Solution Manual for Adaptive Control

Second Edition

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Preface

This Solution Manual contains solutions to selected problems in the second edition of Adaptive Control published by Addison-Wesley 1995, ISBN 0-201-55866-1.

PROBLEM SOLUTIONS

SOLUTIONS TO CHAPTER 1

1.5 Linearization of the valve shows that

$$\Delta v = 4v_0^3 \Delta u$$

The loop transfer function is then

$$G_0(s) G_{PI}(s) 4v_0^3$$

where G_{PI} is the transfer function of a PI controller i.e.

$$G_{PI}(s) = K\left(1 + \frac{1}{sT_i}\right)$$

The characteristic equation for the closed loop system is

$$sT_i(s+1)^3 + K \cdot 4v_0^3(sT_i + 1) = 0$$

with $K = 0.15$ and $T_i = 1$ we get

$$(s+1)\left(s(s+1)^2 + 0.6v_0^3\right) = 0$$

$$(s+1)(s^3 + 2s^2 + s + 0.6v_0^3) = 0$$

The root locus of this equation with respect to v_o is sketched in Fig. 1. According to the Routh Hurwitz criterion the critical case is

$$0.6v_0^3 = 2 \qquad \Rightarrow v_0 = \sqrt[3]{\frac{10}{3}} = 1.49$$

Since the plant G_0 has unit static gain and the controller has integral action the steady-state output is equal to v_0 and the set point y_r. The closed-loop system is stable for $y_r = u_c = 0.3$ and 1.1 but unstable for $y_r = u_c = 5.1$. Compare with Fig. 1.9.

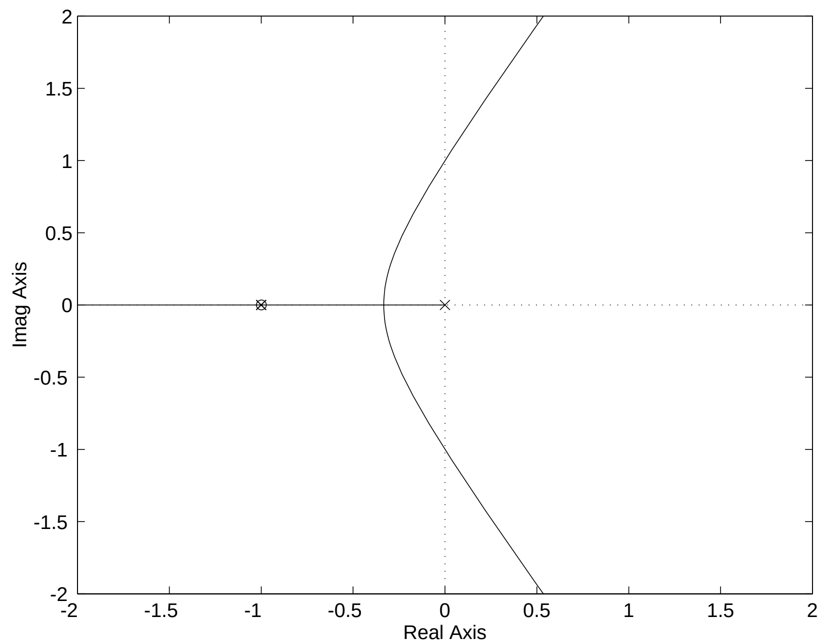

Figure 1. Root locus in Problem 1.5.

1.6 Tune the controller using the Ziegler-Nichols closed-loop method. The frequency ω_u, where the process has 180° phase lag is first determined. The controller parameters are then given by Table 8.2 on page 382 where

$$K_u = \frac{1}{|G_0(i\omega_u)}|$$

we have

$$G_0(s) = \frac{e^{-s/q}}{1 + s/q}$$

$$\arg G_0(i\omega) = -\frac{\omega}{q} - \arctan\frac{\omega}{q} = -\pi$$

q	ω	$G_0(i\omega)$	K	T_i
0.5	1.0	0.45	1	5.24
	2.0	0.45	1	2.62
	4.1	0.45	1	1.3

A simulation of the system obtained when the controller is tuned for the smallest flow $q = 0.5$ is shown Fig. 2. The Ziegler-Nichols method is not the best tuning method in this case. In the Fig. 3 we show results for

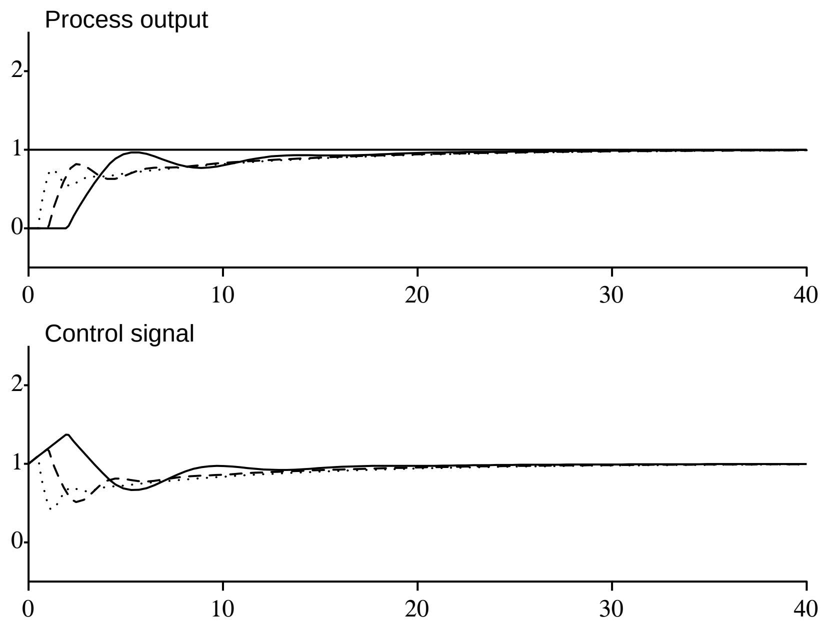

Figure 2. Simulation in Problem 1.6. Process output and control signal are shown for $q = 0.5$ (full), $q = 1$ (dashed), and $q = 2$ (dotted). The controller is designed for $q = 0.5$.

controller designed for $q = 1$ and in Fig. 4when the controller is designed for $q = 2$.

1.7 Introducing the feedback

$$u = -k_2 y_2$$

the system becomes

$$\frac{dx}{dt} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix} x - k_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u_1$$

$$y_1 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} x$$

The transfer function from u_1 to y_1 is

$$G(s) = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} s+1 & 0 & 0 \\ 2k_2 & s+3 & 2k_2 \\ k_2 & 0 & s+1+k_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{s^2 + (4-k_2)s + 3 + k_2}{(s+1)(s+3)(s+1+k_2)}$$

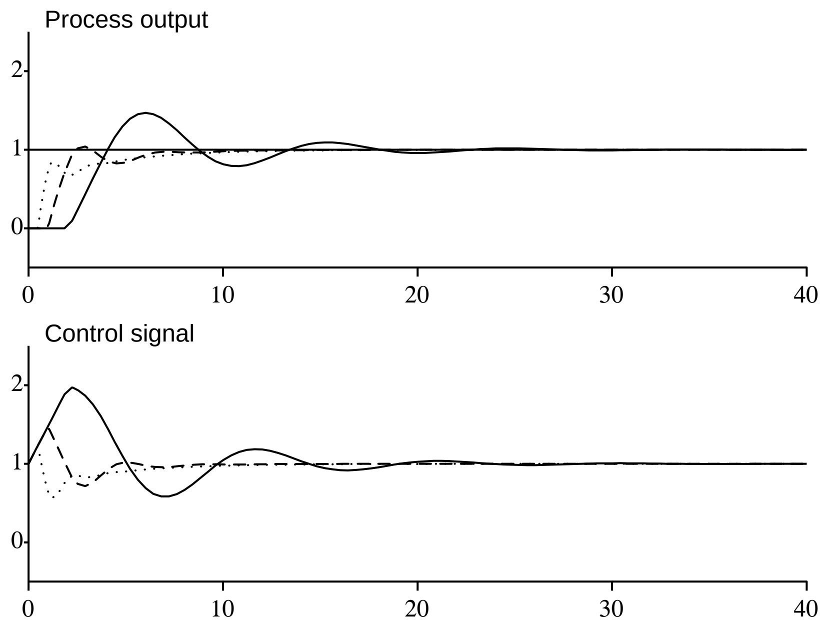

Figure 3. Simulation in Problem 1.6. Process output and control signal are shown for $q = 0.5$ (full), $q = 1$ (dashed), and $q = 2$ (dotted). The controller is designed for $q = 1$.

The static gain is

$$G(0) = \frac{3 + k_2}{3(1 + k_2)}$$

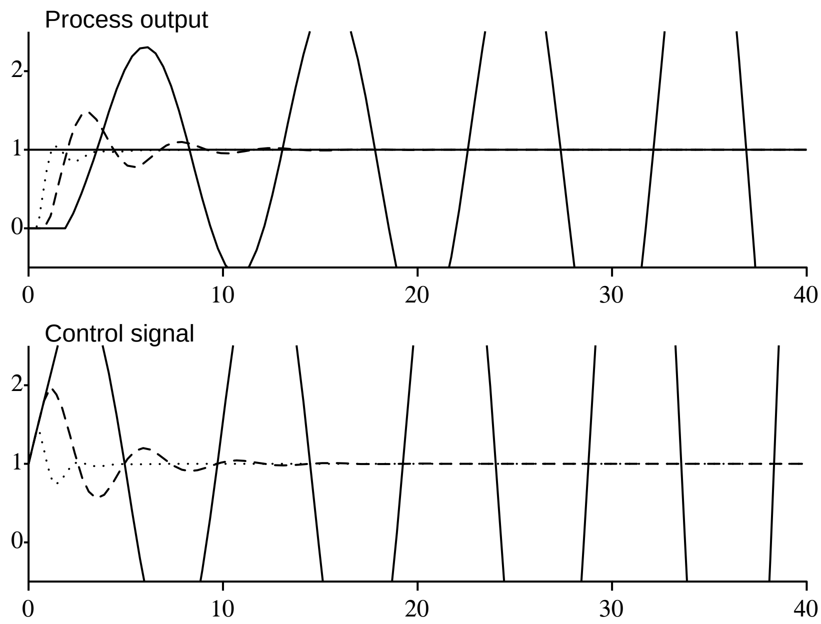

Figure 4. Simulation in Problem 1.6. Process output and control signal are shown for $q = 0.5$ (full), $q = 1$ (dashed), and $q = 2$ (dotted). The controller is designed for $q = 2$.

