+b_1\delta[n-1]$,

$$g[0]-r[0]=4-b_0$$
$$g[1]-r[1]=8-4a-b_1$$
$$g[2]-r[2]=4-8a$$
$$g[3]-r[3]=2-4a$$
$$g[4]-r[4]=1-2a$$
$$g[5]-r[5]=\tfrac{1}{2}-a$$

F. From the equations in part E involving only a, $a=\frac{1}{2}$.
From the equations involving b_0 and b_1, if the system needs to be minimum phase then setting $b_0 = 5$ and $b_1 = 5$ puts the system at the edge of being minimum phase while minimizing $\mathcal{E}_2$. If the system does not need to be minimum phase then $b_0 = 4$ and $b_1 = 6$.

G. If the system needs to be minimum phase, $\mathcal{E}_2 = 2$.
If the system does not need to be minimum phase, $\mathcal{E}_2 = 0$.

11.14

Given $s[n]=\alpha^n, \quad n=0,\ldots,M$. In the following it is assumed that $|\alpha|<1$.

A. Using the autocorrelation method,

$$r_{ss}[m]=r_{ss}[-m]=\begin{cases}\sum_{n=0}^{M-m} s[n]s[n+m], & m=0,\ldots,M\\ 0, & m>M.\end{cases}$$

Then for $m=-M,\ldots,M$,

$$r_{ss}[m]=\sum_{m=0}^{M-|m|}\alpha^n\alpha^{n+|m|}=\sum_{m=0}^{M-|m|}\alpha^{2n+|m|}=\alpha^{|m|}\sum_{m=0}^{M-|m|}\left(\alpha^2\right)^n$$
$$=\alpha^{|m|}\frac{1-\alpha^{2(M-|m|+1)}}{1-\alpha^2}.$$

B. Now $a_1 r_{ss}[0]=r_{ss}[1]$. Then

$$a_1=\frac{\alpha\dfrac{1-\alpha^{2M}}{1-\alpha^2}}{\dfrac{1-\alpha^{2(M+1)}}{1-\alpha^2}}=\alpha\frac{1-\alpha^{2M}}{1-\alpha^{2(M+1)}}.$$

C. Now as $M\to\infty$ we have $\alpha^{2M}\to 0$ and $\alpha^{2(M+1)}\to 0$. Then $a_1\to\alpha$.

D. We have $E=r_{ss}[0]-a_1 r_{ss}[1]$. Then

$$E=\frac{1-\alpha^{2(M+1)}}{1-\alpha^2}-\alpha\frac{1-\alpha^{2M}}{1-\alpha^{2(M+1)}}\alpha\frac{1-\alpha^{2M}}{1-\alpha^2}$$
$$=\frac{1-\alpha^{4M+2}}{1-\alpha^{2M+2}}.$$

As $M \to \infty$ we have $\alpha^{4M+2} \to 0$ and $\alpha^{2M+2} \to 0$, so $E \to 1$.
Using the covariance method,

$$\begin{aligned}
\phi_{ss}[i,k] &= \sum_{n=1}^{M} s[n-i]s[n-k] \\
&= \sum_{n=1}^{M} \alpha^{(n-i)}\alpha^{(n-k)} = \alpha^{-i-k}\sum_{n=1}^{M}\left(\alpha^2\right)^n \\
&= \alpha^{-i-k}\sum_{n=0}^{M}\left(\alpha^2\right)^n - \alpha^{-i-k} \\
&= \alpha^{-i-k}\frac{1-\alpha^{2(M+1)}}{1-\alpha^2} - \alpha^{-i-k}\frac{1-\alpha^2}{1-\alpha^2} \\
&= \alpha^{-i-k}\frac{\alpha^2-\alpha^{2(M+1)}}{1-\alpha^2} = \alpha^{2-i-k}\frac{1-\alpha^{2M}}{1-\alpha^2}.
\end{aligned}$$

Now $a_1\phi_{ss}[1,1] = \phi_{ss}[1,0]$. Then

$$a_1 = \frac{\alpha\dfrac{1-\alpha^{2M}}{1-\alpha^2}}{\dfrac{1-\alpha^{2M}}{1-\alpha^2}} = \alpha.$$

This is identical to the result given in the example in the chapter. The result found in part B approaches this value as M becomes large.

The minimum mean-squared prediction error is given by

$$E = \phi_{ss}[0,0] - a_1\phi_{ss}[0,1].$$

That is,

$$E = \alpha^2\frac{1-\alpha^{2M}}{1-\alpha^2} - \alpha\times\alpha\frac{1-\alpha^{2M}}{1-\alpha^2} = 0.$$

The covariance method produces zero prediction error in this case. This is smaller than the error produced by either the autocorrelation method above or by the example in the chapter.

11.15. Problem 5 in Spring 2002 Final exam.
Appears in: Fall04 PS6, Fall02 PS6.

Problem

(a) Consider the signal

$$p[n] = 3\left(\frac{1}{2}\right)^n u[n] + 4\left(-\frac{2}{3}\right)^n u[n] .$$

(i) We want to use a causal, *second*-order all-pole model, i.e., a model of the form

$$\widehat{P}(z) = \frac{A}{1 - a_1 z^{-1} - a_2 z^{-2}} ,$$

to optimally represent the signal $p[n]$, in the least-square error sense. Find a_1, a_2, and A.

(ii) Now suppose we want to use a causal, *third*-order all-pole model, i.e., a model of the form

$$\widetilde{P}(z) = \frac{B}{1 - b_1 z^{-1} - b_2 z^{-2} - b_3 z^{-3}} ,$$

to optimally represent the signal $p[n]$, in the least-square error sense. Find, b_1, b_2, b_3, and B.

(b) Consider the signal

$$q[n] = 3\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{2}{3}\right)^n u[n] .$$

We want to use a *second*-order linear-predictive model of the form

$$\widehat{q}[n] = c_1 q[n-1] + c_2 q[n-2]$$

to optimally represent the signal $q[n]$, in the least-square error sense. Find c_1 and c_2, and the range of n for which the model is exact, i.e., find c_1, c_2, and the range of n for which $q[n]$ is *exactly* linearly predictable from $q[n-1]$ and $q[n-2]$? **Hint:** Look at the difference equation that $q[n]$ satisfies.

Solution from Fall04 PS6

(a) (i)

$$P(z) = \frac{3}{1-\frac{1}{2}z^{-1}} + \frac{4}{1+\frac{2}{3}z^{-1}} = \frac{3+2z^{-1}+4-2z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{2}{3}z^{-1})}$$

$$P(z) = \frac{7}{1+\frac{1}{6}z^{-1}-\frac{1}{3}z^{-2}}$$

We can perfectly match $\hat{P}(z)$ with $P(z)$ by choosing:

$$a_1 = -\frac{1}{6}, : a_2 = \frac{1}{3}, : and : A = 7$$

$p[n]$ has a second-order all-pole model.

(ii) Increasing the model order does not buy us anything in this case, because $p[n]$ itself has a second-order all-pole model. Hence:

$$b_1 = a_1 = -\frac{1}{6},\ b_2 = a_2 = \frac{1}{3},\ B = A = 7,\ and\, b_3 = 0$$

(b)

$$Q(z) = \frac{3}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{2}{3}z^{-1}} = \frac{4+\frac{3}{2}z^{-1}}{1+\frac{1}{6}z^{-1}-\frac{1}{3}z^{-2}}$$

$q[n]$ has an ARMA model, not an exact all-pole model. However, $q[n]$ satisfies the difference equation:

$$q[n] = -\frac{1}{6}q[n-1] + \frac{1}{3}q[n-2] + 4\delta[n] + \frac{3}{2}\delta[n-1]$$

Clearly, for $n \geq 2$ the last two terms vanish and $q[n]$ becomes perfectly linearly predictable from $q[n-1]$ and $q[n-2]$ for $c_1 = -\frac{1}{6}$ and $c_2 = \frac{1}{3}$.

Solution from Fall02 PS6

(a) (i)

$$P(z) = \frac{3}{1 - \frac{1}{2}z^{-1}} + \frac{4}{1 + \frac{2}{3}z^{-1}} = \frac{3 + 2z^{-1} + 4 - 2z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{2}{3}z^{-1})}$$

$$P(z) = \frac{7}{1 + \frac{1}{6}z^{-1} - \frac{1}{3}z^{-2}}$$

We can perfectly match $\hat{P}(z)$ with $P(z)$ by choosing:

$$a_1 = -\frac{1}{6},\ a_2 = \frac{1}{3},\ and\ A = 7$$

$p[n]$ has a second-order all-pole model.

(ii) Increasing the model order does not buy us anything in this case, because $p[n]$ itself has a second-order all-pole model. Hence:

$$b_1 = a_1 = -\frac{1}{6},\ b_2 = a_2 = \frac{1}{3},\ B = A = 7,\ and\ b_3 = 0$$

(b)

$$Q(z) = \frac{3}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{2}{3}z^{-1}} = \frac{4 + \frac{3}{2}z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{1}{3}z^{-2}}$$

$q[n]$ has an ARMA model, not an exact all-pole model. However, $q[n]$ satisfies the difference equation:

$$q[n] = -\frac{1}{6}q[n-1] + \frac{1}{3}q[n-2] + 4\delta[n] + \frac{3}{2}\delta[n-1]$$

Clearly, for $n \geq 2$ the last two terms vanish and $q[n]$ becomes perfectly linearly predictable from $q[n-1]$ and $q[n-2]$ for $c_1 = -\frac{1}{6}$ and $c_2 = \frac{1}{3}$.

Solution from Spring02 Final

N/A

11.16. Appears in: Fall05 PS6, Fall04 PS6, Fall02 PS6. Note: There are versions of the problem that use different numbers, only solutions for this version are included here.

Problem

Consider the signal

$$s[n] = 2\left(\frac{1}{3}\right)^n u[n] + 3\left(-\frac{1}{2}\right)^n u[n].$$

We wish to model this signal using a second-order ($p = 2$) all-pole model, or equivalently using second-order linear prediction.

For this problem, since we are given an analytical expression for $s[n]$ and $s[n]$ is the impulse response of an all-pole filter, we can obtain the linear prediction coefficients directly from the Z-transform of $s[n]$ (you're asked to do this in part (a)). In practical situations, we are typically given data, i.e. a set of signal values, and not an analytical expression. In this case, even when the signal to be modeled is the impulse response of an all-pole filter, we need to perform some computation on the data, such as the methods discussed in the lecture and in the notes, in order to determine the linear prediction coefficients.

There are also situations in which an analytical expression is available for the signal but the signal is not the impulse response of an all-pole filter and we would like to model it as such, in which case, we again need to carry out computations such as those discussed in lecture and the notes.

(a) Determine the linear prediction coefficients a_1, a_2 directly from the Z-transform of $s[n]$.

(b) Write the normal equations for $p = 2$ to obtain equations for a_1, a_2 in terms of $\phi_s[m]$.

(c) Find the values of $\phi_s[0]$, $\phi_s[1]$, and $\phi_s[2]$ for the signal $s[n]$ given above.

(d) Solve the system of equations from part (a) using the values you found in part (b) to get values for the a_k's. You may find MATLAB or another computer tool useful to solve these equations.

(e) Are the values of a_k from part (c) what you would expect for this signal? Justify your answer clearly.

(f) Suppose you wish to model the signal now with $p = 3$. Write the normal equations for this case.

(g) Find the value of $\phi_s[3]$.

(h) Solve for the values of a_k when $p = 3$. You may find MATLAB or another computer tool useful to solve these equations.

(i) Are the values of a_k found in part (h) what you would expect given $s[n]$? Justify your answer clearly.

(j) Would the values of a_1, a_2 you found in (h) change if we model the signal with $p = 4$?

Solution from Fall05 PS6

(a)

$$S(z) = \frac{2}{1-\frac{1}{3}z^{-1}} + \frac{3}{1+\frac{1}{2}z^{-1}} = \frac{5}{1+\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}}.$$

Thus, $a_1 = -1/6, a_2 = 1/6$.

(b) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{2} a_k \phi_s[i-k], \quad i = 1, 2,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}.$$

(c) Denote $s_1[n] = 2(\frac{1}{3})^n u[n]$, $s_2[n] = 3(-\frac{1}{2})^n u[n]$. Then for $m > 0$,

$$\phi_{s_1}[m] = \frac{9}{2}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_2}[m] = 12\left(-\frac{1}{2}\right)^m$$

$$\phi_{s_{12}}[m] = \frac{36}{7}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_{21}}[m] = \frac{36}{7}\left(-\frac{1}{2}\right)^m.$$

Thus,

$$\phi_s[m] = \frac{135}{14}\left(\frac{1}{3}\right)^{|m|} + \frac{120}{7}\left(-\frac{1}{2}\right)^{|m|}.$$

So, $\phi_s[0] = 26.78$, $\phi_s[1] = -5.36$ and $\phi_s[2] = 5.36$.

(d) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6$.

(e) These values are the same as those we found in part (a), as expected.

(f) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{3} a_k \phi_s[i-k], \quad i = 1, 2, 3,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] & \phi_s[2] \\ \phi_s[1] & \phi_s[0] & \phi_s[1] \\ \phi_s[2] & \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \\ \phi_s[3] \end{bmatrix}.$$

(g) $\phi_s[3] = -1.79$.

(h) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6, a_3 = 0$.

(i) The signal $s[n]$ is the impulse response of an all-pole filter with two poles, i.e. second order. Therefore, $a_k = 0$ for $k > 2$.

(j) No, since the signal corresponds to the impulse response of a second order filter. The higher order coefficients will all be 0.

Solution from Fall04 PS6

(a)

$$S(z) = \frac{2}{1 - \frac{1}{3}z^{-1}} + \frac{3}{1 + \frac{1}{2}z^{-1}} = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}.$$

Thus, $a_1 = -1/6, a_2 = 1/6$.

(b) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{2} a_k \phi_s[i-k], \quad i = 1, 2,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}.$$

(c) Denote $s_1[n] = 2(\frac{1}{3})^n u[n]$, $s_2[n] = 3(-\frac{1}{2})^n u[n]$. Then for $m > 0$,

$$\phi_{s_1}[m] = \frac{9}{2}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_2}[m] = 12\left(-\frac{1}{2}\right)^m$$

$$\phi_{s_{12}}[m] = \frac{36}{7}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_{21}}[m] = \frac{36}{7}\left(-\frac{1}{2}\right)^m.$$

Thus,

$$\phi_s[m] = \frac{135}{14}\left(\frac{1}{3}\right)^{|m|} + \frac{120}{7}\left(-\frac{1}{2}\right)^{|m|}.$$

So, $\phi_s[0] = 26.78$, $\phi_s[1] = -5.36$ and $\phi_s[2] = 5.36$.

(d) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6$.

(e) These values are the same as those we found in part (a), as expected.

(f) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{3} a_k \phi_s[i-k], \quad i = 1, 2, 3,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] & \phi_s[2] \\ \phi_s[1] & \phi_s[0] & \phi_s[1] \\ \phi_s[2] & \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \\ \phi_s[3] \end{bmatrix}.$$

(g) $\phi_s[3] = -1.79$.

(h) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6, a_3 = 0$.

(i) The signal $s[n]$ is the impulse response of an all-pole filter with two poles, i.e. second order. Therefore, $a_k = 0$ for $k > 2$.

(j) No, since the signal corresponds to the impulse response of a second order filter. The higher order coefficients will all be 0.

Solution from Fall02 PS6

(a)

$$S(z) = \frac{2}{1 - \frac{1}{3}z^{-1}} + \frac{3}{1 + \frac{1}{2}z^{-1}} = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}.$$

Thus, $a_1 = -1/6, a_2 = 1/6$.

(b) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{2} a_k \phi_s[i-k], \quad i = 1, 2,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}.$$

(c) Denote $s_1[n] = 2(\frac{1}{3})^n u[n]$, $s_2[n] = 3(-\frac{1}{2})^n u[n]$. Then for $m > 0$,

$$\phi_{s_1}[m] = \frac{9}{2}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_2}[m] = 12\left(-\frac{1}{2}\right)^m$$

$$\phi_{s_{12}}[m] = \frac{36}{7}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_{21}}[m] = \frac{36}{7}\left(-\frac{1}{2}\right)^m.$$

Thus,

$$\phi_s[m] = \frac{135}{14}\left(\frac{1}{3}\right)^{|m|} + \frac{120}{7}\left(-\frac{1}{2}\right)^{|m|}.$$

So, $\phi_s[0] = 26.78$, $\phi_s[1] = -5.36$ and $\phi_s[2] = 5.36$.

(d) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6$.

(e) These values are the same as those we found in part (a), as expected.

(f) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{3} a_k \phi_s[i-k], \quad i = 1, 2, 3,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] & \phi_s[2] \\ \phi_s[1] & \phi_s[0] & \phi_s[1] \\ \phi_s[2] & \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \\ \phi_s[3] \end{bmatrix}.$$

(g) $\phi_s[3] = -1.79$.

(h) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6, a_3 = 0$.

(i) The signal $s[n]$ is the impulse response of an all-pole filter with two poles, i.e. second order. Therefore, $a_k = 0$ for $k > 2$.

(j) No, since the signal corresponds to the impulse response of a second order filter. The higher order coefficients will all be 0.

11.17. Problem 5 in Spring 2003 final exam Appears in: Fall03 PS9.

Problem

The following information is known for $x[n]$ and $y[n]$, which are wide sense stationary, zero mean signals:

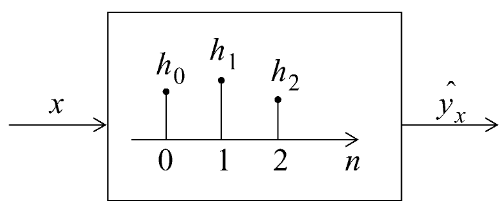
$$\phi_{xx}[m] = \begin{cases} 0 & m \text{ odd} \\ \frac{1}{2^{|m|}} & m \text{ even} \end{cases}$$

$\phi_{yx}[-1] = 2 \qquad \phi_{yx}[0] = 3 \qquad \phi_{yx}[1] = 8 \qquad \phi_{yx}[2] = -3 \qquad \phi_{yx}[3] = 2 \qquad \phi_{yx}[4] = -0.75$

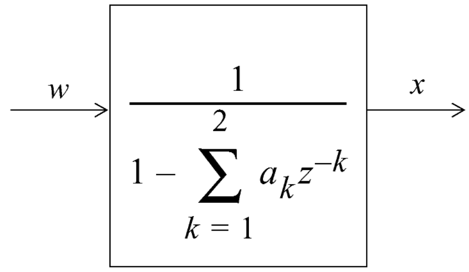
(a) The linear estimate of y given x is $\hat{y}_x$. The estimator of y is designed to minimize

$$\mathcal{J} = E\left(\mid y[n] - \hat{y}_x[n] \mid^2 \right) \tag{1}$$

where the estimate is formed by passing $x[n]$ through $h[n]$, an FIR filter with 3 taps, as shown below.

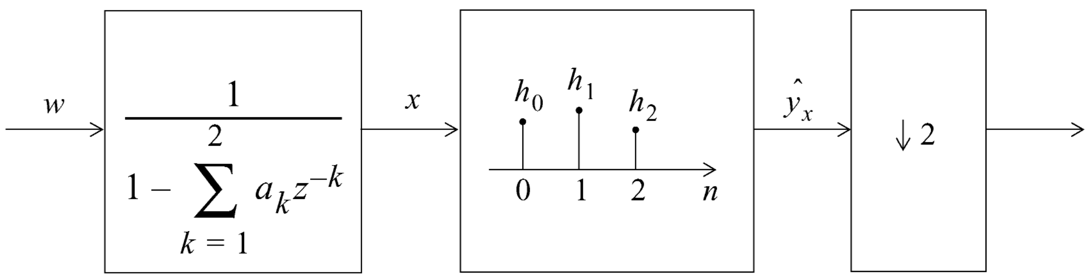


Find $h[n]$.



Find a_1 and a_2.

(d) We want to implement the following system:



where the coefficients a_i are from all-pole modeling in part (c) and the coefficients h_i are the taps of the linear estimator in part (a). Draw an implementation that minimizes the total cost of delays, where the cost of each individual delay is weighted linearly by its clock rate.

(e) Let $\mathcal{J}_a$ be the cost in part (a) and let $\mathcal{J}_b$ be the cost in part (b), where each $\mathcal{J}$ is defined as in equation (**??**). Is $\mathcal{J}_a$ larger than, equal to, or smaller than $\mathcal{J}_b$, or is there not enough information to compare them?

(f) Calculate $\mathcal{J}_a$ and $\mathcal{J}_b$ when $\phi_{yy}[0] = 88$. (Hint: the optimum FIR filters calculated in parts (a) and (b) are such that $E\left[\hat{y}_x[n](y[n] - \hat{y}_x[n])\right] = 0$.)

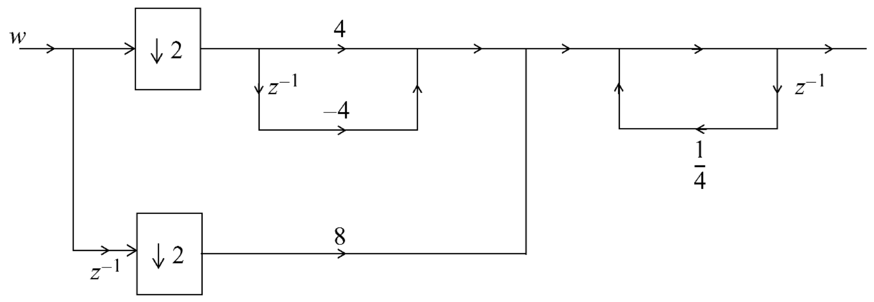
Solution from Fall03 PS9

(a) $h[n] = 4\delta[n] + 8\delta[n-1] - 4\delta[n-2]$

(b) $g[n] = 8\delta[n-1] - 3\delta[n-2]$

(c) $a_1 = 0, \quad a_2 = 1/4$

(d) Below is a realization with a weighted delay cost of 4. This is the same weighted delay cost for the system drawn in direct form II, with the downsampling at the end.



(e) Since $h[n]$ has one more degree of freedom than $g[n]$, $\mathcal{J}_a \leq \mathcal{J}_b$.

(f) $\mathcal{J}_a = 0, \quad \mathcal{J}_b = 15$

Solution from Spring03 Final

N/A

11.18. Problem 6 in Spring2005 final exam.

Problem

A discrete-time communication channel with impulse response $h[n]$ is to be compensated for with a system with impulse response $h_c[n]$ as indicated below:

$$s[n] \longrightarrow \underset{\text{channel}}{\boxed{\; h[n] \;}} \xrightarrow{\;r[n]\;} \underset{\text{compensator}}{\boxed{\; h_c[n] \;}} \longrightarrow \hat{s}[n]$$

The channel $h[n]$ is known to be a one sample delay, *i.e.*,

$$h[n] = \delta[n-1].$$

The compensator $h_c[n]$ is an N-point causal FIR filter, *i.e.*,

$$H_c(z) = \sum_{k=0}^{N-1} a_k z^{-k}.$$

The compensator $h_c[n]$ is designed to invert (or compensate for) the channel. Specifically, $h_c[n]$ is designed so that with $s[n] = \delta[n]$, $\hat{s}[n]$ is as "close" as possible to an impulse. Mathematically, define this as designing $h_c[n]$ so that the error

$$\mathcal{E} = \sum_{n=-\infty}^{\infty} |\hat{s}[n] - \delta[n]|^2$$

is minimized.

Find the optimal compensator of length N, *i.e.*, determine $a_0, a_1, \ldots, a_{N-1}$ to minimize $\mathcal{E}$.

Solution from Spring05 Final

Since $h_c[n]$ is causal and $h[n]$ is a one-sample delay, we know that $\hat{s}[n]$ is zero for $n < 1$. $\mathcal{E}$ then simplifies to

$$\mathcal{E} = 1 + \sum_{n=1}^{\infty} |\hat{s}[n]|^2 = 1 + \sum_{n=1}^{\infty} |h_c[n-1]|^2 .$$

$\mathcal{E}$ is therefore minimized when $h_c[n] = 0$ for all n. (For *any* value of N, the a_ks are all 0.)

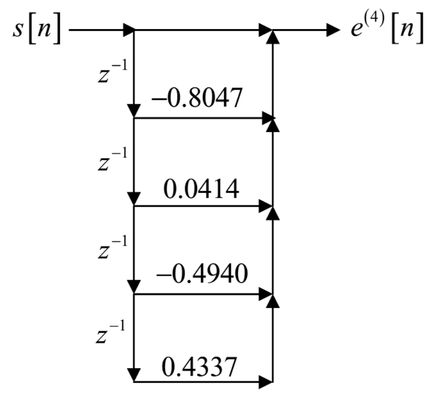
One could also obtain this result by writing out $\mathcal{E}$ in terms of the a_ks and taking partial derivatives with respect to the a_ks.

11.19

A. The coefficients of $A^{(4)}(z)$ can be taken directly from the table provided. We have

$$A^{(4)}(z) = 1 - 0.8047z^{-1} + 0.0414z^{-2} - 0.4940z^{-3} + 0.4337z^{-4}.$$

The direct form implementation is



B. In general, $k_i = a_i^{(i)}$. We have

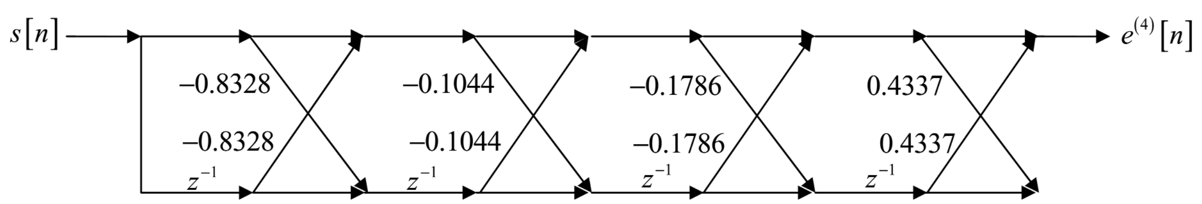
$k_1 = a_1^{(1)} = 0.8328$

$k_2 = a_2^{(2)} = 0.1044$

$k_3 = a_3^{(3)} = 0.1786$

$k_4 = a_4^{(4)} = -0.4337.$

The lattice implementation is



C. We are given that $E^{(2)} = 0.5803$. Since $E^{(i)} = \left(1 - k_i^2\right) E^{(i-1)}$ we have

$$E^{(3)} = \left(1 - (0.1786)^2\right)(0.5803) = 0.5681.$$

Working backwards we have

$$E^{(1)} = \frac{E^{(2)}}{\left(1 - k_2^2\right)} = \frac{0.5803}{\left(1 - (0.1044)^2\right)} = 0.5867$$

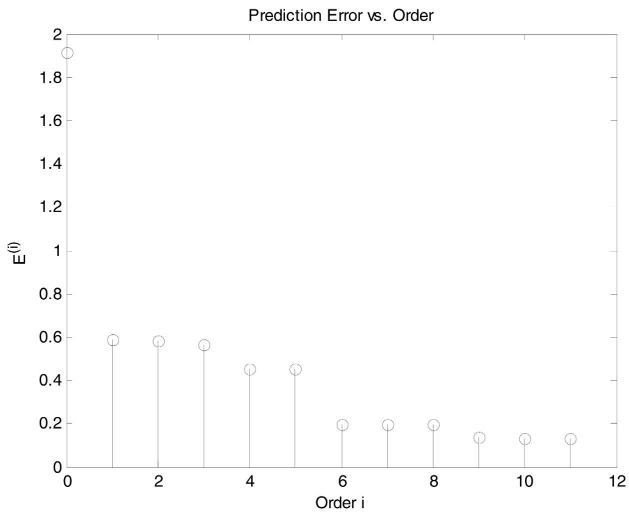
$$E^{(0)} = \frac{E^{(1)}}{\left(1 - k_1^2\right)} = \frac{0.5867}{\left(1 - (0.8328)^2\right)} = 1.915.$$

Since $E^{(0)} = r_{ss}(0)$, the total energy in the signal $s[n]$ is $E^{(0)} = 1.915$.

From the first step of the Levinson-Durbin algorithm we have $k_1 = r_{ss}[1]/r_{ss}[0]$. Then

$$r_{ss}[1] = k_1 r_{ss}[0] = (0.8328)(1.915) = 1.594.$$

D. Minimum mean square prediction error is plotted vs. filter order below.



It can be seen that the error drops abruptly in going from $i = 0$ to $i = 1$ and then makes another sharp decrease in going from $i = 5$ to $i = 6$. This is a consequence of the comparitively large magnitude of the reflection coefficient k_6, i.e. $|k_6| = 0.7505$.

E. The output $e^{(11)}[n]$ of the eleventh-order prediction-error filter should be very nearly white noise.

F. Five of the remaining zeros must be the complex conjugates of the zeros given in the table. The remaining zero must be real-valued. Now

$$\begin{aligned} a_{11}^{(11)} &= -(-1)^{11}\prod_{i=1}^{11} z_i = \prod_{i=1}^{11} z_i \\ &= z_6 (0.2567)^2 (0.9681)^2 (0.9850)^2 (0.8647)^2 (0.9590)^2 \\ &= 0.0371. \end{aligned}$$

Solving gives $z_6 = 0.900$.

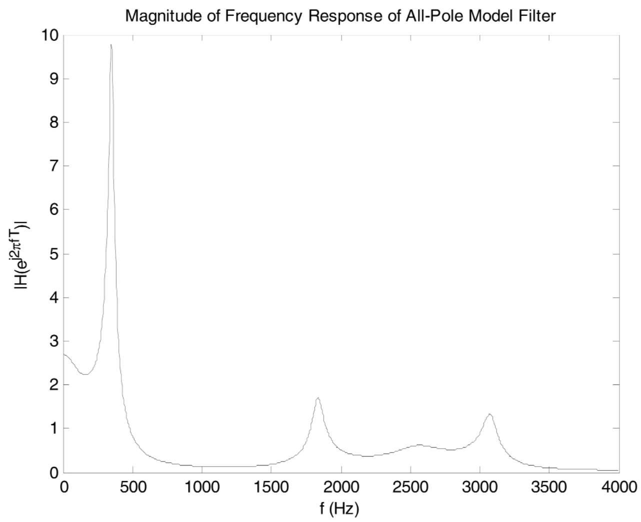
G. For random process modeling, the gain G is chosen so that $\langle \hat{s}^2[n]\rangle = \langle s^2[n]\rangle$. We know from part C that $\langle s^2[n]\rangle = r_{ss}[0] = E^{(0)} = 1.915$. Now if the all-pole model filter is driven by white noise of unit average power, then

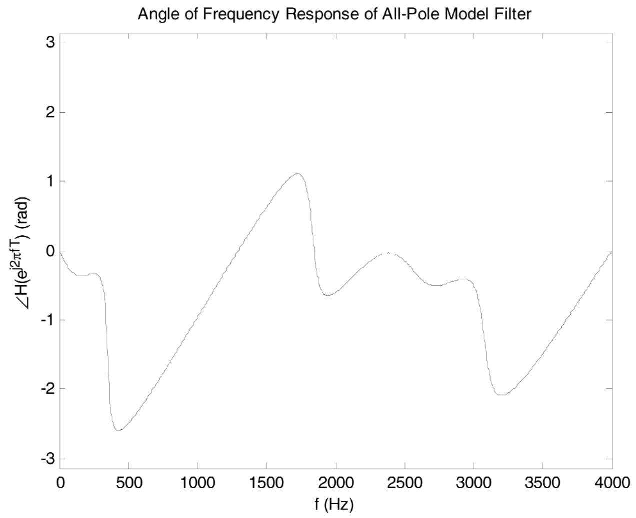
$$\begin{aligned} \langle \hat{s}^2[n]\rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H\left(e^{j\omega}\right)\right|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{G^2}{\left|A^{(11)}\left(e^{j\omega}\right)\right|^2} d\omega \\ &= 1.915. \end{aligned}$$

The frequency response $A^{(11)}\left(e^{j\omega}\right)$ of the prediction-error filter can be written out using the values of $a_1^{(11)},\ldots,a_{11}^{(11)}$ given in the problem statement. The necessary integral can then be evaluated numerically to give $G = 0.363$.

As an alternative and numerically much simpler approach, suppose the input to the all-pole model filter is white noise $v[n]$ of unit variance. The output of this filter is $\hat{s}[n]$. Now suppose $\hat{s}[n]$ is the input to a filter with system function $A^{(11)}(z)/G$. This filter is the exact inverse of the all-pole model filter, so the output of this filter is $v[n]$. This means that if $\hat{s}[n]$ were the input to the prediction-error filter with system function $A^{(11)}(z)$, then the output would be $Gv[n]$, having mean-square value $G^2\langle v^2[n]\rangle = G^2$. Now we do not know the numerical value of the output power of the prediction-error filter when the input is $\hat{s}[n]$, but we do know that when $s[n]$ is the input, the output has mean-square value $E^{(11)} = E^{(2)}\prod_{i=3}^{11}\left(1-k_i^2\right) = 0.1321$. Since $\hat{s}[n] \cong s[n]$, let us take the mean square response to $\hat{s}[n]$ as 0.1321 as well. Then we can write $G^2 = 0.1321$, or $G = 0.363$.

H. The frequency response of the all-pole model filter is plotted below.





Note that the magnitude has a sharp peak at about 350 Hz, corresponding to the sinusoidal component visible in the data segment provided in the problem statement.