SOLUTIONS TO CHAPTER 2

2.1 The function V can be written as

$$V(x_1 \cdots x_m) = \sum_{i,j=1}^{n} x_i x_j (a_{ij} + a_{ji})/2 + \sum_{i=1}^{n} b_i x_i + c$$

Taking derivative with respect to x_i we get

$$\frac{\partial V}{\partial x_i} = \sum_{j=1}^{n} (a_{ij} + a_{ji}) x_j + b_i$$

In vector notation this can be written as

$$\operatorname{grad}_x V(x) = (A + A^T)x + b$$

2.2 The model is

$$y_t = \varphi_t^T \theta + e_t = \begin{pmatrix} u_t & u_{t-1} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} + e_t$$

The least squares estimate is given as the solution of the normal equation

$$\hat{\theta} = (\Phi^T \Phi)^{-1} \Phi^T Y = \begin{pmatrix} \sum u_t^2 & \sum u_t u_{t-1} \\ \sum u_t u_{t-1} & \sum u_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum u_t y_t \\ \sum u_{t-1} y_t \end{pmatrix}$$

(a) Input is a unit step

$$u_t = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Evaluating the sums we get

$$\hat{\theta} = \begin{pmatrix} N & N-1 \\ N-1 & N-1 \end{pmatrix}^{-1} \begin{pmatrix} \sum_1^N y_t \\ \sum_2^N y_t \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & \dfrac{N}{N-1} \end{pmatrix} \begin{pmatrix} \sum_1^N y_t \\ \sum_2^N y_t \end{pmatrix}$$

$$\hat{\theta} = \begin{pmatrix} y_1 \\ \dfrac{1}{N-1} \sum_2^N y_t - y_1 \end{pmatrix}$$

The estimation error is

$$\hat{\theta} - \theta = \begin{pmatrix} e_1 \\ \dfrac{1}{N-1} \sum_2^N e_t - e_1 \end{pmatrix}$$

Hence

$$E(\hat{\theta}-\theta)(\hat{\theta}-\theta)^T = (\Phi^T\Phi)^{-1}\cdot 1 = \begin{pmatrix} 1 & -1 \\ -1 & \dfrac{N}{N-1} \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

when $N \to \infty$. Notice that the variance of the estimates do not go to zero as $N \to \infty$. Consider, however, the estimate of $b_0 + b_1$.

$$E(\hat{b}_0 + \hat{b}_1 - b_0 - b_1) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & \dfrac{N}{N-1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{N-1}$$

With a step input it is thus possible to determine the combination $b_0 + b_1$ consistently. The individual values of b_0 and b_1 can, however, not be determined consistently.

(b) Input u is white noise with $Eu^2 = 1$ and u is independent of e.

$$Eu_t^2 = 1 \qquad Eu_t u_{t-1} = 0$$

$$\mathrm{cov}(\hat{\theta}-\theta) = 1\cdot E(\Phi^T\Phi)^{-1} = \begin{pmatrix} N & 0 \\ 0 & N-1 \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{1}{N} & 0 \\ 0 & \dfrac{1}{N-1} \end{pmatrix}$$

In this case it is thus possible to determine both parameters consistently.

2.3 Data generating process:

$$y(t) = b_0 u(t) + b_1 u(t-1) + e(t) = \varphi^T(t)\theta^0 + \bar{e}(t)$$

where

$$\varphi^T(t) = u(t), \quad \theta^0 = b_0$$

and

$$\bar{e}(t) = b_1 u(t-1) + e(t)$$

Model:

$$\hat{y}(t) = \hat{b}u(t)$$

or

$$y(t) = \hat{b}u(t) + \varepsilon(t) = \varphi^T(t)\hat{\theta} + \varepsilon(t)$$

where

$$\varepsilon(t) = y(t) - \hat{y}(t)$$

The least squares estimate is given by

$$\Phi^T\Phi(\hat{\theta}-\theta^0) = \Phi^T E_d \qquad E_d = \begin{pmatrix} \bar{e}(1) \\ \vdots \\ \bar{e}(N) \end{pmatrix}$$

$$\frac{1}{N}\Phi^T\Phi = \frac{1}{N}\sum_1^N u^2(t) \to Eu^2 \qquad N\to\infty$$

$$\frac{1}{N}\Phi^T E_d = \frac{1}{N}\sum_1^N u(t)\bar{e}(t) = \frac{1}{N}\sum_1^N u(t)\left(b_1 u(t-1) + e(t)\right)$$

$$\to b_1 E\left(u(t)u(t-1)\right) + E\left(u(t)e(t)\right) \qquad N\to\infty$$

(a)

$$u(t) = \begin{cases} 1 & t \geq 1 \\ 0 & t < 1 \end{cases}$$

$$E(u^2) = 1 \qquad E\left(u(t)u(t-1)\right) = 1 \qquad Eu(t)e(t) = 0$$

Hence

$$\hat{b} = \hat{\theta} \to \theta^0 + b_1 = b_0 + b_1 \qquad N\to\infty$$

i.e. $\hat{b}$ converges to the stationary gain

(b)

$$u(t) \in N(0,\sigma) \Rightarrow Eu^2 = \sigma^2 \qquad Eu(t)u(t-1) = 0 \qquad Eu(t)e(t) = 0$$

Hence

$$\hat{b} \to b_0 \qquad N\to\infty$$

2.6 The model is

$$y_t = \varphi_t^T\theta + \varepsilon_t = \underbrace{\begin{pmatrix} -y_{t-1} & u_{t-1}\end{pmatrix}}_{\varphi_t^T}\underbrace{\begin{pmatrix} a \\ b\end{pmatrix}}_{\theta} + \underbrace{\begin{pmatrix} e_t + ce_{t-1}\end{pmatrix}}_{\varepsilon_t}$$

The least squares estimate is given by the solution to the normal equation (2.5). The estimation error is

$$\hat{\theta} - \theta = (\Phi^T\Phi)^{-1}\Phi^T\varepsilon =$$

$$\begin{pmatrix} \sum y_{t-1}^2 & -\sum y_{t-1}u_{t-1} \\ -\sum y_{t-1}u_{t-1} & \sum u_{t-1}^2 \end{pmatrix}^{-1} \begin{pmatrix} -\sum y_{t-1}e_t - c\sum y_{t-1}e_{t-1} \\ \sum u_{t-1}e_t + c\sum u_{t-1}e_{t-1} \end{pmatrix}$$

Notice that Φ^T and ε are not independent. u_t and e_t are independent, y_t depends on e_t, e_{t-1}, $e_{t-2}, \ldots$ and y_t depends on $u_{t-1}, u_{t-2}, \ldots$.
Taking mean values we get

$$E(\hat{\theta} - \theta) = E(\Phi^T\Phi)^{-1}E(\Phi^T\varepsilon)$$

To evaluate this expression we calculate

$$E\begin{pmatrix} \sum y_{t-1}^2 & -\sum y_{t-1}u_{t-1} \\ -\sum y_{t-1}u_{t-1} & \sum u_{t-1}^2 \end{pmatrix} = N\begin{pmatrix} Ey_t^2 & 0 \\ 0 & Eu_t^2 \end{pmatrix}$$

and

$$E\begin{pmatrix} -\sum y_{t-1}e_t - c\sum y_{t-1}e_{t-1} \\ \sum u_{t-1}e_t + c\sum u_{t-1}e_{t-1} \end{pmatrix} = \begin{pmatrix} -cNEy_{t-1}e_{t-1} \\ 0 \end{pmatrix}$$

$$Ey_{t-1}e_{t-1} = E\left(ay_{t-2} + bu_{t-2} + e_{t-1}\right)e_{t-1} = \sigma^2$$

Since

$$y_t = \frac{b}{q+a}u_t + \frac{q+c}{q+a}e_t$$

$$Ey_t^2 = \frac{xb^2}{1-a^2} + \frac{1-2ac+c^2}{1-a^2}\sigma^2$$

$$Eu_t^2 = 1 \qquad Ee_t^2 = \sigma^2$$

we get

$$E(\hat{a}-a)^2 = -\frac{\sigma^2 c(1-a^2)}{b^2 + (1-2ac+c^2)\sigma^2}$$

$$E(\hat{b}-b)^2 = 0$$

2.8 The model is

$$y(t) = a + bt + e(t) = \varphi^T\theta + e(t)$$

$$\varphi = \begin{pmatrix} 1 \\ t \end{pmatrix} \qquad \theta = \begin{pmatrix} a \\ b \end{pmatrix}$$

According to Theorem 2.1 the solution is given by equation (2.6), i.e.

$$\hat{\theta} = (\Phi^T\Phi)^{-1}\Phi^T Y$$

where

$$\Phi^T = \begin{pmatrix} 1 & 1 & 1 & & 1 \\ 1 & 2 & 3 & \cdots & N \end{pmatrix} \qquad Y = \begin{pmatrix} y(1) \\ y(2) \\ \vdots \\ y(N) \end{pmatrix}$$

Hence

$$\hat{\theta} = \begin{pmatrix} \sum_{t=1}^{N} 1 & \sum_{t=1}^{N} t \\ \sum_{t=1}^{N} t & \sum_{t=1}^{N} t^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^{N} y(t) \\ \sum_{t=1}^{N} ty(t) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{2}{N(N-1)}\left((2N+1)s_0 - 3s_1\right) \\ \dfrac{6}{N(N+1)(N-1)}\left(-(N+1)s_0 + 2s_1\right) \end{pmatrix}$$

where we have made use of

$$\sum_{t=1}^{N} t = \frac{N(N+1)}{2} \qquad \sum_{t=1}^{N} t^2 = \frac{N(N+1)(N+2)}{6}$$

and introduced

$$s_0 = \sum_{t=1}^{N} y(t) \qquad s_1 = \sum_{t=1}^{N} ty(t)$$

The covariance of the estimate is given by

$$\operatorname{cov}(\hat{\theta}) = \sigma^2(\Phi^T\Phi)^{-1} = \frac{12}{N(N+1)(N-1)} \begin{pmatrix} \frac{(N+1)(2N+1)}{6} & -\frac{N+1}{2} \\ -\frac{N+1}{2} & 1 \end{pmatrix}$$

Notice that the variance of $\hat{b}$ decreases as N^{-3} for large N but the variance of $\hat{a}$ decreases as N^{-1}. The reason for this is that the regressor associated with a is 1 but the regressor associated with b is t. Notice that there are better numerical methods to solve for $\hat{\theta}$!