11.20

A. Let $s[n] = A\cos(\omega_0 n + \theta)$, where $p_\theta(\phi) = \frac{1}{2\pi}$, $0 \le \phi < 2\pi$.

Then

$$\begin{aligned}
r_{ss}[m] &= E\{s[n]s[n+m]\} \\
&= E\{A\cos(\omega_0 n + \theta)A\cos(\omega_0(n+m)+\theta)\} \\
&= \frac{A^2}{2}E\{\cos(\omega_0 m)\} + \frac{A^2}{2}E\{\cos(\omega_0(2n+m)+2\theta)\} \\
&= \frac{A^2}{2}\cos(\omega_0 m) + \frac{A^2}{2}\int_0^{2\pi}\cos(\omega_0(2n+m)+2\phi)\frac{1}{2\pi}d\phi \\
&= \frac{A^2}{2}\cos(\omega_0 m).
\end{aligned}$$

B. The autocorrelation normal equations are

$$\begin{aligned}
a_1\frac{A^2}{2}\cos(\omega_0 0) + a_2\frac{A^2}{2}\cos(\omega_0 1) &= \frac{A^2}{2}\cos(\omega_0 1) \\
a_1\frac{A^2}{2}\cos(\omega_0 1) + a_2\frac{A^2}{2}\cos(\omega_0 0) &= \frac{A^2}{2}\cos(\omega_0 2),
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
a_1 + a_2\cos(\omega_0) &= \cos(\omega_0) \\
a_1\cos(\omega_0) + a_2 &= \cos(\omega_0 2).
\end{aligned}$$

C. Solving these two equations (and applying a little trigonometry) gives $a_1 = 2\cos(\omega_0)$ ar $a_2 = -1$.

D. We have

$$\begin{aligned}
A(z) &= 1 - 2\cos(\omega_0)z^{-1} + z^{-2} \\
&= (1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1}).
\end{aligned}$$

Note that the zeros lie on the unit circle.

E. The prediction error is

$$\begin{aligned}
E &= r_{ss}[0] - a_1 r_{ss}[1] - a_2 r_{ss}[2] \\
&= \frac{A^2}{2} - 2\cos(\omega_0)\frac{A^2}{2}\cos(\omega_0) + \frac{A^2}{2}\cos(2\omega_0) \\
&= \frac{A^2}{2}(1 - 2\cos^2(\omega_0) + \cos(2\omega_0)) \\
&= 0.
\end{aligned}$$

11.21

The ith-order prediction-error filter has system function given by

$$A^{(i)}(z) = 1 - \sum_{j=1}^{p} a_j^{(i)} z^{-j}.$$

The filter coefficients update according to

$$a_j^{(i)} = a_j^{(i-1)} - k_i a_{i-j}^{(i-1)}, \quad j = 1, 2, \ldots, i-1,$$

$$a_i^{(i)} = k_i.$$

Substituting gives,

$$\begin{aligned} A^{(i)}(z) &= 1 - \sum_{j=1}^{i-1} \left(a_j^{(i-1)} - k_i a_{i-j}^{(i-1)} \right) z^{-j} - k_i z^{-i} \\ &= 1 - \sum_{j=1}^{i-1} a_j^{(i-1)} z^{-j} - k_i \left(\sum_{j=1}^{i-1} a_{i-j}^{(i-1)} z^{-j} + z^{-i} \right) \\ &= A^{(i-1)}(z) - k_i z^{-i} \left(\sum_{j=1}^{i-1} a_{i-j}^{(i-1)} z^{i-j} + 1 \right). \end{aligned}$$

Now let $p = i - j$ in the summation. We have

$$\begin{aligned} A^{(i)}(z) &= A^{(i-1)}(z) - k_i z^{-i} \left(\sum_{p=i-1}^{1} a_p^{(i-1)} z^{p} + 1 \right) \\ &= A^{(i-1)}(z) - k_i z^{-i} \left(\sum_{p=1}^{i-1} a_p^{(i-1)} \left(z^{-1} \right)^{-p} + 1 \right) \\ &= A^{(i-1)}(z) - k_i z^{-i} A^{(i-1)}\left(z^{-1} \right). \end{aligned}$$

11.22. Appears in: Spring05 PS7, Fall04 PS6, Fall02 PS6. Appears in another form(different numbers) in Spring2001 PS9.

The problem has been modified from previous versions in Fall05. The Fall04 is included right after it. **Problem**

This problem considers the construction of lattice filters to implement the inverse filter for the signal

$$s[n] = 2\left(\frac{1}{3}\right)^n u[n] + 3\left(-\frac{1}{2}\right)^n u[n].$$

(a) Find the values of k_1 and k_2 for the second-order $p = 2$ case.

(b) Draw the signal flow graph of a lattice filter implementation of the inverse filter, i.e., the filter which outputs $y[n] = A\delta[n]$ (a scaled impulse) when the input $x[n] = s[n]$.

(c) Verify that the signal flow graph you drew in part (b) has the correct impulse response by showing that the z-transform of this inverse filter is indeed the inverse of $S(z)$.

(d) Draw the signal flow graph for a lattice filter which implements an all-pole system such that when the input is $x[n] = \delta[n]$, the output is the signal $s[n]$ given above.

(e) Derive the system function of the signal flow graph you drew in part (d) and demonstrate that its impulse response $h[n]$ satisfies $h[n] = s[n]$.

Fall04 Version of Problem

This problem considers the construction of lattice filters to implement the inverse filter for the signal

$$s[n] = 2\left(\frac{1}{3}\right)^n u[n] + 3\left(-\frac{1}{2}\right)^n u[n].$$

(a) Find the values of k_1 and k_2 for the second-order $p = 2$ case.

(b) Draw the signal flow graph of a lattice filter implementation of the inverse filter, i.e., the filter which outputs $y[n] = A\delta[n]$ (a scaled impulse) when the input $x[n] = s[n]$.

(c) Verify that the signal flow graph you drew in part (b) has the correct impulse response based on the values of the a_k's found in Problem 6.4.

(d) Draw the signal flow graph for a lattice filter which implements an all-pole system such that when the input is $x[n] = \delta[n]$, the output is the signal $s[n]$ given above.

(e) Derive the system function of the signal flow graph you drew in part (d) and demonstrate that its impulse response $h[n]$ satisfies $h[n] = s[n]$.

Solution from Spring05 PS7

(a) Using the same technique as in Problem 3 part (c), we first find

$$\phi_s[m] = \frac{120}{7}\left(-\frac{1}{2}\right)^{|m|} + \frac{135}{14}\left(\frac{1}{3}\right)^{|m|}.$$

Therefore, $\phi_s[0] \approx 26.79$, $\phi_s[1] \approx -5.357$, and $\phi_s[2] \approx 5.357$.

Substituting $p = 0$ in the Levinson-Durbin recursion we have:

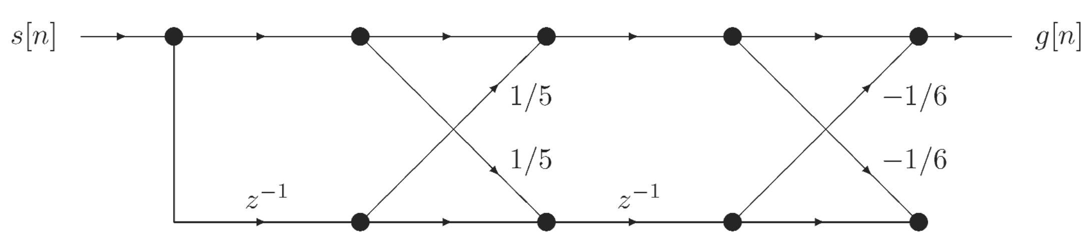
$$k_1 = \frac{\phi_s[1]}{\phi_s[0]} = -\frac{1}{5}.$$

For $p = 1$,

$$a_1 = \frac{\phi_s[1]}{\phi_s[0]} = -\frac{1}{5}$$

$$k_2 = \frac{\phi_s[2] - \phi_s[1]a_1}{\phi_s[0] - \phi_s[1]a_1} = \frac{1}{6}.$$

(b) Signal flow graph for inverse lattice filter:



(c) From the flow graph, $g[n] = s[n] + \frac{1}{6}s[n-1] - \frac{1}{6}s[n-2]$. Therefore,

$$H_{inv}(z) = \frac{G(z)}{S(z)} = 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}.$$

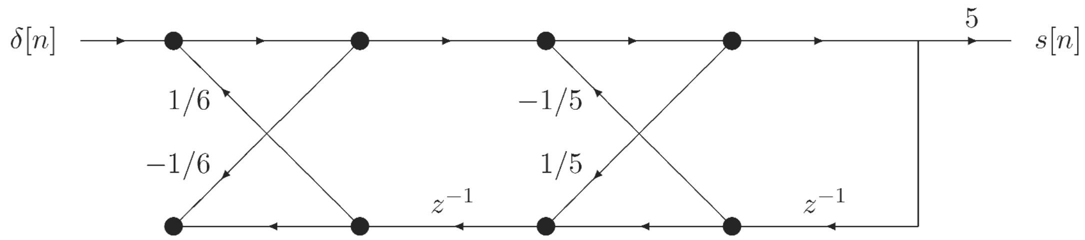
Taking the z-transform of $s[n]$,

$$S(z) = \frac{2}{1 - \frac{1}{3}z^{-1}} + \frac{3}{1 + \frac{1}{2}z^{-1}} = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}},$$

so

$$H_{inv}(z) = AS^{-1}(z).$$

(d) Signal flow graph for forward lattice filter:



(e) The system function is:

$$H(z) = \frac{5}{1 - k_1(1-k_2)z^{-1} - k_2 z^{-2}} = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}.$$

Comparing with $S(z)$ as determined in part (c), we see that $h[n] = s[n]$.

Solution from Fall02 PS6

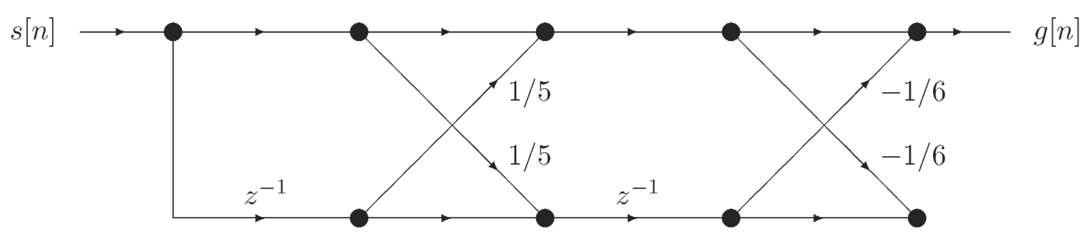
(a) Substituting $p = 0$ in the Levinson-Durbin recursion we have:

$$k_1 = \frac{\phi_s[1]}{\phi_s[0]} = -\frac{1}{5}.$$

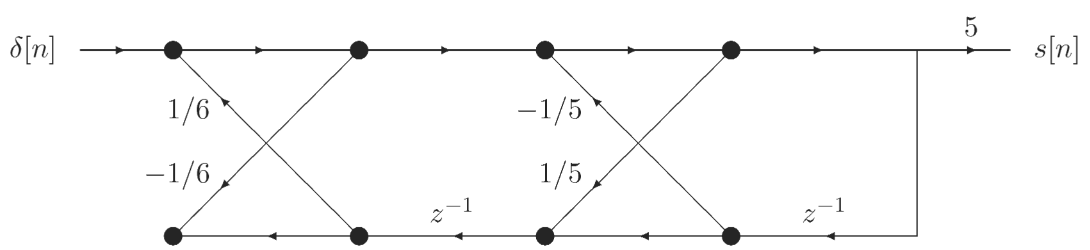
For $p = 1$,

$$a_1 = \frac{\phi_s[1]}{\phi_s[0]} = -\frac{1}{5}$$
$$k_2 = \frac{\phi_s[2] - \phi_s[1]a_1}{\phi_s[0] - \phi_s[1]a_1} = \frac{1}{6}.$$

Figure for Part (b)



(c) From the flow graph, $g[n] = s[n] + \frac{1}{6}s[n-1] - \frac{1}{6}s[n-2]$.
Figure for Part (d):



(e) The system function is:

$$H(z) = \frac{5}{1 - k_1(1-k_2)z^{-1} - k_2 z^{-2}} = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}.$$

Comparing with the answer to part (a) of problem 6.3, we see that $h[n] = s[n]$.

11.23

A.

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}.$$

B.

$$S(z) = \alpha \frac{1}{1-\frac{2}{3}z^{-1}} + \beta \frac{1}{1-\frac{1}{4}z^{-1}} = \frac{\alpha - \frac{\alpha}{4}z^{-1} + \beta - \frac{2\beta}{3}z^{-1}}{\left(1-\frac{2}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

$$= \frac{\alpha + \beta - \frac{\alpha}{4}z^{-1} - \frac{2\beta}{3}z^{-1}}{1-\frac{11}{12}z^{-1} + \frac{1}{6}z^{-2}}.$$

If $a_1 = 11/12$ and $a_2 = -1/6$ then we are modeling a second-order all-pole signal. Therefore the z^{-1} terms in the numerator must cancel. That is, $-\frac{\alpha}{4} = \frac{2\beta}{3}$, or $-3\alpha = 8\beta$.

One possibility is $\alpha = 8, \beta = -3$. The solution is not unique; and pair $c \times 8, c \times (-3)$ with $c \neq 0$ will work.

C. Since $s[n]$ is a second-order all-pole signal, if you were to solve the Levinson recursion for $p = 3$, then $k_3 = a_3^{(3)} = 0$. The k's do not change as the model order increases, therefore $k_3 = 0$ for any p.

11.24

A. Start with $\boldsymbol{\Gamma}_p \mathbf{b}_p = \mathbf{c}_p$. Expanded this is

$$\begin{bmatrix} c[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] \\ c[2] & \phi[1] & \phi[0] & \cdots & \phi[p-2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] \end{bmatrix} \begin{bmatrix} 1 \\ -b_1^p \\ \vdots \\ -b_p^p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Adding an additional row and column gives us

$$\begin{bmatrix} c[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] & \phi[p] \\ c[2] & \phi[1] & \phi[0] & \cdots & \phi[p-2] & \phi[p-1] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] & \phi[1] \\ c[p+1] & \phi[p] & \phi[p-1] & \cdots & \phi[1] & \phi[0] \end{bmatrix} \begin{bmatrix} 1 \\ -b_1^p \\ \vdots \\ -b_p^p \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \delta^{(p)} \end{bmatrix},$$

where the first p rows are the original equations and the last row is valid provided

$$\delta^{(p)} = c[p+1] - \left(\gamma_p^b\right)^T \overline{b}_p.$$

From the proof of the Levinson-Durbin algorithm we have,

$$\begin{bmatrix} \phi[0] & \phi[1] & \phi[2] & \cdots & \phi[p] & \phi[p+1] \\ \phi[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] & \phi[p] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] & \phi[1] \\ \phi[p+1] & \phi[p] & \phi[p-1] & \cdots & \phi[1] & \phi[0] \end{bmatrix} \begin{bmatrix} 0 \\ -a_p^p \\ \vdots \\ -a_1^p \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma^{(p)} \\ 0 \\ \vdots \\ 0 \\ E^{(p)} \end{bmatrix}.$$

Let us disregard the first equation in this system of equations. Then we obtain

$$\begin{bmatrix} \phi[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] & \phi[p] \\ \phi[2] & \phi[1] & \phi[0] & \cdots & \phi[p-2] & \phi[p-1] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] & \phi[1] \\ \phi[p+1] & \phi[p] & \phi[p-1] & \cdots & \phi[1] & \phi[0] \end{bmatrix} \begin{bmatrix} 0 \\ -a_p^p \\ \vdots \\ -a_1^p \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E^{(p)} \end{bmatrix}.$$

Note that entry in the leftmost column of the autocorrelation matrix is multiplied by zero in every equation in this reduced system. The values of the entries in the leftmost column do not matter, then, and they can be replaced by other values. We can write

$$\begin{bmatrix} c[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] & \phi[p] \\ c[2] & \phi[1] & \phi[0] & \cdots & \phi[p-2] & \phi[p-1] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] & \phi[1] \\ c[p+1] & \phi[p] & \phi[p-1] & \cdots & \phi[1] & \phi[0] \end{bmatrix} \begin{bmatrix} 0 \\ -a_p^p \\ \vdots \\ -a_1^p \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E^{(p)} \end{bmatrix}.$$

Let us now construct

$$\begin{bmatrix} c[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] & \phi[p] \\ c[2] & \phi[1] & \phi[0] & \cdots & \phi[p-2] & \phi[p-1] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] & \phi[1] \\ c[p+1] & \phi[p] & \phi[p-1] & \cdots & \phi[1] & \phi[0] \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ -b_1^p \\ \vdots \\ -b_p^p \\ 0 \end{bmatrix} - \hat{k}_{p+1} \begin{bmatrix} 0 \\ -a_p^p \\ \vdots \\ -a_1^p \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \delta^{(p)} \end{bmatrix} - \hat{k}_{p+1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E^{(p)} \end{bmatrix}.$$

To make the right-hand side of the last equation equal to zero, we choose $\hat{k}_{p+1}$ so that

$\delta^{(p)} - \hat{k}_{p+1} E^{(p)} = 0$. That is, $\hat{k}_{p+1} = \dfrac{\delta^{(p)}}{E^{(p)}} = \dfrac{c[p+1] - (\gamma_p^b)^T \bar{b}_p}{E^{(p)}} = \dfrac{c[p+1] - (\gamma_p^b)^T \bar{b}_p}{\phi[0] - (\gamma_p)^T \bar{a}_p}$.

Now let $b_{p+1}^{p+1} = \hat{k}_{p+1}$. Then we have $\mathbf{\Gamma}_{p+1} \mathbf{b}_{p+1} = \mathbf{c}_{p+1}$, where

$$b_m^{p+1} = b_m^p - b_{p+1}^{p+1} a_{p-m+1}^p, \quad m = 1, \ldots, p,$$

QED.

11.25

To solve for the fourth-order system we extend the second-order one using the Levinson recursion. The resulting lattice will have the same k_1 and k_2. However we need to know $\phi_s[m]$ to determine k_3 and k_4. To determine $H_4(z)$, i.e., $a_i^{(4)}$, we need to know all the k_i's. Therefore $k_1 = -2/7$, $k_2 = 1/8$, and the remaining parameters cannot be determined from the information provided.

11.26

A. The prediction-error filter has system function $A(z)=1-\sum_{m=1}^{p}a_m z^{-m}$ and impulse response

$h_A[n]=\delta[n]-\sum_{m=1}^{p}a_m\delta[n-m]$. The function $Q(z)$ is defined as

$$Q(z)=\frac{A(z)A(z^{-1})}{G^2}.$$

Then

$$Q(z)=\frac{\left(1-a_1z^{-1}-\ldots-a_pz^{-p}\right)\left(1-a_1z-\ldots-a_pz^{p}\right)}{G^2}$$
$$=q_pz^p+q_{p-1}z^{p-1}+\ldots+q_1z+q_0+q_1z^{-1}+\ldots+q_{p-1}z^{-(p-1)}+q_pz^{-p}.$$

We make the following observations about the sequence $q[n]$:

i) $q[n]$ can be nonzero only for $-p\le n\le p$. That is, $q[n]$ contains at most $2p+1$ nonzero samples.

ii) $q[n]$ is an even function of n

iii) At $n=\pm p$, $q[\pm p]=q_p=-\frac{a_p}{G^2}$.

B. Given the numerical values

$$C[k]=\left|H\left(e^{j\pi k/p}\right)\right|^2=\left|\frac{G}{1-\sum_{i=1}^{p}a_ie^{-j\pi ki/p}}\right|^2,\quad k=0,1,2,\ldots p,$$

1. Take the reciprocal of the given data to form

$$Q[k]=\frac{1}{G^2}\left|1-\sum_{i=1}^{p}a_ie^{-j\pi ki/p}\right|^2,\quad k=0,1,2,\ldots p.$$

2. Except for the factor of G^2, $Q[k]$ consists of samples of the magnitude-squared of the system function of the prediction-error filter evaluated at equally spaced points around the unit circle from $\omega = 0$ to $\omega = \pi$. Since a magnitude-squared function is an even function of frequency, we can extend $Q[k]$ to $k = 0,1,\ldots,2p-1$. That is

$$Q[k] = Q[2p-k], \quad k = p+1,\ldots 2p-1.$$

3. We now wish to find the sequence $q[n], \quad n = -p,\ldots,p$ that is the inverse Fourier transform of $Q(e^{j\omega})$. Recall from part A that $q[n]$ contains $2p+1$ values. To begin, compute the $2p$-point inverse DFT $\hat{q}[n]$ of $Q[k]$. The sequence $\hat{q}[n], \quad n = 0,1,\ldots,2p-1$ is a time-aliased version of the sequence $q[n]$ that we seek. Fortunately, the period of $\hat{q}[n]$ is $2p$, and the aliasing consists of only one point of overlap. Specifically, the sequence $\hat{q}[n]$ contains

$$\begin{aligned}\hat{q}[n] &= \left[\hat{q}[0],\hat{q}[1],\ldots,\hat{q}[2p-1]\right] \\ &= \left[q[0],q[1],\ldots,q[p-1],2q[p],q[-(p-1)],\ldots,q[-1]\right].\end{aligned}$$

We can make the identification $q[-p] = q[p] = \frac{1}{2}\hat{q}[p]$ and rotate the sequence $\hat{q}[i]$ so that $q[n]$ contains

$$\begin{aligned}q[n] &= \left[q[-p],\ldots,q[-1],q[0],q[1],\ldots,q[p]\right] \\ &= \left[\tfrac{1}{2}\hat{q}[p],\hat{q}[p+1]\ldots,\hat{q}[2p-1],\hat{q}[0],\hat{q}[1],\ldots,\hat{q}[p-1],\tfrac{1}{2}\hat{q}[p]\right].\end{aligned}$$

C.1. The function, $Q(z)$ will have zeros with conjugate-reciprocal symmetry, with half of its zeros inside the unit circle and half outside. Now we know that $H(z)$ is a stable all-pole filter, and therefore its inverse, the prediction-error filter, will have minimum phase. Consequently we can factor $Q(z)$, and group together the p factors corresponding to zeros that are inside the unit circle. The resulting polynomial is $A(z)$.

2. If $A(z)$ is written as a product of terms of the form $(1 - z_i z^{-1})$, we will have

$$A(z) = \left[1 - a_1 z^{-1} - \ldots - a_p z^{-p}\right],$$

giving us the coefficients $a_i, \quad i = 1,\ldots,p$. Now recall that $q[-p] = -a_p / G^2$. Since we now know a_p, we can solve for G.

11.27

A. For the given lattice we have

$$E^{(i)}(z) = A^{(i)}(z)S(z)$$

for the prediction error at stage i. When $i = p$ this is

$$E^{(p)}(z) = A^{(p)}(z)S(z).$$

Similarly, the backward prediction error is given by

$$\begin{aligned}\tilde{E}^{(i)}(z) &= B^{(i)}(z)S(z)\\ &= z^{-i}A^{(i)}(z^{-1})S(z).\end{aligned}$$

The system function $H^{(i)}(z)$ relating $E^{(p)}(z)$ to $\tilde{E}^{(i)}(z)$ is

$$\begin{aligned}H^{(i)}(z) &= \frac{\tilde{E}^{(i)}(z)}{E^{(p)}(z)}\\ &= \frac{z^{-i}A^{(i)}(z^{-1})S(z)}{A^{(p)}(z)S(z)}\\ &= \frac{z^{-i}A^{(i)}(z^{-1})}{A^{(p)}(z)}.\end{aligned}$$

B. We have

$$\begin{aligned}H^{(p)}(z) &= \frac{z^{-p}A^{(p)}(z^{-1})}{A^{(p)}(z)}\\ &= \frac{z^{-p} - a_1^{(p)}z^{-(p-1)} - a_2^{(p)}z^{-(p-2)}\cdots - a_p^{(p)}}{1 - a_1^{(p)}z^{-1} - a_2^{(p)}z^{-2} - \cdots - a_p^{(p)}z^{-p}}\\ &= \frac{1 - a_1^{(p)}z - a_2^{(p)}z^2 - \cdots - a_p^{(p)}z^p}{z^p - a_1^{(p)}z^{p-1} - a_2^{(p)}z^{p-2} - \cdots - a_p^{(p)}}.\end{aligned}$$

Now suppose z_1 is a pole of $H^{(p)}(z)$. (Note that $H^{(p)}(z)$ has no poles at zero or infinity.) Then

$$z_1^p - a_1^{(p)}z_1^{p-1} - a_2^{(p)}z^{p-2} - \cdots - a_p^{(p)} = 0.$$

Multiplying through by z_1^{-p} gives

$$1 - a_1^{(p)}z_1^{-1} - a_2^{(p)}z_1^{-2} - \cdots - a_p^{(p)}z_1^{-p} = 0.$$

This shows that z_1^{-1} is a root of the numerator, and hence is a zero of $H^{(p)}(z)$. A very similar argument shows that if z_2 is a zero of $H^{(p)}(z)$, then z_2^{-1} is a pole. Since the coefficients $a_1^{(p)},\ldots,a_p^{(p)}$ are assumed to be real-valued, the reciprocal pairing of poles and zeros guarantees that $H^{(p)}(z)$ is an all-pass function.

C. From the block diagram we see that

$$\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{Y(z)}{E^{(p)}(z)} \\
&= \frac{\sum_{i=0}^{p} c_i \tilde{E}^{(i)}(z)}{E^{(p)}(z)} \\
&= \sum_{i=0}^{p} c_i \frac{\tilde{E}^{(i)}(z)}{E^{(p)}(z)} \\
&= \sum_{i=0}^{p} c_i \frac{z^{-i}A^{(i)}(z^{-1})}{A^{(p)}(z)} \\
&= \frac{1}{A^{(p)}(z)} \sum_{i=0}^{p} c_i z^{-i} A^{(i)}(z^{-1}) \\
&= \frac{Q(z)}{A^{(p)}(z)},
\end{aligned}$$

where $Q(z) = \sum_{i=0}^{p} c_i z^{-1} A^{(i)}(z^{-1})$. Now $z^{-i}A^{(i)}(z^{-1})$ is a polynomial of degree $-i,\quad i = 0,\ldots,p$. Since $Q(z)$ is a linear combination of such polynomials, $Q(z)$ is a polynomial of degree $-p$. We can write $Q(z) = \sum_{m=0}^{p} q_m z^{-m}$.

Continuing, we have $z^{-0}A^{(0)}(z^{-1}) = 1$ and $z^{-i}A^{(i)}(z^{-1}) = z^{-i} - \sum_{j=1}^{i} a_j^{(i)} z^{-(i-j)},\quad i = 1,\ldots,p$.

Then

$$\begin{aligned} Q(z) = c_0 + c_1\left(z^{-1} - a_1^{(1)}\right) + c_2\left(z^{-2} - a_1^{(2)}z^{-1} - a_2^{(2)}\right) \\ + c_3\left(z^{-3} - a_1^{(3)}z^{-2} - a_2^{(3)}z^{-1} - a_3^{(3)}\right) + \cdots \\ + c_p\left(z^{-p} - a_1^{(p)}z^{-(p-1)} - \cdots - a_p^{(p)}\right). \end{aligned}$$

Collecting terms in like powers of z^{-1} we see that

$$q_p = c_p$$
$$q_{p-1} = c_{p-1} - c_p a_1^{(p)}$$
$$q_{p-2} = c_{p-2} - c_{p-1}a_1^{(p-1)} - c_p a_2^{(p)},$$

etc. That is,

$$q_m = c_m - \sum_{i=m+1}^{p} c_i a_{i-m}^{(i)}, \quad m = p, p-1, \ldots, 0.$$

Rearranging gives

$$c_m = q_m + \sum_{i=m+1}^{p} c_i a_{i-m}^{(i)}, \quad m = p, p-1, \ldots, 0.$$

D. Given a system function $H(z)$, apply the Coefficients-to-k-Parameters algorithm to the coefficients of the denominator to obtain the reflection coefficients of the lattice. Then apply the recursion obtained in the answer to part C above to obtain the coefficients $c_0, \ldots, c_p$.

E. Given

$$H(z) = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-0.9z^{-1}+0.64z^{-2}-0.576z^{-3}}$$

we have $a_1^{(3)} = 0.9, a_2^{(3)} = -0.64, a_3^{(3)} = 0.576$. Then $k_3 = a_3^{(3)} = 0.576$, and

$$a_1^{(2)} = \frac{a_1^{(3)} + k_3 a_2^{(3)}}{1-k_3^2} = \frac{0.9+(0.576)(-0.64)}{1-(0.576)^2} = 0.795$$

$$a_2^{(2)} = \frac{a_2^{(3)} + k_3 a_1^{(3)}}{1-k_3^2} = \frac{-0.64+(0.576)(0.9)}{1-(0.576)^2} = -0.182.$$

Continuing, $k_2 = a_2^{(2)} = -0.182$. Then

$$a_1^{(1)} = \frac{a_1^{(2)} + k_2 a_1^{(1)}}{1-k_2^2} = \frac{0.795+(-0.182)(0.795)}{1-(-0.182)^2} = 0.673$$

and $k_1 = a_1^{(1)} = 0.673$.

We now proceed to find the numerator coefficients as follows: First, $q_0 = 1, q_1 = 3, q_2 = 3, q_3 = 1$. Then

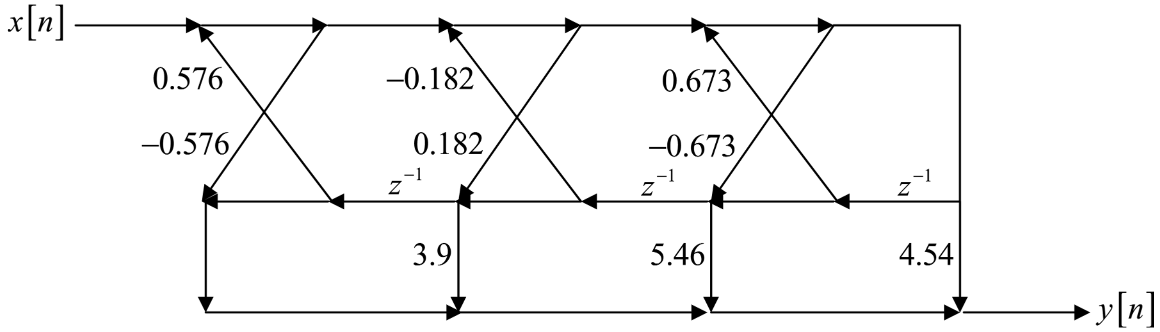
$c_3 = q_3 = 1$

$c_2 = q_2 + c_3 a_1^{(3)} = 3 + (1)(0.9) = 3.9$

$c_1 = q_1 + c_2 a_1^{(2)} + c_3 a_2^{(3)} = 3 + (3.9)(0.795) + (1)(-0.64) = 5.46$

$c_0 = q_0 + c_1 a_1^{(1)} + c_2 a_2^{(2)} + c_3 a_3^{(3)} = 1 + (5.46)(0.673) + (3.9)(-0.182) + (1)(0.576) = 4.54.$

The lattice is as follows:



11.28

Given the relations

$$e^{(i)}[n] = e^{(i-1)}[n] - k_i \tilde{e}^{(i-1)}[n-1]$$
$$\tilde{e}^{(i)}[n] = \tilde{e}^{(i-1)}[n-1] - k_i e^{(i-1)}[n],$$

we can write

$$\begin{aligned}
\mathcal{E}^{(i)} &= \sum_{n=-\infty}^{\infty} \left(e^{(i)}[n]\right)^2 \\
&= \sum_{n=-\infty}^{\infty} \left(e^{(i-1)}[n] - k_i \tilde{e}^{(i-1)}[n-1]\right)^2 \\
&= \sum_{n=-\infty}^{\infty} \left(e^{(i-1)}[n]\right)^2 - 2k_i \sum_{n=-\infty}^{\infty} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1] \\
&\quad + k_i^2 \sum_{n=-\infty}^{\infty} \left(\tilde{e}^{(i-1)}[n-1]\right)^2.
\end{aligned}$$

To maximize over k_i, differentiate and set the derivative equal to zero. That is,

$$\begin{aligned}
\frac{d\left(\mathcal{E}^{(i)}\right)}{dk_i} &= -2\sum_{n=-\infty}^{\infty} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1] + 2k_i \sum_{n=-\infty}^{\infty} \left(\tilde{e}^{(i-1)}[n-1]\right)^2 \\
&= 0.
\end{aligned}$$

Solving gives

$$k_i^f = \frac{\sum_{n=-\infty}^{\infty} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1]}{\sum_{n=-\infty}^{\infty} \left(\tilde{e}^{(i-1)}[n-1]\right)^2}.$$

If, on the other hand, we start with

$$\tilde{\mathcal{E}}^{(i)} = \sum_{n=-\infty}^{\infty} \left(\tilde{e}^{(i)}[n]\right)^2,$$

we can follow an analogous sequence of steps and show that

$$k_i^b = \frac{\sum_{n=-\infty}^{\infty} \tilde{e}^{(i-1)}[n-1]e^{(i-1)}[n]}{\sum_{n=-\infty}^{\infty} \left(e^{(i-1)}[n]\right)^2}.$$

Taking the geometric mean of k_i^f and k_i^b we obtain

$$\begin{aligned}
\sqrt{k_i^f k_i^b} &= \frac{\sum_{n=-\infty}^{\infty} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1]}{\left\{\sum_{n=-\infty}^{\infty} \left(\tilde{e}^{(i-1)}[n-1]\right)^2 \sum_{n=-\infty}^{\infty} \left(e^{(i-1)}[n]\right)^2\right\}^{\frac{1}{2}}} \\
&= k_i^P.
\end{aligned}$$

11.29

We begin with the expression for the PARCOR coefficient,

$$k_i^P = \frac{\sum_{n=-\infty}^{\infty} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1]}{\left\{\sum_{n=-\infty}^{\infty}\left(\tilde{e}^{(i-1)}[n-1]\right)^2 \sum_{n=-\infty}^{\infty}\left(e^{(i-1)}[n]\right)^2\right\}^{\frac{1}{2}}}.$$