2.17

(a) The following derivation gives a formula for the asymptotic LS estimate

$$\begin{aligned}
\hat{b} = (\Phi^T\Phi)^{-1}\Phi Y &= \Big(\sum_{k=1}^{N} \phi(k-1)^2\Big)^{-1}\Big(\sum_{k=1}^{N} \phi(k-1)\bar{y}(k)\Big) \\
&= \Big(\frac{1}{N}\sum_{k=1}^{N} u(k-1)^2\Big)^{-1}\Big(\frac{1}{N}\sum_{k=1}^{N} u(k-1)\bar{y}(k)\Big) \\
&\to \big(E(u(k-1)^2)\big)^{-1} E\left(u(k-1)\bar{y}(k)\right), \quad \text{as} \quad N \to \infty
\end{aligned}$$

The equations for the closed loop system are

$$\begin{aligned}
u(k) &= K\left(u_c(k) - y(k)\right) \\
\bar{y}(k) &= y(k) + ay(k-1) = bu(k-1)
\end{aligned}$$

The signals $u(k)$ and $y(k)$ are stationary signals. This follows since the controller gain is chosen so that the closed loop system is stable. It then follows that $E(u(k-1)^2) = E(u(k)^2)$ and $E(u(k-1)\bar{y}(k)) = E(bu(k-1)^2) = bE(u(k)^2)$ exist and the asymptotic LS estimate becomes $\hat{b} = b$, i.e. we have an unbiased estimate.

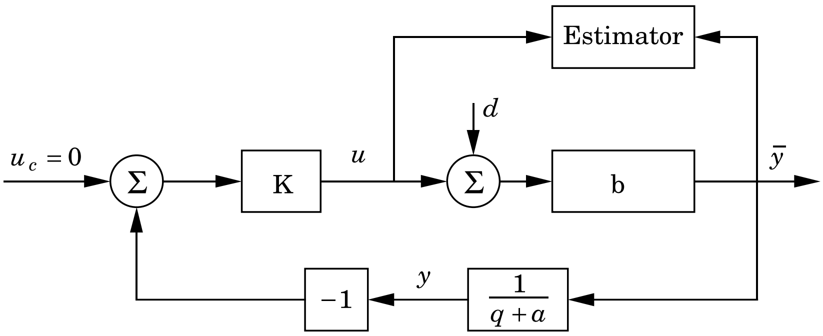

Figure 5. The system redrawn.

(b) Similarly to (a), we get

$$\left(\frac{1}{N}\sum_{k=1}^{N} u(k-1)^2\right)^{-1}\left(\frac{1}{N}\sum_{k=1}^{N} u(k-1)\bar{y}(k)\right)$$
$$\to \left((u^2(k-1))_0\right)^{-1}(u(k-1)\bar{y}(k))_0, \quad \text{as} \quad N\to\infty$$

where $(\cdot)_0$ denote the stationary value of the argument. We have

$$\begin{aligned}(u^2(k-1))_0 &= ((u(k))_0)^2\\ (u(k-1)\bar{y}(k))_0 &= (u(k))_0\, b\,((u(k)_0 + d_0)\\ (u(k))_0 &= H_{ud}(1)d_0 = -\frac{Kb}{1+a+Kb}\,d_0\end{aligned}$$

and the asymptotic LS estimate becomes

$$\begin{aligned}\hat{b} &= \left((u^2(k-1))_0\right)^{-1}(u(k-1)\bar{y}(k))_0 = b\left(1+\frac{d_0}{(u(k))_0}\right)\\ &= b\left(1-\frac{1+a+Kb}{Kb}\right) = \frac{1+a}{K}\end{aligned}$$

How do we interpret this result? The system may be redrawn as in Figure 5. Since $U_c = 0$, we have that $u = \frac{K}{q+a}\,\bar{y}$, and we can regard $\frac{K}{q+a}$ as the controller for the system in Figure 5. It is then obvious that we have estimated the negative inverse of the static controller gain.

(c) Introduction of high pass regressor filters as in Figure 6 eliminates or at least reduces the influence from the disturbance d on the estimate of b. One choice of regressor filter could be $H_f(q^{-1}) = 1 - q^{-1}$, i.e. a differentiator. Another possibility would be to introduce a constant in

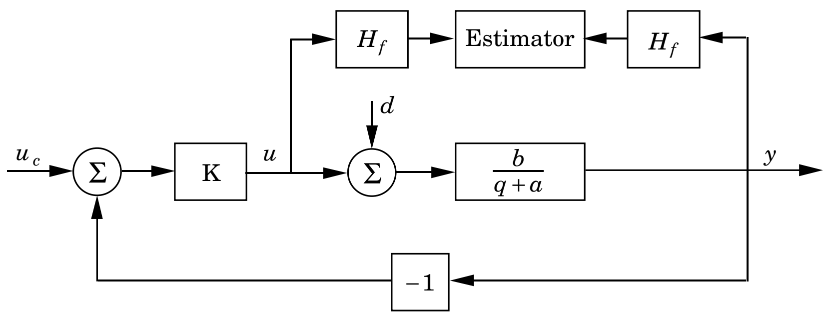

Figure 6. Introduction of regressor filters.

the regressor and then estimate both b and bd. The regression model is in this case

$$\bar{y}(t) = \begin{pmatrix} u(t-1) & 1 \end{pmatrix} \begin{pmatrix} b \\ bd \end{pmatrix} = \phi(t)^T\theta$$

2.18 The equations for recursive least squares are

$$\begin{aligned} y(t) &= \varphi^T(t-1)\theta^0 \\ \hat{\theta}(t) &= \hat{\theta}(t-1) + K(t)\varepsilon(t) \\ \varepsilon(t) &= y(t) - \varphi^T(t-1)\hat{\theta}(t-1) \\ K(t) &= P(t)\varphi(t-1) \\ &= \frac{P(t-1)\varphi(t-1)}{\lambda + \varphi^T(t-1)P(t-1)\varphi(t-1)} \\ P(t) &= \big(I - K(t)\varphi^T(t-1)\big)P(t-1)/\lambda \end{aligned}$$

Since the quantity $P(t)\varphi(t-1)$ appears in many places it is convenient to introduce it as an auxiliary variable $w = P\varphi$. The following computer code is then obtained:

```
Input
          u,y: real
Parameter lambda: real
State     phi, theta: vector
          P: symmetric matrix

Local variables w: vector, den : real
"Compute residual
e=y-phi^T*theta
"update estimate
w=P*phi
den=w^T*phi+lambda
```

```
theta=theta+w*e/den
"Update covariance matrix
P=(P-w*w^T/den)/lambda
"Update regression vectors
phi=shift(phi)
phi(1)=-y
phi(n+1)=u
```

SOLUTIONS TO CHAPTER 3

3.1 Given the process

$$H(z) = \frac{B(z)}{A(z)} = \frac{z + 1.2}{z^2 - z + 0.25}$$

Design specification: The closed system should have a pole that correspond to following characteristic polynomial in continuous time

$$s^2 + 2s + 1 = (s + 1)^2$$

This corresponds to

$$A_m(z) = z^2 + a_{m1}z + a_{m2}$$

with

$$\begin{cases} a_{m1} = & -(e^{-1} + e^{-1}) = -2e^{-1} \\ a_{m2} = & e^{-2} \end{cases}$$

(a) Determine an indirect STR of minimal order. The controller should have an integrator and the stationary gain should be 1.
Solution:
Choose B_m such that

$$\frac{B_m(1)}{A_m(1)} = 1$$

The integrator condition gives

$$R = R^{'}(z - 1)$$

We get the following conditions

(1) $$BT = B_m A_o$$

(2) $$AR + BS = A_m A_o$$

As B is unstable must $B_m = BB^{'}_m$. This makes (1) $\Leftrightarrow$ $BT = BB^{'}_m A_o \Leftrightarrow T = B^{'}_m A_o$. Choose $B^{'}_m$ such that

$$\frac{B(1)B^{'}_m(1)}{A(1)} = 1 \Rightarrow B^{'}_m(1) = \frac{A(1)}{B(1)}$$

The simplest way is to choose

$$B^{'}_m = b^{'}_m = \frac{A(1)}{B(1)} = \frac{0.25}{2.2}$$

Further we have

$$(z^2 + a_1z + a_2)(z - 1)(z + r) + (b_0z + b_1)(s_0z^2 + s_1z + s_2)$$
$$= (z^2 + a_{m1}z + a_{m2})(z^2 + a_{o1}z + a_{o2})$$

with $a_1 = -1$, $a_2 = 0.25$ and a_{o1} and a_{o2} chosen so that A_o is stable. Equating coefficients give

$$\begin{cases} r - 1 + a_1 + b_0 s_0 = a_{o1} + a_{m1} \\ -r + a_1(r-1) + a_2 + b_0 s_1 + b_1 s_0 = a_{o2} + a_{m1}a_{o1} + a_{m2} \\ -ar + a_2(r-1) + b_0 s_2 + b_1 s_1 = a_{m1}a_{o2} + a_{m2}a_{o1} \\ -a_2 r + b_1 s_2 = a_{m2}a_{o2} \end{cases}$$

or

$$\begin{pmatrix} 1 & b_0 & 0 & 0 \\ a_1 - 1 & b_1 & b_0 & 0 \\ a_2 - a_1 & 0 & b_1 & b_0 \\ -a_2 & 0 & 0 & b_1 \end{pmatrix} \begin{pmatrix} r \\ s_0 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} a_{o1} + a_{m1} + 1 - a_1 \\ a_{a2} + a_{m1}a_{o1} + a_{m2} + a_1 - a_2 \\ a_{m1}a_{o2} + a_{m2}a_{o1} + a_2 \\ a_{m2}a_{o2} \end{pmatrix}$$

Now choose to estimate

$$\theta = \begin{pmatrix} b_1 & b_1 & a_1 & a_2 \end{pmatrix}^T$$

by equation 3.22 in the textbook.