Substitute

$$e^{(i)}[n] = s[n] - \sum_{k=1}^{i} a_k^{(i)} s[n-k]$$

and

$$\tilde{e}^{(i)}[n] = s[n-i] - \sum_{j=1}^{i} a_j^{(i)} s[n-i+j]$$

to give

$$k_i^P = \frac{\sum_{n=-\infty}^{\infty}\left(s[n]-\sum_{k=1}^{i-1} a_k^{(i-1)} s[n-k]\right)\left(s[n-i]-\sum_{j=1}^{i-1} a_j^{(i-1)} s[n-i+j]\right)}{\left\{\sum_{n=-\infty}^{\infty}\left(s[n-i]-\sum_{j=1}^{i-1} a_j^{(i-1)} s[n-i+j]\right)^2 \sum_{n=-\infty}^{\infty}\left(s[n]-\sum_{k=1}^{i-1} a_k^{(i-1)} s[n-k]\right)^2\right\}^{\frac{1}{2}}}.$$

To make the expressions easier to read we will expand the numerator and denominator separately. First, the numerator can be expanded to give

$$\begin{aligned}
num = &\sum_{n=-\infty}^{\infty}\Bigg(s[n]s[n-i] - s[n-i]\sum_{k=1}^{i-1} a_k^{(i-1)} s[n-k] \\
&-s[n]\sum_{j=1}^{i-1} a_j^{(i-1)} s[n-i+j] + \sum_{k=1}^{i-1} a_k^{(i-1)} s[n-k]\sum_{j=1}^{i-1} a_j^{(i-1)} s[n-i+j]\Bigg) \\
= &\sum_{n=-\infty}^{\infty} s[n]s[n-i] - \sum_{k=1}^{i-1} a_k^{(i-1)} \sum_{n=-\infty}^{\infty} s[n-i]s[n-k] - \sum_{j=1}^{i-1} a_j^{(i-1)} \sum_{n=-\infty}^{\infty} s[n]s[n-i+j] \\
&+\sum_{k=1}^{i-1}\sum_{j=1}^{i-1} a_k^{(i-1)} a_j^{(i-1)} \sum_{n=-\infty}^{\infty} s[n-k]s[n-i+j].
\end{aligned}$$

Now we will use the fact that $r_{ss}[m] = r_{ss}[-m] = \sum^{\infty} s[n]s[n+m]$. Substituting gives

$$num = r_{ss}[i] - \sum_{k=1}^{i-1} a_k^{(i-1)} r_{ss}[i-k] - \sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[i-j]$$
$$+ \sum_{k=1}^{i-1}\sum_{j=1}^{i-1} a_k^{(i-1)} a_j^{(i-1)} r_{ss}[j-i+k].$$

The double sum in the second line can be reduced if we recall the autocorrelation normal equations, which specify

$$\sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[m-j] = r_{ss}[m], \quad m = 1, \ldots, i-1.$$

Then we have

$$\sum_{k=1}^{i-1}\sum_{j=1}^{i-1} a_k^{(i-1)} a_j^{(i-1)} r_{ss}[j-i+k] = \sum_{k=1}^{i-1} a_k^{(i-1)} \sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[j-i+k]$$
$$= \sum_{k=1}^{i-1} a_k^{(i-1)} \sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[-j+i-k]$$
$$= \sum_{k=1}^{i-1} a_k^{(i-1)} r_{ss}[i-k].$$

The numerator is therefore

$$num = r_{ss}[i] - \sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[i-j].$$

Each term in the denominator can be expanded using a sequence of steps analogous to those used to expand the numerator. We obtain

$$\sum_{n=-\infty}^{\infty}\left(s[n-i]-\sum_{j=1}^{i-1}a_j^{(i-1)}s[n-i+j]\right)^2 = r_{ss}[0]-\sum_{j=1}^{i-1}a_j^{(i-1)}r_{ss}[j]$$
$$= \mathcal{E}^{(i-1)}$$

and

$$\sum_{n=-\infty}^{\infty}\left(s[n]-\sum_{k=1}^{i-1}a_k^{(i-1)}s[n-k]\right)^2 = r_{ss}[0]-\sum_{k=1}^{i-1}a_k^{(i-1)}r_{ss}[k]$$
$$= \mathcal{E}^{(i-1)},$$

which we can substitute into the denominator to obtain

$$den = \left\{\mathcal{E}^{(i-1)}\mathcal{E}^{(i-1)}\right\}^{\frac{1}{2}} = \mathcal{E}^{(i-1)}.$$

The PARCOR coefficient can now be written as

$$k_i^P = \frac{r_{ss}[i]-\sum_{j=1}^{i-1}a_j^{(i-1)}r_{ss}[i-j]}{\mathcal{E}^{(i-1)}} = k_i,$$

where the last equality recognizes that the expression obtained for k_i^P is precisely the expression for the reflection coefficient k_i obtained in the Levinson-Durbin algorithm.

11.30

Given

$$\mathcal{E}^{(i)} = \sum_{n=i}^{M}\left[\left(e^{(i)}[n]\right)^2 + \left(\tilde{e}^{(i)}[n]\right)^2\right].$$

A. Substitute $e^{(i)}[n] = e^{(i-1)}[n] - k_i\tilde{e}^{(i-1)}[n-1]$ and $\tilde{e}^{(i)}[n] = \tilde{e}^{(i-1)}[n-1] - k_i e^{(i-1)}[n]$. This gives

$$\mathcal{E}^{(i)} = \sum_{n=i}^{M}\left[\left(e^{(i-1)}[n] - k_i\tilde{e}^{(i-1)}[n-1]\right)^2 + \left(\tilde{e}^{(i-1)}[n-1] - k_i e^{(i-1)}[n]\right)^2\right].$$

Now set the derivative with respect to k_i equal to zero. We obtain

$$\begin{aligned}\frac{d\left(\mathcal{E}^{(i)}\right)}{dk_i} = \sum_{n=i}^{M}\Big[&-2\left(e^{(i-1)}[n] - k_i\tilde{e}^{(i-1)}[n-1]\right)\tilde{e}^{(i-1)}[n-1] \\ &-2\left(\tilde{e}^{(i-1)}[n-1] - k_i e^{(i-1)}[n]\right)e^{(i-1)}[n]\Big] \\ = 0.&\end{aligned}$$

That is,

$$\begin{aligned}&\sum_{i=1}^{M} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1] - k_i\sum_{i=1}^{M}\left(\tilde{e}^{(i-1)}[n-1]\right)^2 \\ &+\sum_{i=1}^{M}\tilde{e}^{(i-1)}[n-1]e^{(i-1)}[n] - k_i\sum_{i=1}^{M}\left(e^{(i-1)}[n]\right)^2 = 0.\end{aligned}$$

Solving for k_i gives the result,

$$k_i^B = \frac{2\sum_{n=i}^{M} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1]}{\left\{\sum_{n=i}^{M}\left(e^{(i-1)}[n]\right)^2 + \sum_{n=i}^{M}\left(\tilde{e}^{(i-1)}[n-1]\right)^2\right\}}.$$

B. Start with the inequality

$$0 \le \sum_{n=i}^{M} \left(e^{(i-1)}[n] \pm \tilde{e}^{(i-1)}[n-1] \right)^2 .$$

This inequality is true because the sum of squares cannot be negative. Now expand giving

$$\begin{aligned} 0 &\le \sum_{n=i}^{M} \left[\left(e^{(i-1)}[n] \right)^2 \pm 2e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1] + \left(\tilde{e}^{(i-1)}[n-1] \right)^2 \right] \\ &= \sum_{n=i}^{M} \left(e^{(i-1)}[n] \right)^2 \pm 2\sum_{n=i}^{M} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1] + \sum_{n=i}^{M} \left(\tilde{e}^{(i-1)}[n-1] \right)^2 . \end{aligned}$$

If we divide through by $\sum_{n=i}^{M} \left(e^{(i-1)}[n] \right)^2 + \sum_{n=i}^{M} \left(\tilde{e}^{(i-1)}[n-1] \right)^2$ we obtain

$$0 \le 1 \pm k_i^B .$$

That is, $-1 \le k_i^B \le 1$, as was to have been shown.

C. To find $A^{(p)}(z)$, follow the steps of the Levinson-Durbin algorithm. That is, given

$k_i^B, \quad i = 1, \ldots, p,$

for $i = 1, 2, \ldots, p$

  $a_i^{(i)} = k_i^B$

  if $i > 1$ then for $j = 1, 2, \ldots, i-1$

    $a_j^{(i)} = a_j^{(i-1)} - k_i^B a_{i-j}^{(i-1)}$

  end

end

$a_j = a_j^{(p)}, \quad j = 1, 2, \ldots, p.$

Finally, $A^{(p)}(z) = 1 - \sum_{j=1}^{p} a_j z^{-j}$.

12.1. Using the fact that $x_e[n]$ is the inverse transform of $\mathcal{R}e$ we get

$$\begin{aligned}\mathcal{R}e\{X(e^{j\omega})\} &= 2 - ae^{j\omega} - ae^{-j\omega} \\ x_e[n] &= 2\delta[n] - a\delta[n+1] - a\delta[n-1]\end{aligned}$$

Since $x[n]$ is causal, we can recover it from $x_e[n]$

$$x[n] = 2x_e[n]u[n] - x_e[0]\delta[n] = 2\delta[n] - 2a\delta[n-1]$$

This implies that

$$x_o[n] = \frac{x[n] - x[-n]}{2} = a\delta[n+1] - a\delta[n-1]$$

and since $j\mathcal{I}m\{X(e^{j\omega})\}$ is the transform of $x_o[n]$ we find

$$\mathcal{I}m\{X(e^{j\omega})\} = 2a\sin\omega$$

12.2. Taking the inverse transform of $\mathcal{R}e\{X(e^{j\omega})\} = 5/4 - \cos\omega$, we get

$$x_e[n] = \frac{5}{4}\delta[n] - \frac{1}{2}\delta[n+1] - \frac{1}{2}\delta[n-1]$$

Since $x[n]$ is causal, we can recover it from $x_e[n]$

$$x[n] = 2x_e[n]u[n] - x_e[0]\delta[n] = \frac{5}{4}\delta[n] - \delta[n-1],$$

12.3. Note that

$$\begin{aligned} |X(e^{j\omega})|^2 &= \frac{5}{4} - \cos\omega \\ &= \left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{2}e^{j\omega}\right) \\ &= X(e^{j\omega})X^*(e^{j\omega}) \end{aligned}$$

If $X(e^{j\omega}) = (1 - \frac{1}{2}e^{-j\omega})$ we get

$$x[n] = \delta[n] - \frac{1}{2}\delta[n-1]$$

but this does not satisfy the conditions on $x[n]$ given in the problem statement.
However, if we let $X(e^{j\omega}) = (1 - \frac{1}{2}e^{-j\omega})e^{-j\omega}$ we get

$$x[n] = \delta[n-1] - \frac{1}{2}\delta[n-2]$$

which satisfies all the constraints. The idea behind this choice is that cascading a signal with an allpass system does not change the magnitude squared response.
Another choice that works is $X(e^{j\omega}) = \frac{1}{2}(1 - 2e^{-j\omega})e^{-j\omega}$ for which we get

$$x[n] = \frac{1}{2}\delta[n-1] - \delta[n-2]$$

The idea behind this choice was to flip the zero to its reciprocal location outside the unit circle. This has the same magnitude squared response up to a scaling factor; hence, the $\frac{1}{2}$ term.

12.4. Take the DTFT of $x_r[n]$ to get

$$x_r[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n+2] - \frac{1}{4}\delta[n-2]$$

$$X_r(e^{j\omega}) = \frac{1}{2} - \frac{1}{2}\cos 2\omega.$$

where $X_r(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega}) + X^*(e^{j\omega})]$ is the *conjugate symmetric* part of $X(e^{j\omega})$. Since $X(e^{j\omega}) = 0$ for $-\pi \leq \omega < 0$ we have

$$\begin{aligned} X(e^{j\omega}) &= \begin{cases} 2X_r(e^{j\omega}), & 0 \leq \omega < \pi \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 - \cos 2\omega, & 0 \leq \omega < \pi \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

Thus,

$$\mathcal{Re}\{X(e^{j\omega})\} = \begin{cases} 1 - \cos 2\omega, & 0 \leq \omega < \pi \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathcal{Im}\{X(e^{j\omega})\} = 0.$$

About Notation: $X_R(e^{j\omega})$ with a capital R is the real part of $X(e^{j\omega})$. $X_r(e^{j\omega})$ with a small r is the conjugate symmetric part of $X(e^{j\omega})$ which is complex-valued in general.

12.5. The Hilbert transform can be viewed as a filter with frequency response

$$H(e^{j\omega}) = \begin{cases} -j, & 0 < \omega < \pi, \\ j, & -\pi < \omega < 0. \end{cases}$$

(a) First, take the transform of $x_r[n]$

$$X_r(e^{j\omega}) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0).$$

Now, filter with $H(e^{j\omega})$ and take the inverse transform to get $x_i[n]$

$$\begin{aligned} X_i(e^{j\omega}) &= H(e^{j\omega})X_r(e^{j\omega}) \\ &= -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0) \\ x_i[n] &= \sin\omega_0 n \end{aligned}$$

(b) Similarly, $x_i[n] = -\cos\omega_0 n$.

(c) $x_r[n]$ is the ideal low pass filter

$$x_r[n] = \frac{\sin(\omega_c n)}{\pi n} \longleftrightarrow \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

After filtering with the Hilbert tranformer we get

$$X_i(e^{j\omega}) = \begin{cases} -j, & 0 \le \omega \le \omega_c \\ j, & -\omega_c \le \omega \le 0 \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

Taking the inverse transform yields

$$x_i[n] = \frac{1}{2\pi}\int_{-\omega_c}^{0} je^{j\omega n}\,d\omega - \frac{1}{2\pi}\int_{0}^{\omega_c} je^{j\omega n}\,d\omega = \frac{1 - \cos\omega_c n}{\pi n}$$

12.6. Using Euler's identity,

$$\begin{aligned} jX_I(e^{j\omega}) &= j(2\sin\omega - 3\sin 4\omega) \\ &= 2\left(\frac{e^{j\omega} - e^{-j\omega}}{2}\right) - 3\left(\frac{e^{j4\omega} - e^{-j4\omega}}{2}\right) \\ &= -\frac{3}{2}e^{j4\omega} + e^{j\omega} - e^{-j\omega} + \frac{3}{2}e^{-j4\omega} \end{aligned}$$

Since $x_o[n]$ is the inverse transform of $jX_I(e^{j\omega})$ we get

$$x_o[n] = -\frac{3}{2}\delta[n+4] + \delta[n+1] - \delta[n-1] + \frac{3}{2}\delta[n-4]$$

Because $x[n]$ is real and causal we can recover most of x[n], i.e.,

$$\begin{aligned} x[n] &= 2x_o[n]u[n] + x[0]\delta[n] \\ &= x[0]\delta[n] - 2\delta[n-1] + 3\delta[n-4] \end{aligned}$$

The extra information given to us allows us to find $x[0]$,

$$\begin{aligned} 6 &= X(e^{j\omega})\big|_{\omega=0} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega(0)} \\ &= x[0] - 2 + 3 \end{aligned}$$

Plugging this into our equation for $x[n]$ we find

$$x[n] = 5\delta[n] - 2\delta[n-1] + 3\delta[n-4]$$

12.7. (a) Given the imaginary part of $X(e^{j\omega})$, we can take the inverse DTFT to find the odd part of $x[n]$, denoted $x_o[n]$.

$$\begin{aligned}
\mathcal{Im}\left\{X(e^{j\omega})\right\} &= \sin\omega + 2\sin 2\omega \\
&= \frac{1}{2j}e^{j\omega} - \frac{1}{2j}e^{-j\omega} + \frac{1}{j}e^{j2\omega} - \frac{1}{j}e^{-j2\omega} \\
&= \frac{1}{j}e^{j2\omega} + \frac{1}{2j}e^{j\omega} - \frac{1}{2j}e^{-j\omega} - \frac{1}{j}e^{-j2\omega}
\end{aligned}$$

$$\begin{aligned}
x_o[n] &= \mathcal{DFT}^{-1}\left[j\mathcal{Im}\left\{X(e^{j\omega})\right\}\right] \\
&= \mathcal{DFT}^{-1}\left[e^{j2\omega} + \frac{1}{2}e^{j\omega} - \frac{1}{2}e^{-j\omega} - e^{-j2\omega}\right] \\
&= \delta[n+2] + \frac{1}{2}\delta[n+1] - \frac{1}{2}\delta[n-1] - \delta[n-2]
\end{aligned}$$

Using the formula $x[n] = 2x_o[n]u[n] + x[0]\delta[n]$, we find

$$x[n] = -\delta[n-1] - 2\delta[n-2] + x[0]\delta[n]$$

Any $x[0]$ will result in a correct solution to this problem. Setting $x[0] = 0$ gives the result

$$x[n] = -\delta[n-1] - 2\delta[n-2]$$

(b) No, the answer to part (a) is not unique, since any choice for $x[0]$ will result in a correct solution.