(b) As $H(z)$ is not minimum phase we cancel $B^- = B$ between $\bar{R}$ and $\bar{S}$. This is difficult, see page 118 in the textbook. An indirect STR is given by Eq. 3.24.

$$A_o A_m y = \bar{R}u + \bar{S}y$$

with $\bar{R} = B^- R$, $\bar{S} = B^- S$, $T = B'_m A_o$. Furthermore we have

$$A_o A_m y_m = A_o B_m u_c = A_o B B'_m u_c = BTu_c = \bar{T}u_c$$

$$y = \bar{R} \underbrace{\frac{1}{A_o A_m} u}_{=u_f} + \bar{S} \underbrace{\frac{1}{A_o A_m} y}_{=y_f}$$

$$y_m = \bar{T} \underbrace{\frac{1}{A_o A_m} u_c}_{=u_{cf}}$$

$$\varepsilon = y - y_m = \bar{R}u_f + \bar{S}y_f - \bar{T}u_{cf}$$

Now estimate $\bar{R}$, $\bar{S}$ and $\bar{T}$ with a recursive method in the above equation. Then cancel B and calculate the control signal.

(c) Take $a = 0$ in Example 5.7, page 206 in the textbook. This gives

$$\begin{cases} \dfrac{dt_0}{dt} = -\gamma u_c e \\ \dfrac{ds_0}{dt} = -\gamma y e \end{cases}$$

with $e = y - y_m = y - G_m u_c$.

3.3 The process has the transfer function

$$G(s) = \frac{b}{s+a} \cdot \frac{q}{s+p}$$

where p and q are known. The desired closed loop system has the transfer function

$$G_m(s) = \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2}$$

Since a discrete time controller is used the transfer functions are sampled. We get

$$H(z) = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2}$$
$$H_m(z) = \frac{b_{m0} z + b_{m1}}{z^2 + a_{m1} z + a_{m2}}$$

The fact that p and q are known implies that one of the poles of H is known. This information will be disregarded. In the following we will assume

$$G(s) = \frac{1}{s(s+1)}$$

With $h = 0.2$ this gives

$$H(z) = \frac{0.0187(z+0.936)}{(z-1)(z-0.819)}$$

Furthermore we will assume that $\omega = 2$ and $\zeta = 0.7$.
Indirect STR:
The parameters of a general pulse transfer function of second order is estimated by recursive least squares (See page 51). We have

$$\theta = \begin{pmatrix} b_o & b_1 & a_1 & a_2 \end{pmatrix}^T$$
$$\varphi(t) = \begin{pmatrix} u(t-1) & u(t-2) & -y(t-1) & -y(t-2) \end{pmatrix}$$

The controller is calculated by solving the Diophantine equation. We look at two cases
1.B canceled:

$$(z^2 + a_1 z + a_2)1 + b_0(s_0 z + s_1) = z^2 + a_{m1} z + a_{m2}$$
$$z^1 : a_1 + b_0 s_0 = a_{m1} \qquad s_0 = \frac{a_{m1} - a_1}{b_0}$$
$$z^0 : a_2 + b_0 s_1 = a_{m2} \qquad s_1 = \frac{a_{m2} - a_2}{b_0}$$

The controller is thus given by

$$R(z) = z + b_1/b_0$$
$$S(z) = s_0 z + s_1$$
$$T(z) = t_0 z \qquad \text{where} \qquad t_0 = \frac{1 + a_{m1} + a_{m2}}{b_0}$$

2. B not canceled:
The design equation becomes

$$(z^2 + a_1 z + a_2)(z + r_1) + (b_0 z + b_1)(s_0 z + s_1) = (z^2 + a_{m1} z + a_{m2})(z + a_{o1})$$

Identification of coefficients of equal powers of z gives

$$z^2 : a_1 + r_1 + b_0 s_0 = a_{m1} + a_{o1}$$
$$z^1 : a_2 + a_1 r_1 + b_1 s_0 + b_0 s_1 = a_{m1} a_{o1} + a_{m2}$$
$$z^0 : a_2 r_1 + b_1 s_1 = a_{m2} a_{o1}$$

The solution to these linear equations is

$$r_1 = \frac{b_1^2 n_1 - b_0 b_1 n_2 + a_{o1} a_{m2} b_0^2}{b_0^2 a_2 - a_1 b_0 b_1 + b_1^2}$$
$$s_0 = \frac{n_1 - r_1}{b_0}$$
$$s_1 = \frac{b_0 n_2 - b_1 n_1 - r_1(a_1 b_0 - b_1)}{b_0^2}$$

where

$$n_1 = a_1 - a_{m1} - a_{o1}$$
$$n_2 = a_{m1} a_{o1} + a_{m2} - a_2$$

The solution exists if the denominator of r_1 is different from zero, which means that there is no pole-zero cancellation. It is helpful to have access to computer algebra for this problems e.g. Macsyma, Maple or Matematica! Figure 7 shows a simulation of the controller obtained when the polynomial B is canceled. Notice the "ringing" of the control signal which is typical for cancellation of a poorly damped zero. In this case we have $z = -0.936$. In Figure 8 we show a simulation of the controller with no cancellation. This is clearly the way to solve the problem.
Direct STR:
To obtain a direct self-tuning regulator we start with the design equation

$$AR + BS = A_m A_o B^+$$

Hence

$$B^+ A_m A_o y = ARy + BSy = BRu + BSy$$
$$y = R \underbrace{\left(\frac{B^-}{A_o A_m} u\right)}_{u_f} + S \underbrace{\left(\frac{B^-}{A_o A_m} y\right)}_{y_f}$$

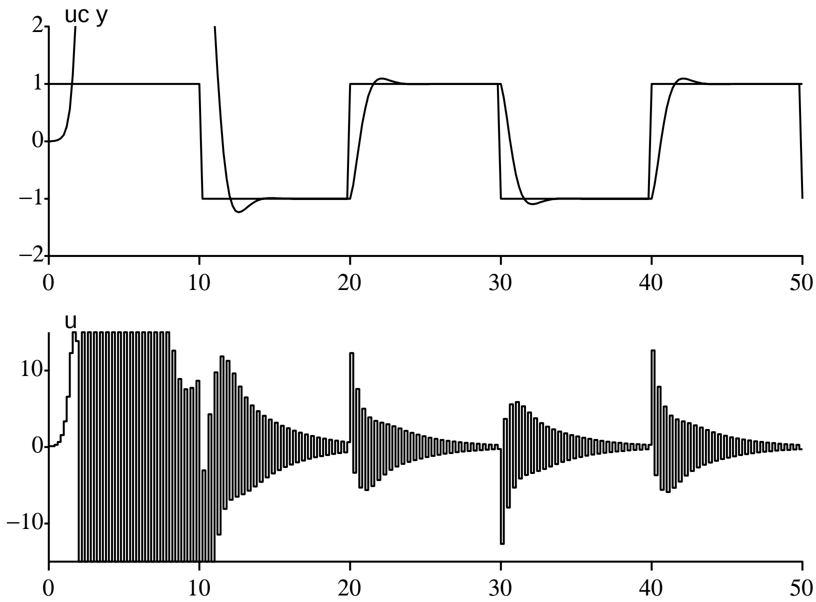

Figure 7. Simulation in Problem 3.3. Process output and control signal are shown for the indirect self-tuning regulator when the process zero is canceled.

From this model R and S can be estimated. The polynomial T is then given by

$$T = \frac{t_o A_o B_m}{B^-}$$

where t_o is chosen to give the correct steady state gain. Again we separate two cases.

1. Cancel B:

If the polynomial B is canceled we have $B^+ = z + b_1/b_0$, $B^- = b_0$. From the analysis of the indirect STR we know that no observer is needed in this case and that the controller has the structure $\deg R = \deg S = 1$. Hence

$$y(t) = R\left(\frac{b_o}{A_m}u(t)\right) + S\left(\frac{b_o}{A_m}y(t)\right)$$

Since b_o is not known we include it in the polynomial R and S and estimate it. The polynomial R then is not monic. We have

$$y(t) = (r_0 q + r_1)\left(\frac{1}{A_m}u(t)\right) + (s_0 q + s_1)\left(\frac{1}{A_m}y(t)\right)$$

To obtain a direct STR we thus estimate

$$\theta = \begin{pmatrix} r_0 & r_1 & s_0 & s_1 \end{pmatrix}^T$$

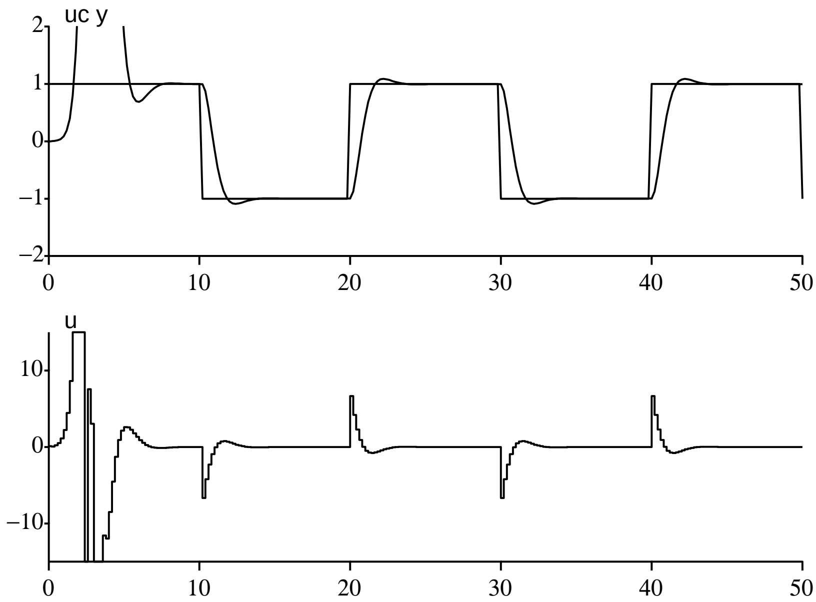

Figure 8. Simulation in Problem 3.3. Process output and control signal are shown for the indirect self-tuning regulator when the process zero is not canceled.

by RLS. The case $r_0 = 0$ must be taken care of separately. Furthermore T has the form $T(q) = t_0 q$ where

$$\frac{BT}{AR + BS} = \frac{Bt_o q}{\underbrace{B^+ b_o}_{B} A_m} = \frac{t_o q}{A_m}$$

To get unit steady state gain choose

$$t_o = A_m(1)$$

A simulation of the system is shown in Fig. 9. We see the typical ringing phenomena obtained with a controller that cancels a poorly damped zero. To avoid this we will develop an algorithm where the process zero is not canceled.

2. No cancellation of process zero:

We then have $B^+ = 1$ and $B^- = b_0 q + b_1$. From the analysis of the indirect STR we know that a first order observer is required, i.e. $A_0 = q + a_{o1}$. We have as before

$$y = R \underbrace{\left(\frac{B^-}{A_o A_m} u \right)}_{u_f} + S \underbrace{\left(\frac{B^-}{A_o A_m} y \right)}_{y_f} \qquad (*)$$

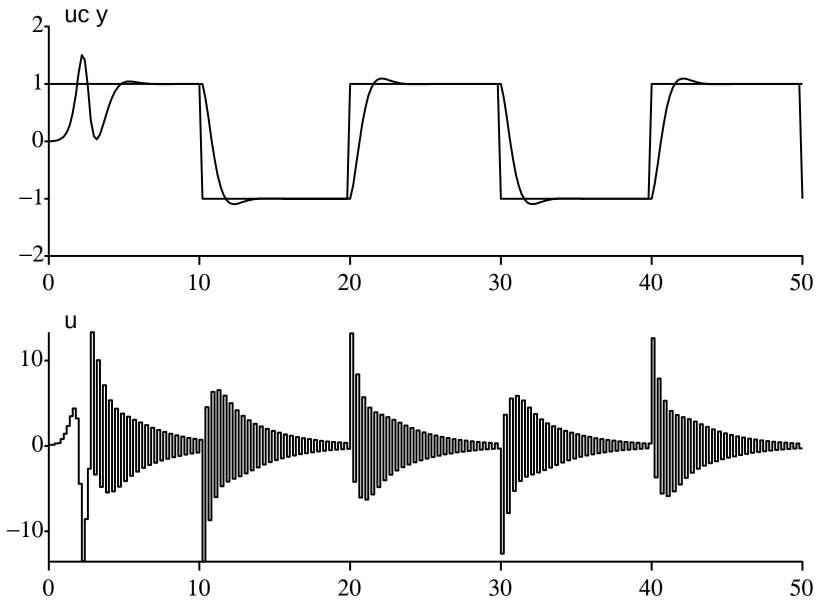

Figure 9. Simulation in Problem 3.3. Process output and control signal are shown for the direct self-tuning regulator when the process zero is canceled.

Since B^- is not known we can, however, not calculate u_f and y_f. One possibility is to rewrite (*) as

$$y = \underbrace{RB^-}_{R'}\left(\frac{1}{A_oA_m}u\right) + \underbrace{SB^-}_{S'}\left(\frac{1}{A_oA_m}y\right)$$

and to estimate R' and S' as second order polynomials and to cancel the common factor B^- from the estimated polynomials. This is difficult because there will not be an exact cancellation. Another possibility is to use some estimate of B^-. A third possibility is to try to estimate B^-R and B^-S as a bilinear problem. In Fig. 8–11 we show simulation when the model (*) is used with

$$B^- = 1$$

$$B^- = q$$

$$B^- = \frac{q + 0.4}{1.4}$$

$$B^- = \frac{q - 0.4}{0.6}$$

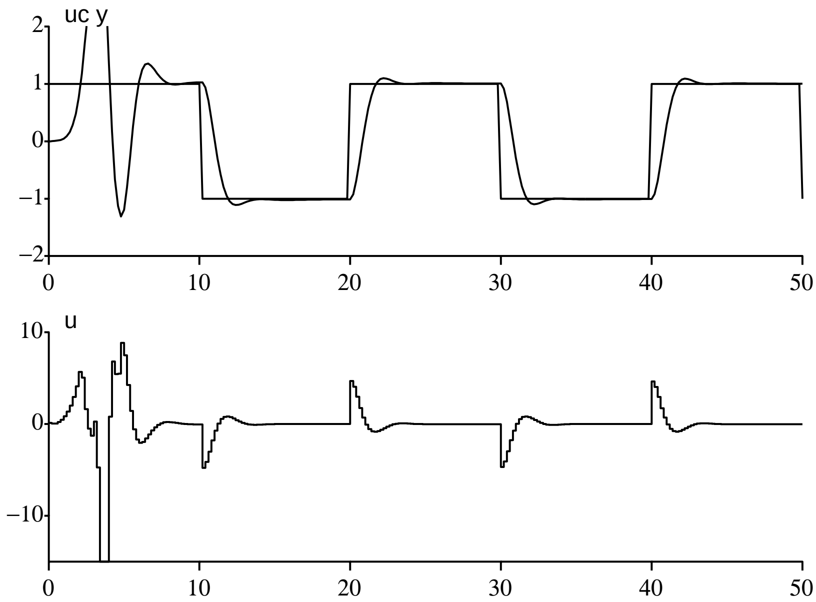

Figure 10. Simulation in Problem 3.3. Process output and control signal are shown for the direct self-tuning regulator when the process zero is not canceled and when $B^- = 1$.

3.4 The process has the transfer function

$$G(s) = \frac{b}{s(s+1)}$$

with proportional feedback

$$u = k(u_c - y)$$

we get the closed loop transfer function

$$G_{cl}(s) = \frac{kb}{s^2 + s + kb}$$

The gain $k = 1/b$ gives the desired result. Idea for STR: Estimate b and use $k = 1/\hat{b}$. To estimate b introduce

$$s(s+1)y = bu$$

$$\underbrace{\frac{s(s+1)}{(s+a)^2}\, y}_{y_f} = b \underbrace{\frac{1}{(s+a)^2}\, u}_{\varphi}$$

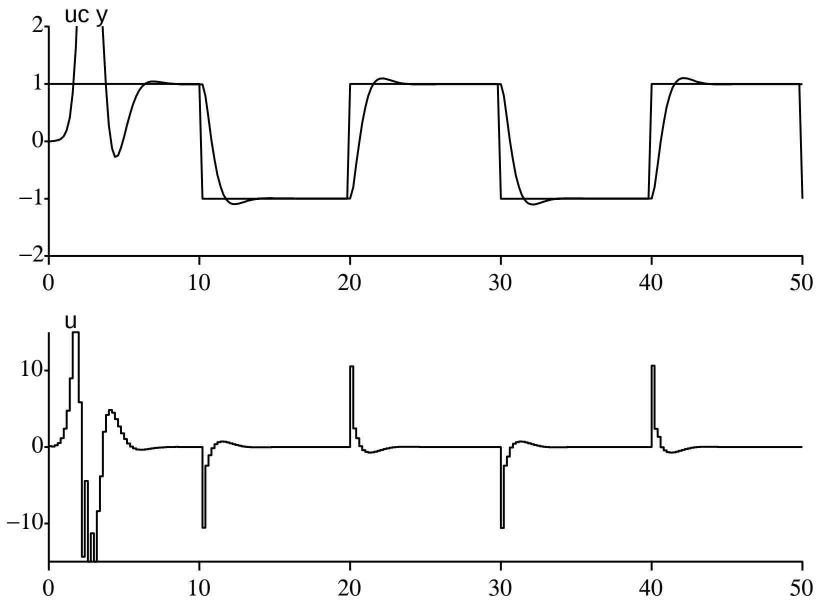

Figure 11. Simulation in Problem 3.3. Process output and control signal are shown for the direct self-tuning regulator when the process zero is not canceled and when $B^- = q$.

The equations on page 56 gives

$$\frac{d\hat{b}}{dt} = P\varphi e = P\varphi(y_f - \hat{b}\varphi)$$

$$\frac{dP}{dt} = \alpha P - P\varphi\varphi^T P = \alpha P - P^2\varphi^2$$

With $\hat{b}(0) = 1$, $P(0) = 100$, $\alpha = -0.1$, and $a = 1$ we get the result shown in Fig. 14.

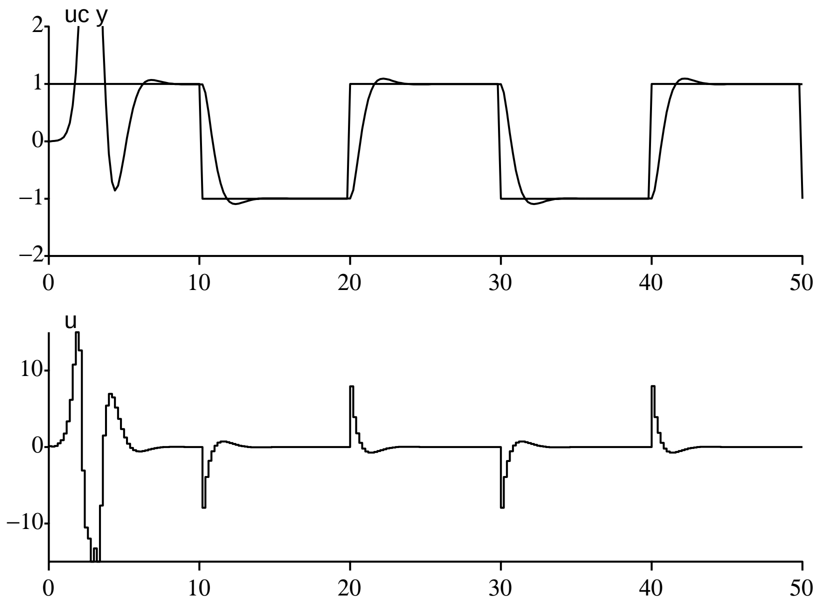

Figure 12. Simulation in Problem 3.3. Process output and control signal are shown for the direct self-tuning regulator when the process zero is not canceled and when $B^- = (q + 0.4)/1.4$.