12.8. Using Euler's identity and the fact that $x_o[n]$ is the inverse transform of $jX_I(e^{j\omega})$ we find

$$\begin{aligned} jX_I(e^{j\omega}) &= 3j\sin 2\omega \\ &= 3\left(\frac{e^{j2\omega} - e^{-j2\omega}}{2}\right) \\ x_o[n] &= \frac{3}{2}(\delta[n+2] - \delta[n-2]) \end{aligned}$$

Because $x[n]$ is real and causal we can recover all of $x[n]$ except at $n = 0$,

$$\begin{aligned} x[n] &= 2x_o[n]u[n] + x[0]\delta[n] \\ &= -3\delta[n-2] + x[0]\delta[n] \end{aligned}$$

Therefore,

$$\begin{aligned} x_e[n] &= \frac{x[n] + x[-n]}{2} \\ &= \frac{(-3\delta[n-2] + x[0]\delta[n]) + (-3\delta[n+2] + x[0]\delta[n])}{2} \\ &= -\frac{3}{2}\delta[n+2] + x[0]\delta[n] - \frac{3}{2}\delta[n-2] \end{aligned}$$

Using the fact that $X_R(e^{j\omega})$ is the transform of $x_e[n]$ we find

$$\begin{aligned} X_R(e^{j\omega}) &= -\frac{3}{2}e^{j2\omega} + x[0] - \frac{3}{2}e^{-j2\omega} \\ &= x[0] - 3\cos 2\omega \end{aligned}$$

Thus, $X_{R2}(e^{j\omega})$ and $X_{R3}(e^{j\omega})$ are possible if $x[0] = -1$ and $x[0] = 0$ respectively.

12.9. (a) Given the imaginary part of $X(e^{j\omega})$, we can take the inverse DTFT to find the odd part of $x[n]$, denoted $x_o[n]$.

$$\begin{aligned}\mathcal{Im}\left\{X(e^{j\omega})\right\} &= 3\sin\omega + \sin 3\omega \\ &= \frac{3}{2j}e^{j\omega} - \frac{3}{2j}e^{-j\omega} + \frac{1}{2j}e^{j3\omega} - \frac{1}{2j}e^{-j3\omega} \\ &= \frac{1}{2j}e^{j3\omega} + \frac{3}{2j}e^{j\omega} - \frac{3}{2j}e^{-j\omega} - \frac{1}{2j}e^{-j3\omega}\end{aligned}$$

$$\begin{aligned}x_o[n] &= \mathcal{DFT}^{-1}\left[j\mathcal{Im}\left\{X(e^{j\omega})\right\}\right] \\ &= \mathcal{DFT}^{-1}\left[\frac{1}{2}e^{j3\omega} + \frac{3}{2}e^{j\omega} - \frac{3}{2}e^{-j\omega} - \frac{1}{2}e^{-j3\omega}\right] \\ &= \frac{1}{2}\delta[n+3] + \frac{3}{2}\delta[n+1] - \frac{3}{2}\delta[n-1] - \frac{1}{2}\delta[n-3]\end{aligned}$$

Using the formula $x[n] = 2x_o[n]u[n] + x[0]\delta[n]$, we find

$$x[n] = -3\delta[n-1] - \delta[n-3] + x[0]\delta[n]$$

Taking the DTFT of $x[n]$ gives

$$X(e^{j\omega}) = -3e^{-j\omega} - e^{-j3\omega} + x[0]$$

Evaluating this at $\omega = \pi$ gives

$$X(e^{j\omega})\Big|_{\omega=\pi} = -3e^{-j\pi} - e^{-j3\pi} + x[0] = 3$$

$$\begin{aligned}3 + 1 + x[0] &= 3 \\ x[0] &= -1\end{aligned}$$

Therefore,

$$x[n] = -3\delta[n-1] - \delta[n-3] - \delta[n]$$

(b) Yes, the answer to part (a) is unique. The specification of $X(e^{j\omega})$ at $\omega = \pi$ allowed us to find a unique $x[n]$.

12.10. Factoring the magnitude squared response we get

$$|H(e^{j\omega})|^2 = \frac{\frac{5}{4} - \cos\omega}{5 + 4\cos\omega} = \frac{1 - \cos\omega + \frac{1}{4}}{1 + 4\cos\omega + 4} = \frac{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{2}e^{j\omega})}{(1 + 2e^{-j\omega})(1 + 2e^{j\omega})}$$

$$\begin{aligned} |H(z)|^2 &= \frac{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{2}z)}{(1 + 2z^{-1})(1 + 2z)} \\ &= H(z)H^*(1/z^*) \end{aligned}$$

Since $h[n]$ is stable and causal and has a stable and causal inverse, it must be a minimum phase system. It therefore has all its poles and zeros inside the unit circle which allows us to uniquely identify $H(z)$ from $|H(z)|^2$.

$$\begin{aligned} H(z) &= \frac{1 - \frac{1}{2}z^{-1}}{1 + 2z} \\ &= \frac{1}{2}z^{-1}\left(\frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{2}z^{-1}}\right) \\ &= \frac{1}{2}z^{-1}\left(-1 + \frac{2}{1 + \frac{1}{2}z^{-1}}\right), \qquad |z| > \frac{1}{2} \end{aligned}$$

$$h[n] = -\frac{1}{2}\delta[n-1] + \left(-\frac{1}{2}\right)^{(n-1)} u[n-1]$$

12.11. Note that $x_i[n]$ can be written as

$$x_i[n] = -4\delta[n+3] + 4\delta[n-3]$$

Taking the DTFT of $x_i[n]$ gives

$$\begin{aligned} X_i(e^{jw}) &= -4e^{j3\omega} + 4e^{-j3\omega} \\ &= -4(2j\sin 3\omega) \\ &= -8j\sin 3\omega \end{aligned}$$

Since $X(e^{jw}) = 0$ for $-\pi \le \omega < 0$, we can find $X(e^{jw})$ using the relation

$$X(e^{jw}) = \begin{cases} 2jX_i(e^{jw}), & 0 < \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}$$

Thus,

$$X(e^{jw}) = \begin{cases} 16\sin 3\omega, & 0 < \omega < \pi \\ 0, & -\pi \le \omega < 0 \end{cases}$$

Therefore, the real part of $X(e^{jw})$ is

$$\begin{aligned} X_r(e^{j\omega}) &= \frac{1}{2}\left[X(e^{j\omega}) + X^*(e^{-j\omega})\right] \\ &= \begin{cases} 8\sin 3\omega, & 0 < \omega < \pi \\ -8\sin 3\omega, & -\pi \le \omega < 0 \end{cases} \end{aligned}$$

12.12. (a) Factoring the magnitude squared response we get

$$\begin{aligned} |H(e^{j\omega})|^2 &= \frac{10}{9} - \frac{2}{3}\cos\omega = 1 - \frac{2}{3}\cos\omega + \frac{1}{9} = \left(1 - \frac{1}{3}e^{-j\omega}\right)\left(1 - \frac{1}{3}e^{j\omega}\right) \\ &= H(e^{j\omega})H^*(e^{j\omega}) \end{aligned}$$

Thus, one choice for $H(e^{j\omega})$ and $h[n]$ is

$$H(e^{j\omega}) = 1 - \frac{1}{3}e^{-j\omega}$$

$$h[n] = \delta[n] - \frac{1}{3}\delta[n-1]$$

(b) *No.* We can find a new system by taking the zero from the original system and flipping it to its reciprocal location. This only changes the magnitude squared response by a scaling factor. If we compensate for the scaling factor the two magnitude squared responses will be the same. Thus, we find

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{3}(1 - 3e^{-j\omega}) \\ h[n] &= \frac{1}{3}\delta[n] - 3\delta[n-1] \end{aligned}$$

satisifies the given conditions.

12.13. Expressing $X_R(e^{j\omega})$ in terms of complex exponentials gives

$$\begin{aligned} X_R(e^{j\omega}) &= 1+\cos\omega+\sin\omega-\sin 2\omega \\ &= 1+\frac{1}{2}e^{j\omega}+\frac{1}{2}e^{-j\omega}+\frac{1}{2j}e^{j\omega}-\frac{1}{2j}e^{-j\omega}-\frac{1}{2j}e^{j2\omega}+\frac{1}{2j}e^{-j2\omega} \\ &= -\frac{1}{2j}e^{j2\omega}+\frac{1}{2}e^{j\omega}+\frac{1}{2j}e^{j\omega}+1+\frac{1}{2}e^{-j\omega}-\frac{1}{2j}e^{-j\omega}+\frac{1}{2j}e^{-j2\omega} \end{aligned}$$

Taking the inverse DTFT of $X_R(e^{j\omega})$ gives the conjugate-symmetric part of $x[n]$, denoted as $x_e[n]$.

$$x_e[n] = -\frac{1}{2j}\delta[n+2]+\frac{1}{2}\delta[n+1]+\frac{1}{2j}\delta[n+1]+\delta[n]+\frac{1}{2}\delta[n-1]-\frac{1}{2j}\delta[n-1]+\frac{1}{2j}\delta[n-2]$$

Using the relation $x[n] = 2x_e[n]u[n] - x_e[0]\delta[n]$,

$$x[n] = \delta[n]+\delta[n-1]+j\delta[n-1]-j\delta[n-2]$$

We then find the conjugate-antisymmetric part, $x_o[n]$ as

$$\begin{aligned} x_o[n] &= \frac{1}{2}\left(x[n]-x^*[-n]\right) \\ &= \frac{1}{2}\left(\delta[n]+\delta[n-1]+j\delta[n-1]-j\delta[n-2]-\delta[n]-\delta[-n-1]+j\delta[-n-1]-j\delta[-n-2]\right) \\ &= \frac{1}{2}\left(\delta[n-1]+j\delta[n-1]-j\delta[n-2]-\delta[n+1]+j\delta[n+1]-j\delta[n+2]\right) \\ &= -\frac{1}{2}\left(\delta[n+1]-\delta[n-1]\right)+\frac{j}{2}\left(\delta[n+1]+\delta[n-1]\right)-\frac{j}{2}\left(\delta[n+2]+\delta[n-2]\right) \end{aligned}$$

Taking the DTFT of $x_o[n]$ gives $jX_I(e^{jw})$.

$$\begin{aligned} jX_I(e^{jw}) &= -\frac{1}{2}\left(e^{j\omega}-e^{-j\omega}\right)+\frac{j}{2}\left(e^{j\omega}+e^{-j\omega}\right)-\frac{j}{2}\left(e^{j2\omega}+e^{-j2\omega}\right) \\ &= -j\sin\omega+j\cos\omega-j\cos 2\omega \end{aligned}$$

So

$$X_I(e^{jw}) = -\sin\omega+\cos\omega-\cos 2\omega$$

12.14. First note that,

(a) The inverse transform of $X_R(e^{j\omega})$ is $x_e[n]$, the even part of $x[n]$. This is true for any sequence whether it is causal, anticausal, or neither.

(b) $jX_I(e^{j\omega})$ is the transform of $x_o[n]$, the odd part of $x[n]$. This is true for any sequence whether it is causal, anticausal, or neither.

(c) For an anticausal sequence

$$x[n] = 2x_e[n]u[-n] - x_e[0]\delta[n]$$

Using Euler's identity and (a),

$$\begin{aligned} X_R(e^{j\omega}) &= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \cos(k\omega) \\ &= 1 + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k (e^{jk\omega} + e^{-jk\omega}) \\ x_e[n] &= \delta[n] + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k (\delta[n+k] + \delta[n-k]) \end{aligned}$$

Using (c) and then taking the odd part we get,

$$\begin{aligned} x[n] &= 2x_e[n]u[-n] - x_e[0]\delta[n] \\ &= \delta[n] + \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \delta[n+k] \\ x_o[n] &= \frac{x[n] - x[-n]}{2} \\ &= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k (\delta[n+k] - \delta[n-k]) \end{aligned}$$

Now taking the DTFT and using (b),

$$\begin{aligned} jX_I(e^{j\omega}) &= \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k (e^{jk\omega} - e^{-jk\omega}) \\ &= j\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k \sin(k\omega) \\ &= j\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \sin(k\omega) \end{aligned}$$

Thus,

$$X_I(e^{j\omega}) = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k \sin(k\omega)$$

12.15. Given $X_i(e^{j\omega})$, we can take the inverse DTFT of $jX_i(e^{j\omega})$ to find the odd part of $x[n]$, denoted $x_o[n]$.

$$\begin{aligned}\mathcal{Im}\left\{X(e^{j\omega})\right\} &= \sin\omega \\ &= \frac{1}{2j}e^{j\omega} - \frac{1}{2j}e^{-j\omega}\end{aligned}$$

$$\begin{aligned}x_o[n] &= \mathcal{DFT}^{-1}\left[j\mathcal{Im}\left\{X(e^{j\omega})\right\}\right] \\ &= \mathcal{DFT}^{-1}\left[\frac{1}{2}e^{j\omega} - \frac{1}{2}e^{-j\omega}\right] \\ &= \frac{1}{2}\delta[n+1] - \frac{1}{2}\delta[n-1]\end{aligned}$$

Using the formula $x[n] = 2x_o[n]u[n] + x[0]\delta[n]$,

$$x[n] = -\delta[n-1] + x[0]\delta[n]$$

Since

$$\sum_{n=-\infty}^{\infty} x[n] = 3$$

$$\begin{aligned}-1 + x[0] &= 3 \\ x[0] &= 4\end{aligned}$$

Therefore,

$$x[n] = 4\delta[n] - \delta[n-1]$$

12.16. Using Euler's identity and the fact that $x_e[n]$ is the inverse transform of $X_R(e^{j\omega})$ we have

$$\begin{aligned} X_R(e^{j\omega}) &= 2 - 4\cos(3\omega) \\ &= 2 - 2(e^{j\omega} + e^{-j\omega}) \end{aligned}$$

$$x_e[n] = -2\delta[n+3] + 2\delta[n] - 2\delta[n-3]$$

Since $x[n]$ is real and causal, it is fully determined by its even part $x_e[n]$,

$$\begin{aligned} x[n] &= 2x_e[n]u[n] - x_e[0]\delta[n] \\ &= 4\delta[n] - 4\delta[n-3] - 2\delta[n] \\ &= 2\delta[n] - 4\delta[n-3] \end{aligned}$$

Using this information in the second condition we find

$$\begin{aligned} X(e^{j\omega})|_{\omega=\pi} &= \sum_{n=-\infty}^{\infty} x[n]e^{j\pi n} \\ &= \sum_{n=-\infty}^{\infty} x[n](-1)^n \\ &= 2+4 \\ &\neq 7 \end{aligned}$$

Thus, there is no real, causal sequence that satisfies both conditions.

12.17. There is more than one way to solve this problem. Two solutions are presented below.

Solution 1: Yes, it is possible to determine $x[n]$ uniquely. Note that $X[k]$, the 2 point DFT of a real signal $x[n]$, is also real, as demonstrated below.

$$\begin{aligned} X[k] &= \sum_{n=0}^{1} x[n]e^{-j2\pi nk/2} \\ X[k] &= \sum_{n=0}^{1} x[n](-1)^{nk} \end{aligned}$$

Thus,

$$\begin{aligned} X[0] &= x[0] + x[1] \\ X[1] &= x[0] - x[1] \end{aligned}$$

Clearly, if $x[n]$ is real, then $X[k]$ is real. Therefore, we can conclude that the imaginary part $X_I[k]$ is zero.

Therefore, the inverse DFT of $X_R[k]$ is $x[n]$, computed below.

$$\begin{aligned} x[n] &= \frac{1}{2}\sum_{k=0}^{1} X_R[k]e^{j2\pi nk/2} \\ x[n] &= \frac{1}{2}\sum_{k=0}^{1} X_R[k](-1)^{nk} \end{aligned}$$

$$\begin{aligned} x[0] &= \frac{1}{2}(X_R[0] + X_R[1]) \\ &= -1 \\ x[1] &= \frac{1}{2}(X_R[0] - X_R[1]) \\ &= 3 \end{aligned}$$

Thus,

$$x[n] = -\delta[n] + 3\delta[n-1]$$

Solution 2: Start by making the assumption that $X[k]$ is complex, i.e., $X_I[k]$ is nonzero and $X_R[k] = 2\delta[k] - 4\delta[k-1]$. Then, because $x_{ep}[n]$ is the inverse DFT of $X_R[k]$ we find

$$\begin{aligned} x_{ep}[n] &= \frac{1}{2}\sum_{k=0}^{1} X_R[k]e^{j2\pi nk/2} \\ &= \frac{1}{2}\sum_{k=0}^{1} X_R[k](-1)^{nk} \end{aligned}$$

and

$$\begin{aligned} x_{ep}[0] &= \frac{1}{2}(X_R[0] + X_R[1]) \\ &= -1 \\ x_{ep}[1] &= \frac{1}{2}(X_R[0] - X_R[1]) \\ &= 3 \end{aligned}$$

$$x_{ep}[n] = -\delta[n] + 3\delta[n-1]$$

Because $x[n]$ is real and causal, we can determine it from $x_{ep}[n]$

$$x[n] = \begin{cases} x_{ep}[n], & n = 0 \\ 2x_{ep}[n], & 0 < n < N/2 \\ x_{ep}[N/2], & n = N/2 \\ 0, & \text{otherwise} \end{cases}$$

With $N = 2$ we have

$$x[n] = -\delta[n] + 3\delta[n-1]$$

If we began by making the assumption that $X[k]$ was real, i.e., $X_I[k] = 0$ and $X[k] = X_R[k] = 2\delta[k] - 4\delta[k-1]$ than by taking the inverse transform we find that

$$x[n] = x_{ep}[n] = -\delta[k] + 3\delta[k-1]$$

This is the same answer we got before. Since there was no ambiguities in our determination of $x[n]$, we conclude that $x[n]$ can be uniquely determined.
The next problem shows that when $N > 2$, we cannot necessarily uniquely determine $x[n]$ from $X_R[k]$ unless we make additional assumptions about $x[n]$ such as periodic causality. When $N > 2$ the two assumptions we used above leads to two different sequences with the same $X_R[k]$.