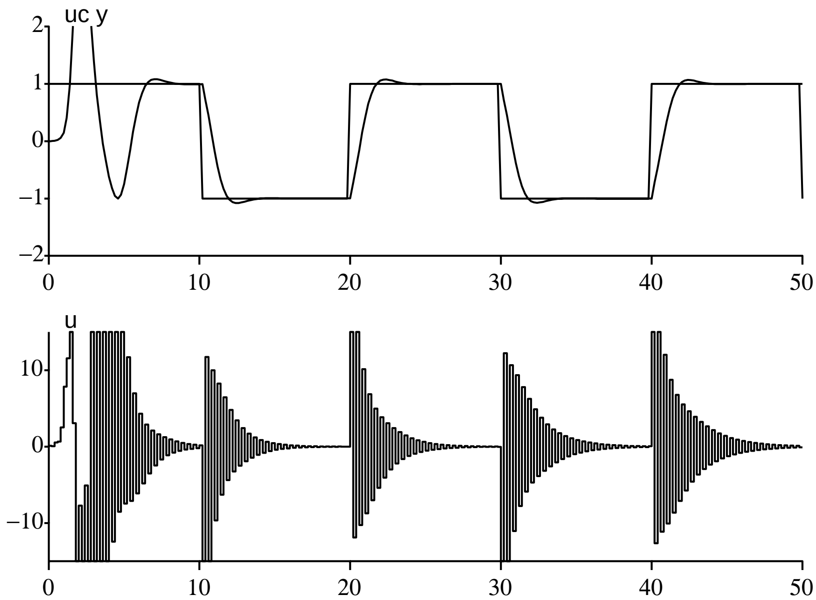

Figure 13. Simulation in Problem 3.3. Process output and control signal are shown for the direct self-tuning regulator when the process zero is not canceled and when $B^- = (q - 0.4)/0.6$.

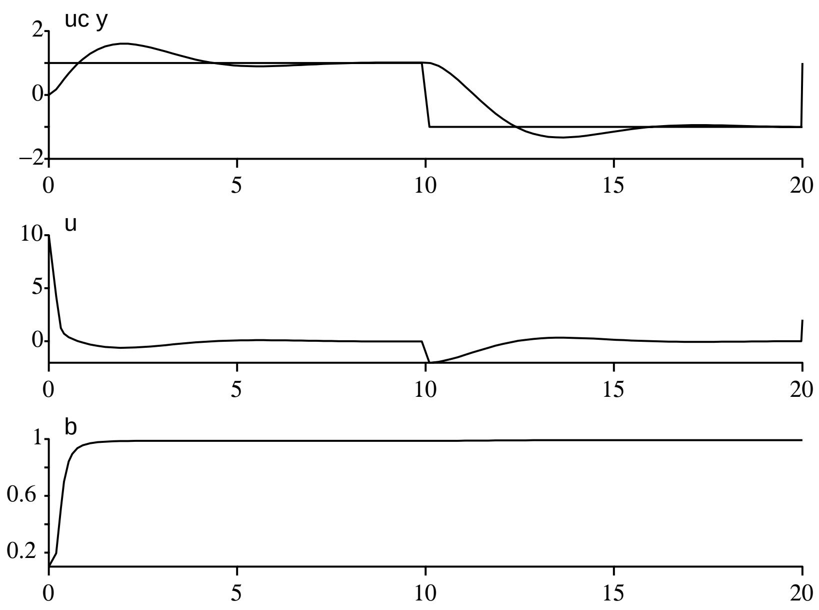

Figure 14. Simulation in Problem 3.4. Process output, control signal and estimated parameter b are shown for the indirect continuous-time self-tuning regulator.

SOLUTIONS TO CHAPTER 4

4.1 The estimate $\hat{b}$ may be small because of a poor estimate. One possibility is to use a projection algorithm where the estimate is restricted to be in a given range, $b_o \le \hat{b} \le b_1$. This requires prior information about b:s values. Another possibility is to replace

$$\frac{1}{\hat{b}} \qquad by \qquad \frac{\hat{b}}{\hat{b}^2 + P}$$

where P is the variance of the estimate. Compare with the discussion of cautious control on pages 356–358.

4.10 Using (4.21) the output can be written as

$$\begin{aligned} y_{t+d} &= \frac{R^*}{C^*} u_t + \frac{S^*}{C^*} y_t + \frac{R_1^*}{C^*} e_{t+d_o} \\ &= r_o u_t + f_t \end{aligned} \qquad (*)$$

Consider minimization of

$$J = y_{t+d}^2 + \rho u_t^2 \qquad (+)$$

Introduce the expression (*)

$$\begin{aligned} J &= (r_o u_t + f_t)^2 + \rho u_t^2 \\ &= (r_o^2 + \rho) u_t^2 + 2 r_o u_t f_t + f_t^2 \\ &= (r_o^2 + \rho)\left(u_t^2 + \frac{2 r_o u_t f_t}{r_o^2 + \rho}\right) + f_t^2 \\ &= (r_o^2 + \rho)\left(u_t + \frac{r_o f_t}{r_o^2 + \rho}\right)^2 - \frac{r_o^2 f_t^2}{r_o^2 + \rho} + f_t^2 \end{aligned}$$

Hence

$$\begin{aligned} J &= \frac{1}{r_o^2 + \rho}\left(f_t + \frac{r_o^2 + \rho}{r_o} u_t\right)^2 + \frac{\rho}{r_o^2 + \rho} f_t^2 \\ &= \frac{1}{r_o^2 + \rho}\left(f_t + \left(r_o + \frac{\rho}{r_o}\right) u_t\right)^2 + \frac{\rho}{r_o^2 + \rho} f_t^2 \\ &= \frac{1}{r_o^2 + \rho}\left(y_{t+d} + \frac{\rho}{r_o} u_t\right)^2 + \frac{\rho}{r_o^2 + \rho} f_t^2 \end{aligned}$$

Since $r_o^2 + \rho$ is a constant we find that minimizing (+) is the same as to minimize

$$J_1 = y_{t+d} + \frac{\rho}{r_o} u_t = f_t + \left(r_o + \frac{\rho}{r_o}\right) u_t$$

SOLUTIONS TO CHAPTER 5

5.1 The plant is

$$G(s) = \frac{1}{s(s+a)} = \frac{B}{A}$$

The desired response is

$$G_m(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2} = \frac{B_m}{A_m}$$

(a) Gradient method or MIT rule. Use formulas similar to (5.9). In this case $B^+ = 1$ and A_o is of first order. The regulator structure is

$$R = s + r_1 \quad S = s_0 s + s_1 \quad T = t_0 A_o$$

This gives the updating rules

$$\frac{dr_1}{dt} = \gamma e \left(\frac{1}{A_o A_m} u \right)$$

$$\frac{ds_0}{dt} = \gamma e \left(\frac{p}{A_o A_m} y \right)$$

$$\frac{ds_1}{dt} = \gamma e \left(\frac{1}{A_o A_m} y \right)$$

$$\frac{dt_0}{dt} = -\gamma e \left(\frac{1}{A_o A_m} u_c \right)$$

(b) Stability theory approach 1. First derive the error equation. If all process zeros are cancelled we have

$$\begin{aligned} A_o A_m y &= A R_1 y + b_o S y = B R_1 u + b_o S y \\ &= b_o (Ru + Sy) \end{aligned}$$

Further

$$A_o A_m y_m = A_o B_m u_c = b_o T u_c$$

Hence

$$A_o A_m e = A_o A_m (y - y_m) = b_o (Ru + Sy - Tu_c)$$

$$e = \frac{b}{A_o A_m} (Ru + Sy - Tu_c)$$

Since $1/A_o A_m$ is not SPR we introduce a polynomial D such that $D/A_o A_m$ is SPR. We then get

$$e = \frac{bD}{A_o A_m} \left(Ru_f + Sy_f - Tu_{cf} \right)$$

where

$$u_f = \frac{1}{D} u \qquad y_f = \frac{1}{D} y \qquad u_{cf} = \frac{1}{D} u_c$$

(c) Stability theory approach 2. In this case we assume that all states measurable. Process model:

$$\dot{x} = \underbrace{\begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix}}_{A_p} x + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B_p}$$

$$y = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{C} x$$

Control law

$$u = L_r u_c - Lx = \theta_3 u_c - \theta_1 x_1 - \theta_2 x_2$$

The closed loop system is

$$\begin{aligned} \dot{x} &= (A_p - B_p L)x + B L_r u_c = Ax + Bu_c \\ y &= Cx \end{aligned}$$

where

$$\begin{aligned} A(\theta) &= A_p - B_p L = \begin{pmatrix} -a - \theta_1 & -\theta_2 \\ 1 & 0 \end{pmatrix} \\ B(\theta) &= B_p L_r = \begin{pmatrix} \theta_3 \\ 0 \end{pmatrix} \end{aligned}$$

The desired response is given by

$$\dot{x}_m = A_m x_m + B_m u_c$$

where

$$A_m = \begin{pmatrix} -2\zeta\omega & -\omega^2 \\ 1 & 0 \end{pmatrix} \qquad B_m = \begin{pmatrix} \omega^2 \\ 0 \end{pmatrix}$$

We have

$$\begin{aligned} (A - A_m)^T &= \begin{pmatrix} 2\zeta\omega - a - \theta_1 & 0 \\ \omega^2 - \theta_2 & 0 \end{pmatrix} \\ (B - B_m)^T &= \begin{pmatrix} \theta_3 - \omega^2 & 0 \end{pmatrix} \end{aligned}$$

Introduce the state error

$$e = x - x_m$$

we get

$$\begin{aligned} \dot{e} &= x - x_m = Ax + Bu_c - A_m x_m - B_m u_c \\ &= A_m e + (A - A_m)x + (B - B_m)u_c \end{aligned} \qquad (*)$$

The error goes to zero if A_m is stable and

$$A(\theta) - A_m = 0 \qquad (*)$$

$$B(\theta) - B_m = 0 \qquad (+)$$

It is thus necessary that $A(\theta)$ and $B(\theta)$ are such that there is a θ for which (*) and (+) hold. Introduce the Lyapunov function

$$V = e^T Pe + tr(A - A_m)^T Q_a (A - A_m) + tr(B - B_m)^T Q_b (B - B_m)$$

Notice that

$$tr(A + B) = trA + trB$$
$$x^T Ax = tr(xx^T A) = tr(Axx^T)$$
$$tr(AB) = tr(BA)$$

we get

$$\frac{dV}{dt} = tr\Big(P\dot{e}e^T + Pe\dot{e}^T + \dot{A}^T Q_a (A - A_m) + (A - A_m)^T Q_a \dot{A} + \dot{B} Q_b (B - B_m) + (B - B_m)^T Q_b \dot{B}\Big) \qquad (+)$$

But from (*)

$$P\dot{e}e^T = P\left(A_m e + (A - A_m)x + (B - B_m)u_c\right)e^T$$
$$Pe\dot{e}^T = Pe\left(A_m e + (A - A_m)x + (B - B_m)u_c\right)^T$$

Introducing this into (+) and collecting terms proportional to $(A - A_m)^T$ we find that they are

$$2tr(A - A_m)^T \left(Q_a \dot{A} + Pex^T\right)$$

Similarly we find that terms proportional to $(B - B_m)$ are

$$2tr(B - B_m)^T \left(Q_b \dot{B} + Peu_c^T\right)$$

Hence

$$\frac{dV}{dt} = e^T PA_m e + e^T A_m^T Pe + 2tr(A - A_m)^T \left(Q_a \dot{A} + Pex^T\right) + 2tr(B - B_m)^T \left(Q_b \dot{B} + Peu_c^T\right)$$

Hence if the symmetric matrix P is chosen so that

$$A_m^T P + PA_m = -Q$$

where Q is positive definite (can always be done if A_m is stable!) and parameters are updated so that

$$\begin{cases} (A - A_m)^T \left(Q_a \dot{A} + Pex^T\right) = 0 \\ (B - B_m)^T \left(Q_b \dot{B} + Peu_c^T\right) = 0 \end{cases} \qquad (+)$$

we get

$$\frac{dV}{dt} = -e^T Qe$$

The equations for updating the parameters derived above can now be used. This gives

$$\begin{pmatrix} 2\zeta\omega - a - \theta_1 & 0 \\ \omega^2 - \theta_2 & 0 \end{pmatrix} \left(Q_a \begin{pmatrix} -\dot{\theta}_1 & -\dot{\theta}_2 \\ 0 & 0 \end{pmatrix} + Pex^T \right) = 0$$

$$\begin{pmatrix} \theta_3 - \omega^2 & 0 \end{pmatrix} \left(Q_b \begin{pmatrix} \dot{\theta}_3 \\ 0 \end{pmatrix} + Peu_c \right) = 0$$

Hence with $Q_a = I$ and $Q_b = 1$

$$\frac{d\theta_1}{dt} = p_{11}e_1x_1 + p_{12}e_2x_1 = (p_{11}e_1 + p_{12}e_2)x_1$$

$$\frac{d\theta_2}{dt} = p_{11}e_1x_2 + p_{12}e_2x_2 = (p_{11}e_1 + p_{12}e_2)x_2$$

$$\frac{d\theta_3}{dt} = -(p_{11}e_1 + p_{12}e_2)u_c$$

where $e_1 = x_1 - x_{m1}$ and $e_2 = x_2 - x_{m2}$. It now remains to determine P such that

$$A_m^T P + PA_m = -Q$$

Choosing $\zeta = 0.707, \omega = 2$ and

$$Q = \begin{pmatrix} 41.2548 & 11.3137 \\ 11.3137 & 16.0000 \end{pmatrix}$$

we get

$$P = \begin{pmatrix} 4 & 2 \\ 2 & 16 \end{pmatrix}$$

The parameter update laws become

$$\frac{d\theta_1}{dt} = (4e_1 + 2e_2)x_1$$

$$\frac{d\theta_2}{dt} = (4e_1 + 2e_2)x_2$$

$$\frac{d\theta_3}{dt} = -(4e_1 + 2e_2)u_c$$

A simulation of the system is given in Fig. 15.

5.2 The block diagram of the system is shown in Fig. 16. The PI version of the SPR rule is

$$\frac{d\theta}{dt} = -\gamma_1 \frac{d}{dt}(u_c e) - \gamma_2 u_c e \qquad (*)$$

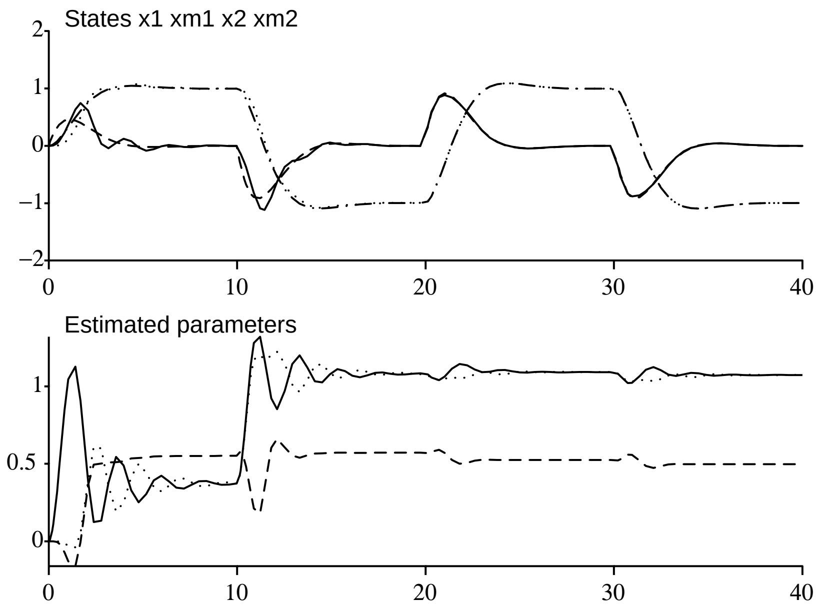

Figure 15. Simulation in Problem 5.1. Top: Process and model states, x_1 (full), x_{m1} (dashed), x_2 (dotted), and x_{m2} (dash-dotted). Bottom: Controller parameters θ_3 (full), θ_1 (dashed), and θ_2 (dotted).

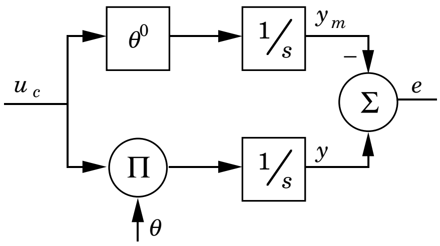

Figure 16. Block diagram in Problem 5.2.

To derive the error equation we notice that

$$\frac{dy_m}{dt} = \theta^0 u_c$$
$$\frac{dy}{dt} = \theta u_c$$

Hence

$$\frac{de}{dt} = (\theta - \theta^0)u_c$$

we get

$$\frac{d^2e}{dt^2} = \frac{d\theta}{dt}u_c + (\theta - \theta^0)\frac{du_c}{dt}$$

Inserting the parameter update law (*) into this we get

$$\frac{d^2e}{dt^2} = -\gamma_1\left(\frac{du_c}{dt}e + u_c\frac{de}{dt}\right)u_c - \gamma_2 u_c^2 e + (\theta - \theta^0)\frac{du_c}{dt}$$

Hence

$$\frac{d^2e}{dt^2} + \gamma_1 u_c^2\frac{de}{dt} + \left(\gamma_1 u_c\frac{du_c}{dt} + \gamma_2 u_c^2\right)e = (\theta - \theta^0)\frac{du_c}{dt}$$

Assuming that u_c is constant we get the following error equation

$$\frac{d^2e}{dt^2} + \gamma_1 u_c^2\frac{de}{dt} + \gamma_2 u_c^2 e = 0$$

Assuming that we want this to be a second order system with ω and ζ we get

$$\begin{cases} \gamma_1 u_c^2 = 2\zeta\omega \\ \gamma_2 u_c^2 = \omega^2 \end{cases} \qquad \begin{aligned} \gamma_1 &= 2\zeta\omega/u_c^2 \\ \gamma_2 &= \omega^2/u_c^2 \end{aligned}$$

This gives an indication of how the parameters γ_1 and γ_2 should be selected. The analysis was based on the assumption that u_c was constant. To get some insight into what happens when u_c changes we will give a simulation where u_c is a triangular wave with varying period. The adaptation gains are chosen for different ω and ζ. Figure 17 shows what happens when the period of the square wave is 20 and $\omega = 0.5$, 1 and 2. Corresponding to the periods 12, 6 and 3. Figure 18 show what happen when u_c is changed more rapidly.

5.6 The transfer function is

$$G(s) = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$$

The transfer function has no poles and zeros in the right half plane if $a_1 \geq 0$, $a_2 \geq 0$, $b_0 \geq 0$, $b_1 \geq 0$, and $b_2 \geq 0$. Consider

$$G(i\omega) = \frac{B(i\omega)}{A(i\omega)} \cdot \frac{A(-i\omega)}{A(-i\omega)}$$

The condition $\operatorname{Re} G(i\omega) \geq 0$ is equivalent to $\operatorname{Re}\left(B(i\omega)A(-i\omega)\right) \geq 0$. But

$$\begin{aligned} g(\omega) &= \operatorname{Re}\left((-b_0\omega^2 + i\omega b_1 + b_2)(-\omega^2 - i\omega a_1 + a_2)\right) \\ &= b_0\omega^4 + (a_1 b_1 - b_0 a_2 - b_2)\omega^2 + a_2 b_2 \end{aligned}$$

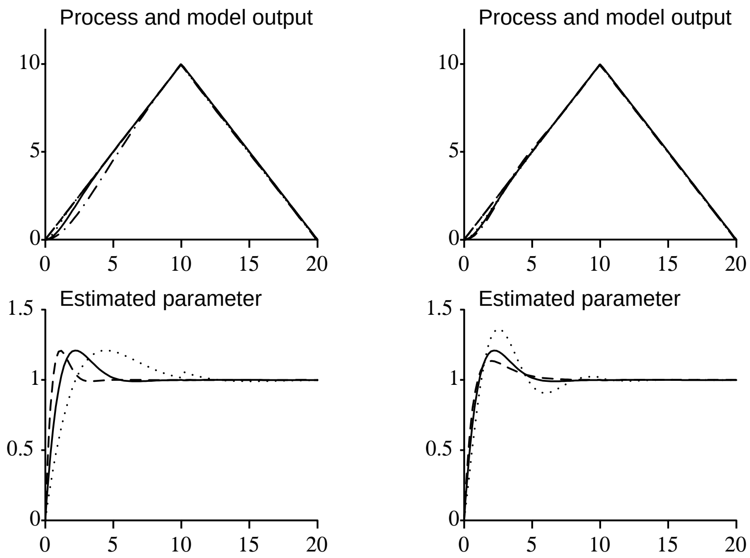

Figure 17. Simulation in Problem 5.2 for a triangular wave of period 20. Left top: Process and model outputs, Left bottom: Estimated parameter θ when $\omega = 0.5$ (full), 1 (dashed), and 2 (dotted) for $\zeta = 0.7$. Right top: Process and model outputs, Right bottom: Estimated parameter θ when $\zeta = 0.4$ (full), 0.7 (dashed), and 1.0 (dotted) for $\omega = 1$.