12.18. Sequence 1: For $k = 0, 1, 2$ we have

$$X_R[k] = 9\delta[k] + 6\delta[k-1] + 6\delta[((k+1))_3]$$

and $X_R[k] = 0$ for any other k. Using the DFT properties and taking the inverse DFT we find for $n = 1, 2, 3$

$$\begin{aligned} x_{ep}[n] &= 3 + 2\left(e^{j(2\pi/3)n} + e^{-j(2\pi/3)n}\right) \\ &= 3 + 4\cos(2\pi n/3) \\ &= 7\delta[n] + \delta[n-1] + \delta[n-2] \end{aligned}$$

If we let $x[n] = x_{ep}[n]$ we have the desired sequence.

Sequence 2: If we assume $x[n]$ is periodically causal, we can use the following property to solve for $x[n]$ from $x_{ep}[n]$:

$$x[n] = \begin{cases} x_{ep}[0], & n = 0 \\ 2x_{ep}[n], & 0 < n < \frac{N}{2} \\ 0, & \text{otherwise} \end{cases}$$

Note that this is only true for odd N. For even N, we would also need to handle the $n = N/2$ point as shown in the chapter. We have

$$\begin{aligned} x[n] &= \begin{cases} x_{ep}[0], & n = 0 \\ 2x_{ep}[n], & n = 1 \\ 0, & \text{otherwise} \end{cases} \\ &= 7\delta[n] + 2\delta[n-1] \end{aligned}$$

12.19. Given the real part of $X[k]$, we can take the inverse DFT to find the even periodic part of $x[n]$, denoted $x_{ep}[n]$.

Using the inverse DFT relation,

$$x_{ep}[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_R[k]W^{-nk}$$

we find

$$\begin{aligned}
x_{ep}[0] &= \frac{1}{4}(4+1+2+1) = 2\\
x_{ep}[1] &= \frac{1}{4}(4+j-2-j) = \frac{1}{2}\\
x_{ep}[2] &= \frac{1}{4}(4-1+2-1) = 1\\
x_{ep}[3] &= \frac{1}{4}(4-j-2+j) = \frac{1}{2}
\end{aligned}$$

Thus,

$$x_{ep}[n] = 2\delta[n] + \frac{1}{2}\delta[n-1] + \delta[n-2] + \frac{1}{2}\delta[n-3]$$

Next, we can relate the odd periodic and even periodic parts of $x[n]$ using

$$x_{op}[n] = \begin{cases} x_{ep}[n], & 0 < n < N/2\\ -x_{ep}[n], & N/2 < n \leq N-1\\ 0, & \text{otherwise}\end{cases}$$

Performing this operation gives

$$x_{op}[n] = \frac{1}{2}\delta[n-1] - \frac{1}{2}\delta[n-3]$$

Taking the DFT of $x_{op}[n]$ yields $jX_I[k]$. Using the DFT relation,

$$jX_I[k] = \sum_{n=0}^{N-1} x_{op}[n]W^{nk}$$

we find

$$\begin{aligned}
jX_I[0] &= \left(0+\frac{1}{2}+0-\frac{1}{2}\right) = 0\\
jX_I[1] &= \left(0-\frac{j}{2}+0-\frac{j}{2}\right) = -j\\
jX_I[2] &= \left(0+\frac{1}{2}+0-\frac{1}{2}\right) = 0\\
jX_I[3] &= \left(0+\frac{j}{2}+0+\frac{j}{2}\right) = j
\end{aligned}$$

Thus,

$$jX_I[k] = -j\delta[k-1] + j\delta[k-3]$$

12.20. As the following shows, the second condition implies $x[0] = 1$.

$$\begin{aligned} x[0] &= \frac{1}{6}\sum_{k=0}^{5} X[k]e^{j(2\pi/6)kn}\bigg|_{n=0} \\ &= \frac{1}{6}\sum_{k=0}^{5} X[k] \\ &= 1 \end{aligned}$$

This condition eliminates all choices except $x_2[n]$ and $x_3[n]$.
The odd periodic parts of $x_2[n]$ and $x_3[n]$ for $n = 0, \dots, 5$ are

$$\begin{aligned} x_{op_2}[n] &= \frac{x_2[n] - x_2^*[((-n))_6]}{2} \\ &= \frac{1}{3}\left(\delta[n-4] - \delta[((n+4))_6]\right) - \frac{1}{3}\left(\delta[n-5] - \delta[((n+5))_6]\right) \end{aligned}$$

$$\begin{aligned} x_{op_3}[n] &= \frac{x_3[n] - x_3^*[((-n))_6]}{2} \\ &= \frac{1}{3}\left(\delta[n-1] - \delta[((n+1))_6]\right) - \frac{1}{3}\left(\delta[n-2] - \delta[((n+2))_6]\right) \end{aligned}$$

For $n < 0$ or $n > 5$, these sequences are zero. Since the transform of $x_{op}[n]$ is $jX_I[k]$ we find for $k = 0, \dots, 5$

$$\begin{aligned} jX_{I_2}[k] &= \frac{1}{3}\left(e^{-j(2\pi/6)4k} - e^{j(2\pi/6)4k}\right) - \frac{1}{3}\left(e^{-j(2\pi/6)5k} - e^{j(2\pi/6)5k}\right) \\ &= -\frac{2}{3}j\sin(4\pi k/3) + \frac{2}{3}j\sin(5\pi k/3) \\ &= j\frac{2}{\sqrt{3}}\left(-\delta[k-2] + \delta[k-4]\right) \end{aligned}$$

$$\begin{aligned} jX_{I_3}[k] &= \frac{1}{3}\left(e^{-j(2\pi/6)k} - e^{j(2\pi/6)k}\right) - \frac{1}{3}\left(e^{-j(2\pi/6)2k} - e^{j(2\pi/6)2k}\right) \\ &= -\frac{2}{3}j\sin(\pi k/3) + \frac{2}{3}j\sin(2\pi k/3) \\ &= j\frac{2}{\sqrt{3}}\left(-\delta[k-2] + \delta[k-4]\right) \end{aligned}$$

Thus, both $x_2[n]$ and $x_3[n]$ are consistent with the information given.

12.21. (a) *Method 1*:

We are given

$$\begin{aligned} X_R(\rho e^{j\omega}) &= U(\rho,\omega) \\ &= 1+\rho^{-1}\alpha\cos\omega \end{aligned}$$

Since $\frac{\partial U}{\partial \rho} = \frac{1}{\rho}\frac{\partial V}{\partial \omega}$ we have,

$$\begin{aligned} \frac{\partial V}{\partial \omega} &= -\alpha\rho^{-1}\cos\omega \\ V &= -\alpha\rho^{-1}\sin\omega + K(\rho) \end{aligned}$$

Since $\frac{\partial V}{\partial \rho} = -\frac{1}{\rho}\frac{\partial U}{\partial \omega}$ we have,

$$\overbrace{\alpha\rho^{-2}\sin\omega + K'(\rho)}^{\frac{\partial V}{\partial \rho}} = \overbrace{\alpha\rho^{-2}\sin\omega}^{-\frac{1}{\rho}\frac{\partial U}{\partial \omega}}$$

Thus,

$$\begin{aligned} K'(\rho) &= 0 \\ K(\rho) &= C \end{aligned}$$

Since $x[n]$ is real $V(\rho,\omega)$ is an odd function of ω. Hence, $V(\rho,0)=0$, implying that $C=0$. Therefore,

$$\begin{aligned} X(\rho e^{j\omega}) &= U(\rho,\omega) + jV(\rho,\omega) \\ &= 1+\rho^{-1}\alpha\cos\omega - j\rho^{-1}\alpha\sin\omega \\ &= 1+\alpha\rho^{-1}(\cos\omega - j\sin\omega) \\ &= 1+\alpha\rho^{-1}e^{-j\omega} \\ X(z) &= 1+\alpha z^{-1} \end{aligned}$$

(b) *Method 2*: Since $X_R(e^{j\omega})$ is the transform of $x_e[n]$ we have

$$\begin{aligned} X_R(e^{j\omega}) &= 1+\alpha\cos\omega \\ &= 1+\frac{\alpha}{2}e^{j\omega}+\frac{\alpha}{2}e^{-j\omega} \\ x_e[n] &= \delta[n]+\frac{\alpha}{2}\delta[n+1]+\frac{\alpha}{2}\delta[n-1] \end{aligned}$$

Because $x[n]$ is real and causal, we can recover $x_o[n]$ from $x_e[n]$ as follows

$$\begin{aligned} x_o[n] &= \begin{cases} x_e[n], & n>0 \\ 0, & n=0 \\ -x_e[n], & n<0 \end{cases} \\ &= -\frac{\alpha}{2}\delta[n+1]+\frac{\alpha}{2}\delta[n-1] \end{aligned}$$

Thus,

$$\begin{aligned} x[n] &= x_e[n]+x_o[n] \\ &= \delta[n]+\alpha\delta[n-1] \\ X(z) &= 1+\alpha z^{-1} \end{aligned}$$

Note that we could have obtained $x[n]$ directly from $x_e[n]$ as follows

$$\begin{aligned} x[n] &= 2x_e[n]u[n] - x_e[0]\delta[n] \\ &= (2\delta[n]+\alpha\delta[n+1]+\alpha\delta[n-1])u[n]-\delta[n] \\ &= \delta[n]+\alpha\delta[n-1] \end{aligned}$$

12.22. Problem 3 in Fall2002 final Appears in: Fall05 PS11.
The problem has been modified from the exam version in Fall05. The Fall02 final exam version is included right after it.

Problem

(a) $x[n]$ is a causal, real-valued sequence with Fourier Transform $X(e^{j\omega})$. It is known that

$$\mathrm{Re}\{X(e^{j\omega})\} = 1 + 3\cos\omega + \cos 3\omega.$$

Determine a choice for $x[n]$ consistent with this information and specify whether or not your choice is unique.

Fall02 Final Exam Version of Problem

8% (a) $x[n]$ is a causal real-valued sequence with Fourier Transform $X(e^{j\omega})$. It is known that

$$Re\{X(e^{j\omega})\} = 1 + 3\cos\omega + \cos 3\omega.$$

Determine a choice for $x[n]$ consistent with this information and specify whether or not your choice is unique.

Solution from Fall05 PS11

(a) Similar to Problem 11.2, the inverse DTFT of $\mathrm{Re}\{X(e^{j\omega})\}$ is the even part $x_e[n]$ of $x[n]$.

$$\begin{aligned} x_e[n] &= \mathrm{DTFT}^{-1}[1 + 3\cos\omega + \cos 3\omega] \\ &= \delta[n] + \frac{1}{2}\left(3\delta[n+1] + 3\delta[n-1] + \delta[n+3] + \delta[n-3]\right) \end{aligned}$$

Since $x[n]$ is real and causal, it can be *uniquely* determined from its even part $x_e[n]$:

$$\begin{aligned} x[n] &= 2x_e[n]u[n] - x_e[0]\delta[n] \\ x[n] &= \delta[n] + 3\delta[n-1] + \delta[n-3] \end{aligned}$$

Solution from Fall02 Final

N/A

12.23

Since $x[n]$ is real, $j\,\mathrm{Im}\{X(e^{j\omega})\}$ is the Fourier transform of $x_o[n]$, the odd part of $x[n]$.

$$j\,\mathrm{Im}\{X(e^{j\omega})\} = j\left[\frac{3}{j2}\left(e^{j2\omega} - e^{-j2\omega}\right) - \frac{2}{j2}\left(e^{j3\omega} - e^{-j3\omega}\right)\right]$$

$$= \frac{3}{2}e^{j2\omega} - \frac{3}{2}e^{-j2\omega} - e^{j3\omega} + e^{-j3\omega}.$$

Inverse transforming gives

$$x_o[n] = \frac{3}{2}\delta[n+2] - \frac{3}{2}\delta[n-2] - \delta[n+3] + \delta[n-3].$$

Now $x[n]$ is also causal, so it can be recovered by doubling $x_o[n]$ for $n > 0$ and setting it to zero for $n < 0$. The only value that cannot be determined is $x[0]$, so we leave it as arbitrary.

$$x[n] = 2\delta[n-3] - 3\delta[n-2] + x[0]\delta[n].$$

12.24. Taking the z-transform of $u_N[n]$ we get

$$\begin{aligned} U_N(z) &= \frac{2}{1-z^{-1}} - \frac{2z^{-N/2}}{1-z^{-1}} - 1 + z^{-N/2} \\ &= \frac{1 - z^{-N/2} + z^{-1} - z^{-1-N/2}}{1-z^{-1}}, \qquad |z| \neq 0 \end{aligned}$$

Sampling this we find

$$\begin{aligned} \tilde{U}_N[k] &= U_N(e^{2\pi k/N}) \\ &= \frac{1 - (-1)^k + e^{-j2\pi k/N} - e^{-j2\pi k/N}(-1)^k}{1 - e^{-j2\pi k/N}} \end{aligned}$$

When k is even but $k \neq 0$ we see that $\tilde{U}_N[k] = 0$. For k odd, we get

$$\begin{aligned} \tilde{U}_N[k] &= \frac{2 + 2e^{-j2\pi k/N}}{1 - e^{-j2\pi k/N}} \\ &= \frac{2e^{-j\pi k/N}(e^{j\pi k/N} + e^{-j\pi k/N})}{e^{-j\pi k/N}(e^{j\pi k/N} - e^{-j\pi k/N})} \\ &= -2j\cot(\pi k/N) \end{aligned}$$

When $k = 0$ we get $0/0$ which, if the function was continuous, you would use l'Hôpital's rule. In this case the function is discrete so that is not available to us. One route to the answer is to use the definition of the DFS

$$\begin{aligned} \tilde{U}_N[0] &= \sum_{k=0}^{N} \tilde{u}_N[n] e^{-j\frac{2\pi}{N}kn}\Bigg|_{k=0} \\ &= \sum_{k=0}^{N} \tilde{u}_N[n] \\ &= N \end{aligned}$$

Putting it all together gives us the desired answer

$$\tilde{U}_N[k] = \begin{cases} N, & k = 0, \\ -2j\cot(\pi k/N), & k \text{ odd}, \\ 0, & k \text{ even},\ k \neq 0 \end{cases}$$

12.25. (a) Because $x_{ep}[n]$ is the inverse DFT of $X_R[k]$ we have for $n = 0, \ldots, N-1$ and $k = 0, \ldots, N-1$

$$\begin{aligned} X_R[k] &= \frac{X[k] + X^*[k]}{2} \\ x_{ep}[n] &= \frac{x[n] + x^*[((-n))_N]}{2} \end{aligned}$$

or equivalently, if we periodically extend these sequences with period N

$$\tilde{x}_e[n] = \frac{\tilde{x}[n] + \tilde{x}[-n]}{2}$$

Note that since the signal is real $\tilde{x}^*[-n] = \tilde{x}[-n]$.
The first period of $\tilde{x}[n]$ is zero from $n = M$ to $n = N-1$. If $N = 2(M-1)$ there is no overlap of $\tilde{x}[n]$ and $\tilde{x}[-n]$ except at $n = 0$ and $n = N/2$. We can therefore recover $\tilde{x}[n]$ from $\tilde{x}_e[n]$ with the following:

$$\tilde{x}[n] = \begin{cases} 2\tilde{x}_e[n], & n = 1, \ldots, N/2-1 \\ \tilde{x}_e[n], & n = 0, N/2 \\ 0, & n = M, \ldots, N-1 \end{cases}$$

If we tried to make N any smaller, the overlap of $\tilde{x}[n]$ and $\tilde{x}[-n]$ would prevent the recovery of $x[n]$. Consequently, the smallest value of N we can use to recover $X[k]$ from $X_R[k]$ is $N = 2(M-1)$.

(b) If $N = 2(M-1)$,

$$x[n] = x_{ep}[n]u_N[n] = \begin{cases} 2x_{ep}[n] & n = 1, \ldots, N/2-1 \\ x_{ep}[n], & n = 0, N/2 \\ 0, & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} u_N[n] &= \begin{cases} 2, & n = 1, 2, \ldots, N/2-1 \\ 1, & n = 0, N/2 \\ 0, & \text{otherwise} \end{cases} \\ &= 2u[n] - 2u[n - N/2] - \delta[n] + \delta[n - N/2] \end{aligned}$$

Taking the DFT of $x[n]$ we find

$$X[k] = X_R[k] \otimes U_N[k]$$

where

$$\begin{aligned} U_N[k] &= DFT\,\{2u[n] - 2u[n-N/2] - \delta[n] + \delta[n-N/2]\} \\ &= \frac{1 - (-1)^k + e^{-j2\pi k/N} - e^{-j2\pi k/N}(-1)^k}{1 - e^{-j2\pi k/N}}, \qquad k = 0, \ldots, N-1 \\ U_N[k] &= \begin{cases} N, & k = 0, \\ -2j\cot(\pi k/N), & 0 < k < N-1,\ k \text{ odd} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

12.26

$$\begin{aligned} Y\left(e^{j\omega}\right) &= Y_r\left(e^{j\omega}\right) + jY_i\left(e^{j\omega}\right) \\ &= Y_r\left(e^{j\omega}\right)\left(1 + jH\left(e^{j\omega}\right)\right). \end{aligned}$$

To satisfy the constraint,

$$1 + jH\left(e^{j\omega}\right) = \begin{cases} 1, & -\pi < \omega < 0 \\ 0, & 0 < \omega < \pi. \end{cases}$$

Then

$$H\left(e^{j\omega}\right) = \begin{cases} 0, & -\pi < \omega < 0 \\ j, & 0 < \omega < \pi. \end{cases}$$

12.27. We are given

$$\begin{aligned} H_R(e^{j\omega}) &= H_{ER}(e^{j\omega}) + H_{OR}(e^{j\omega}) \\ H_I(e^{j\omega}) &= H_{EI}(e^{j\omega}) + H_{OI}(e^{j\omega}) \end{aligned}$$

$$\begin{aligned} h_r[n] &\longleftrightarrow H_r(e^{j\omega}) = H_A(e^{j\omega}) + jH_B(e^{j\omega}) \\ h_i[n] &\longleftrightarrow H_i(e^{j\omega}) = H_C(e^{j\omega}) + jH_D(e^{j\omega}) \end{aligned}$$

where $h_r[n]$, $h_i[n]$, $H_R(e^{j\omega})$, $H_I(e^{j\omega})$, $H_{ER}(e^{j\omega})$, $H_{OR}(e^{j\omega})$, $H_{EI}(e^{j\omega})$, and $H_{OI}(e^{j\omega})$ are real. Begin by breaking $H(e^{j\omega})$ into its real and imaginary parts $H_R(e^{j\omega})$ and $H_I(e^{j\omega})$

$$\begin{aligned} H(e^{j\omega}) &= H_R(e^{j\omega}) + jH_I(e^{j\omega}) \\ &= [H_{ER}(e^{j\omega}) + H_{OR}(e^{j\omega})] + j[H_{EI}(e^{j\omega}) + H_{OI}(e^{j\omega})] \end{aligned}$$

Now solve for the conjugate symmetric and conjugate antisymmetric parts of $H(e^{j\omega})$

$$\begin{aligned} H_r(e^{j\omega}) &= \frac{H(e^{j\omega}) + H^*(e^{-j\omega})}{2} \\ &= \frac{[H_{ER}(e^{j\omega}) + H_{OR}(e^{j\omega})] + j[H_{EI}(e^{j\omega}) + H_{OI}(e^{j\omega})]}{2} \\ &\quad + \frac{[H_{ER}(e^{j\omega}) - H_{OR}(e^{j\omega})] - j[H_{EI}(e^{j\omega}) - H_{OI}(e^{j\omega})]}{2} \\ &= H_{ER}(e^{j\omega}) + jH_{OI}(e^{j\omega}) \end{aligned}$$

$$\begin{aligned} H_i(e^{j\omega}) &= \frac{H(e^{j\omega}) - H^*(e^{-j\omega})}{2j} \\ &= \frac{[H_{ER}(e^{j\omega}) + H_{OR}(e^{j\omega})] + j[H_{EI}(e^{j\omega}) + H_{OI}(e^{j\omega})]}{2j} \\ &\quad - \frac{[H_{ER}(e^{j\omega}) - H_{OR}(e^{j\omega})] - j[H_{EI}(e^{j\omega}) - H_{OI}(e^{j\omega})]}{2j} \\ &= H_{EI}(e^{j\omega}) - jH_{OR}(e^{j\omega}) \end{aligned}$$

Thus,

$$\begin{aligned} H_A(e^{j\omega}) &= H_{ER}(e^{j\omega}) \qquad & H_C(e^{j\omega}) &= H_{EI}(e^{j\omega}) \\ H_B(e^{j\omega}) &= H_{OI}(e^{j\omega}) \qquad & H_D(e^{j\omega}) &= -H_{OR}(e^{j\omega}) \end{aligned}$$

12.28. (a) By inspection,

$$H(e^{j\omega}) = j(2H_{\text{lp}}(e^{j(\omega+\frac{\pi}{2})}) - 1)$$

$$H_{\text{lp}}(e^{j\omega}) = \frac{1 - jH(e^{j(\omega-\frac{\pi}{2})})}{2}$$

(b) **Find h[n]:**
Taking the inverse DTFT of $H(e^{j\omega})$ yields

$$\begin{aligned} h[n] &= j\left[2e^{-j(\pi/2)n}\, h_{\text{lp}}[n] - \delta[n]\right] \\ &= j\left[2\cos(\pi n/2)h_{\text{lp}}[n] - j2\sin(\pi n/2)h_{\text{lp}}[n] - \delta[n]\right] \\ &= 2\sin(\pi n/2)h_{\text{lp}}[n] \end{aligned}$$

The simplification in the last step used the fact that $h_{\text{lp}}[n] = \frac{\sin(\pi n/2)}{\pi n}$ is zero for even n and equals $1/2$ for $n = 0$.

Find $\mathbf{h_{lp}[n]}$:
Taking the inverse DTFT of $H_{\text{lp}}(e^{j\omega})$ yields

$$\begin{aligned} h_{\text{lp}}[n] &= \frac{\delta[n] - je^{j(\pi/2)n}\, h[n]}{2} \\ &= \frac{1}{2}\delta[n] - \frac{1}{2}(j)^{n+1}h[n] \end{aligned}$$

Using the fact that $h[n]$ is zero for $n = 0$ and n even we can reduce this to

$$h_{\text{lp}}[n] = \frac{\sin(\pi n/2)}{2}h[n] + \frac{1}{2}\delta[n]$$

(c) The linear phase causes a delay of $n_d = M/2$ in the responses. If n_d is not an integer, then we interpret $h_{\text{lp}}[n]$ and $h[n]$ as

$$\begin{aligned} h_{\text{lp}}[n - n_d] &= \frac{\sin(\pi(n - n_d)/2)}{\pi(n - n_d)} \\ h[n - n_d] &= \frac{2}{\pi}\frac{\sin^2(\pi(n - n_d)/2)}{(n - n_d)} \end{aligned}$$

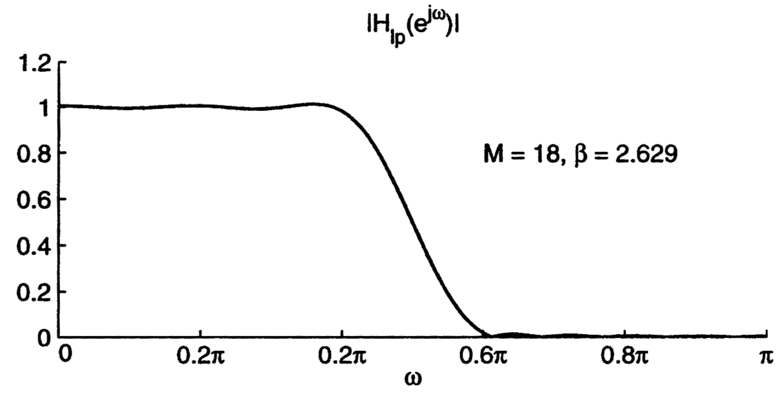
Then,

$$\begin{aligned} \hat{h}[n] &= h[n - n_d]w[n] \\ &= 2\sin(\pi(n - n_d)/2)\, h_{\text{lp}}[n - n_d]w[n] \\ &= 2\sin(\pi(n - n_d)/2)\, \hat{h}_{\text{lp}}[n] \end{aligned}$$

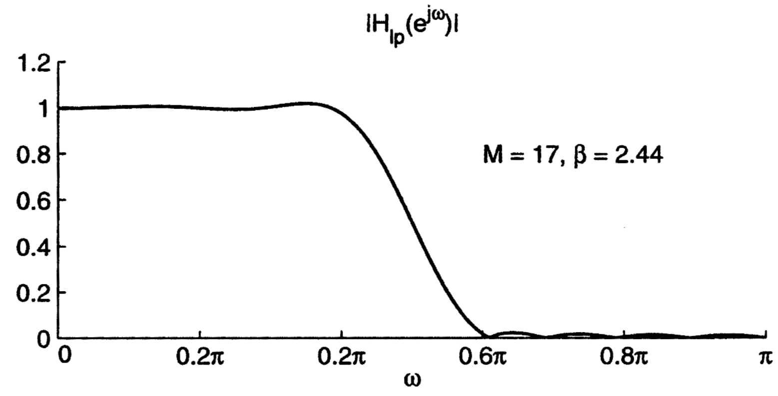
where $\hat{h}[n]$ and $\hat{h}_{\text{lp}}[n]$ are the causal FIR approximations to $h[n]$ and $h_{\text{lp}}[n]$. Similarly,

$$\hat{h}_{\text{lp}}[n] = \begin{cases} \dfrac{\sin(\pi(n - n_d)/2)}{2}\hat{h}[n] + \dfrac{1}{2}\delta[n - n_d]w[n], & M \text{ even} \\ \sin(\pi(n - n_d)/2)\hat{h}[n], & M \text{ odd} \end{cases}$$

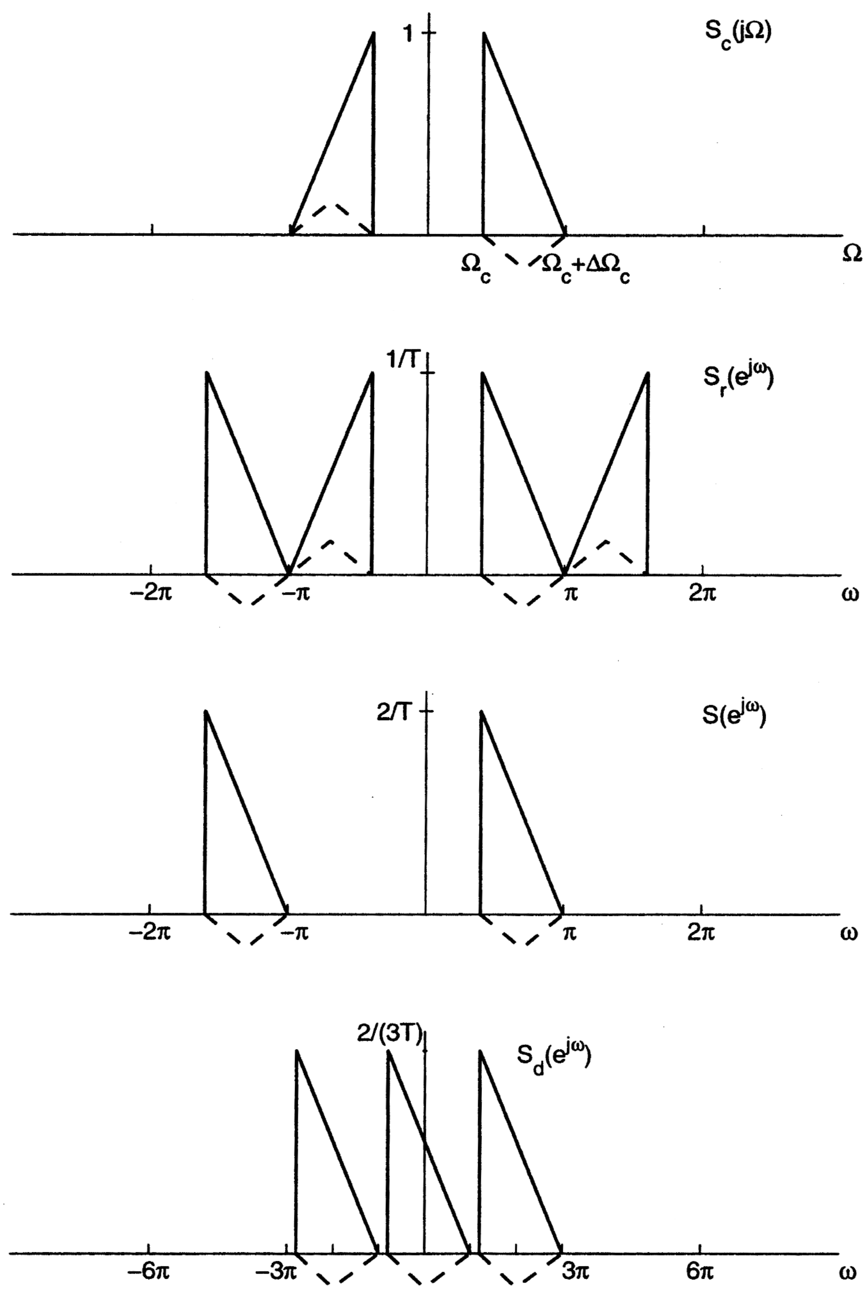
(d) The lowpass filter corresponding to the first filter in the example looks like



The lowpass filter corresponding to the second filter in the example looks like



12.29. (a) The example shown here samples at the Nyquist rate of $T = \pi/(\Omega_c + \Delta\Omega)$ as in the chapter's example, but the bandpass signal is such that $\Delta\Omega/(\Omega_c + \Delta\Omega) = 3/5$. Then, $2\pi/(\Delta\Omega T) = 10/3$.



(b) If $2\pi/(\Delta\Omega T) = M + e$, where M is an integer and e some fraction, then using the Nyquist rate of $2\pi/T = 2(\Omega_c + \Delta\Omega)$ will force decimation by M. As just shown, this choice for T causes $S_d(e^{j\omega})$ to have intervals of zero. Instead, choose T such that $2\pi/(\Delta\Omega T)$ is the next highest integer

$$\frac{2\pi}{\Delta\Omega T} = M + 1.$$

Then decimating by $(M + 1)$ produces the desired result.

12.30. *Yes*, it is possible to always uniquely recover the system input from the system output. Although $Y(e^{j\omega})$ contains roughly half the frequency spectrum as $X(e^{j\omega})$, we can reconstruct $X(e^{j\omega})$ from $Y(e^{j\omega})$. We can accomplish this by recognizing that since $x[n]$ is real, $X(e^{j\omega})$ must be conjugate symmetric.

The output of the system, $y[n]$, has a Fourier transform $Y(e^{j\omega})$ that is the product of $X(e^{j\omega})$ and $H(e^{j\omega})$. Therefore, $Y(e^{j\omega})$ will correspond to

$$Y(e^{j\omega}) = \begin{cases} X(e^{j\omega}), & 0 \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

At first glance, it may seem like $X(e^{j\omega}) = Y(e^{j\omega}) + Y^*(e^{-j\omega})$. This is close to the right answer, but it doesn't take into consideration the fact that $Y(e^{j\omega})$ is non-zero at $\omega = 0$ and $\omega = \pi$. Thus, the solution $X(e^{j\omega}) = Y(e^{j\omega}) + Y^*(e^{-j\omega})$, will be incorrect at $\omega = 0$ and $\omega = \pi$, since $Y(e^{j\omega})$ and $Y^*(e^{j\omega})$ will overlap at these frequencies. It is necessary to pay special attention to these frequencies to get the right answer. Let

$$Z(e^{j\omega}) = \begin{cases} 0, & \omega = 0, \omega = \pi \\ Y(e^{j\omega}), & \text{otherwise} \end{cases}$$

Alternatively, we can express $Z(e^{j\omega})$ with the constants a and b defined as

$$\begin{aligned} a &= Y(e^{j\omega})\big|_{\omega=0} &= \sum_{n=-\infty}^{\infty} y[n] \\ b &= Y(e^{j\omega})\big|_{\omega=\pi} &= \sum_{n=-\infty}^{\infty} y[n](-1)^n \end{aligned}$$

$$Z(e^{j\omega}) = Y(e^{j\omega}) - a\delta(\omega) - b\delta(\omega - \pi)$$

We can construct a conjugate symmetric $X(e^{j\omega})$ from $Y(e^{j\omega})$ and $Z(e^{j\omega})$ as

$$X(e^{j\omega}) = Y(e^{j\omega}) + Z^*(e^{-j\omega})$$

In the time domain, this is

$$x[n] = y[n] + z^*[n]$$

Or, since

$$\begin{aligned} z[n] &= y[n] - \frac{a}{2\pi} - \frac{b(-1)^n}{2\pi} \\ x[n] &= y[n] + y^*[n] - \frac{a}{2\pi} - \frac{b(-1)^n}{2\pi} \end{aligned}$$