Completing the squares the function can be written as

$$g(\omega) = b_0 \left(\omega^2 + \frac{a_1b_1 - b_0a_2 - b_2}{2b_0}\right)^2 + a_2b_2 - \frac{(a_1b_1 - b_0a_2 - b_2)^2}{4b_0}$$

When $b_0 = 0$ the condition for g to be positive is that

$$a_1b_1 - b_0a_2 - b_2 \geq 0 \qquad (i)$$

If $b_0 > 0$ the function $g(\omega)$ is nonnegative for all ω if either (i) holds or if

$$a_1b_1 - b_0a_2 - b_2 < 0$$

and

$$a_2b_2 > \frac{(a_1b_1 - b_0a_2 - b_2)^2}{4b_0}$$

Example 1. Consider

$$G(s) = \frac{s^2 + 6s + 8}{s^2 + 4s + 3}$$

We have $a_1b_1 - b_0a_2 - b_2 = 24 - 3 - 8 = 13 > 0$. Hence the transfer function $G(s)$ is SPR. Example 2.

$$G(s) = \frac{3s^2 + s + 1}{s^2 + 3s + 4}$$

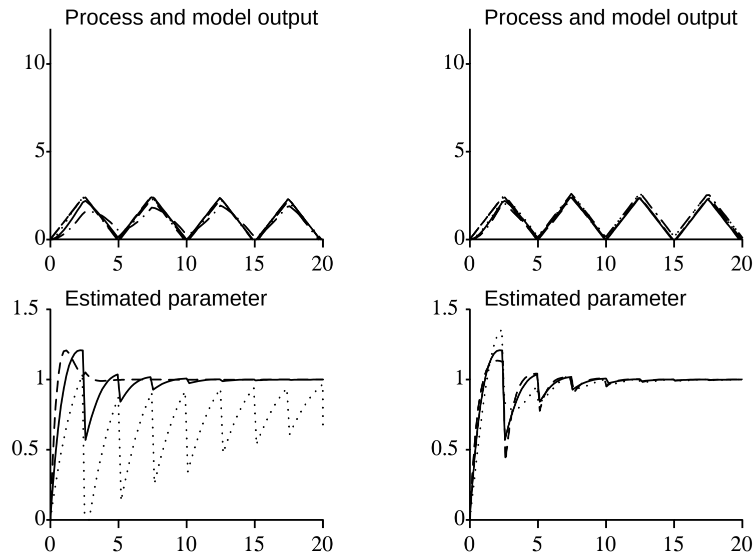

Figure 18. Simulation in Problem 5.2 for a triangular wave of period 5. Left top: Process and model outputs, Left bottom: Estimated parameter θ when $\omega = 0.5$ (full), 1 (dashed), and 2 (dotted) for $\zeta = 0.7$. Right top: Process and model outputs, Right bottom: Estimated parameter θ when $\zeta = 0.4$ (full), 0.7 (dashed), and 1.0 (dotted) for $\omega = 1$.

we have $a_1b_1 - a_2b_0 - b_2 = 3 - 12 - 1 = -10$. Furthermore

$$a_2b_2 = 4$$

$$\frac{(a_1b_1 - a_2b_0 - b_2)^2}{4b_0} = \frac{100}{12}$$

Hence the transfer function $G(s)$ is neither PR nor SPR.

5.7 Consider the system

$$\frac{dx}{dt} = Ax + B_1u$$
$$y = C_1x$$

where

$$B_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let Q be positive and let P be the solution of the Lyapunov equation

$$A^TP + PA = -Q \qquad (*)$$

Define C_1 as

$$C_1 = B^T P = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \end{pmatrix}$$

According to the Kalman-Yacobuvich Lemma the transfer function

$$G_1(s) = C_1(sI - A)^{-1} B_1$$

is then positive definite. This transfer function can, however, be written as

$$G(s) = \frac{p_{11}s^{n-1} + p_{12}s^{n-2} + \dots + p_{1n}}{s^n + a_1 s^{n-1} + \dots + a_n}$$

Since there are good numerical algorithms for solving the Lyapunov equation we can use this result to construct transfer functions that are SPR. The method is straightforward.

1. Choose A stable.
2. Solve (*) for given Q positive.
3. Choose B as

$$B(s) = p_{11}s^{n-1} + p_{12}s^{n-2} + \dots + p_{1n}$$

5.11 Let us first solve the underlying design problem for systems with known parameters. This can be done using pole placement. Let the plant dynamics be

$$y = \frac{B}{A} u$$

and let the controller be

$$Ru = Tu_c - sy$$

The basic design equation is then

$$AR + BS = A_m A_o \qquad (*)$$

In this case we have

$$\begin{aligned} A &= (s+a)(s+p) \\ B &= bq \\ A_m &= s^2 + 2\zeta\omega s + \omega^2 \end{aligned}$$

We need an observer of at least first order. The design equation (*) then becomes

$$(s+a)(s+p)(s+r_1) + bq(s_0 s + s_1) = (s^2 + 2\zeta\omega s + \omega^2)(s + a_0) \qquad (+)$$

where $A_o = s + a_o$ is the observer polynomial. The controller is thus of the form

$$\frac{du}{dt} + r_1 u = t_0 u_c - s_0 y - s_1 \frac{dy}{dt}$$

It has four parameters that have to be updated r_1, t_0, s_0, and s_1. If no prior knowledge is available we thus have to update these four parameters. When parameter p is known it follows from the design equation (+) that there is a relation between the parameters and it then suffices to estimate three parameters. This is particularly easy to see when the observer polynomial is chosen as $A_o(s) = s + p$. Putting $s = -p$ in (+) gives

$$-s_0 p + s_1 = 0$$

Hence

$$s_1 = ps_0 \qquad (**)$$

In this particular case we can thus update t_0, s_0, and r_1 and compute s_1 from (**). Notice, however, that the knowledge of q is of no value since q always appear in combination with the unknown parameter b. The equations for updating the parameters are derived in the usual way. With $a_0 = p$ equation (+) simplifies to

$$(s + a)(s + r_1) + bqs_0 = s^2 + 2\zeta\omega s + \omega^2$$

Introducing

$$A'(s) = s + a \qquad S'(s) = s_0 \qquad T' = t_0$$

we get

$$y = \frac{BT'}{A'R + BS'} u_c$$

$$\begin{aligned}
\frac{\partial e}{\partial t_0} &= \frac{B}{A'R + BS'} u_c \approx \frac{B}{A_m} u_c \\
\frac{\partial e}{\partial s_0} &= -\frac{BT'B}{(A'R + BS')^2} u_c = -\frac{B}{A'R + BS'} y \approx -\frac{B}{A_m} y \\
\frac{\partial e}{\partial r_1} &= -\frac{A'BT'}{(A'R + BS')^2} u_c = -\frac{A'}{A'R + BS'} y \approx -\frac{A'}{A_m} y \\
&= -\frac{A'}{A_m}\frac{B}{A} u = -\frac{B}{A_m(s + p)} u
\end{aligned}$$

The MIT rule then gives

$$\begin{aligned}
\frac{dr_1}{dt} &= \gamma \left(\frac{1}{(s + p)A_m} u \right) e \\
\frac{ds_0}{dt} &= \gamma \left(\frac{1}{A_m} y \right) e \\
\frac{dt_0}{dt} &= -\gamma \left(\frac{1}{A_m} u_c \right) e
\end{aligned}$$

A simulation of the controller is given in Fig. 19.

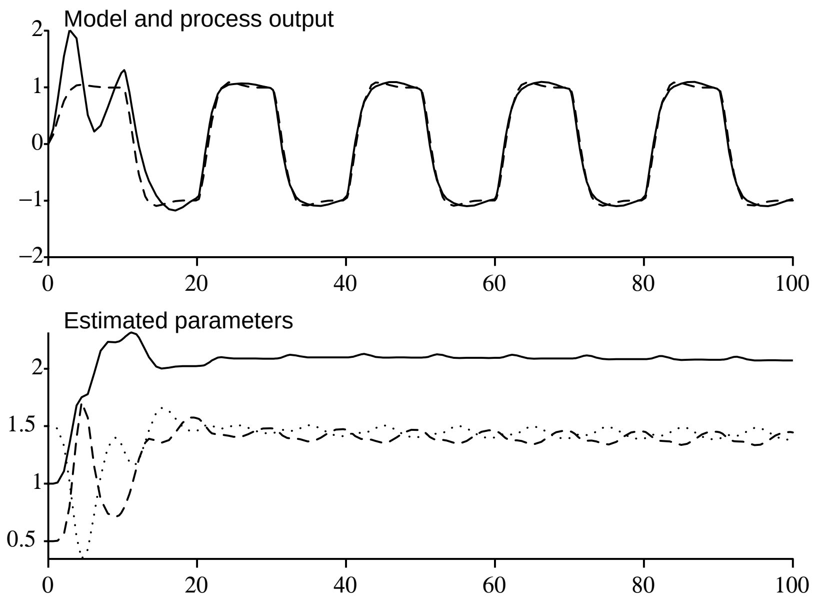

Figure 19. Simulation in Problem 5.11. Top: Process (full) and model (dashed) output. Bottom: Estimated parameters r_1 (full), s_0 (dashed), and t_0 (dotted).

5.12 The closed loop transfer function is

$$G_{cl}(s) = \frac{kb}{s^2 + s + kb}$$

This is compatible with the desired dynamics. The error is

$$e = y - y_m = \frac{kb}{p^2 + p + kb} u_c - y_m$$

Hence

$$\begin{aligned} \frac{\partial e}{\partial k} &= \frac{b}{p^2 + p + kb} u_c - \frac{b^2 k}{(p^2 + p + kb)^2} u_c \\ &= \frac{b}{p^2 + p + kb} (u_c - y) \\ &\approx b \frac{1}{p^2 + p + 1} (u_c - y) \qquad p = \frac{d}{dt} \end{aligned}$$

The following adjustment rule is obtained from the MIT rule

$$\frac{dk}{dt} = -\gamma' \frac{\partial e}{\partial k} e = \underbrace{-\gamma' b}_{\gamma} \left(\frac{1}{p^2 + p + 1} (u_c - y) \right) e$$

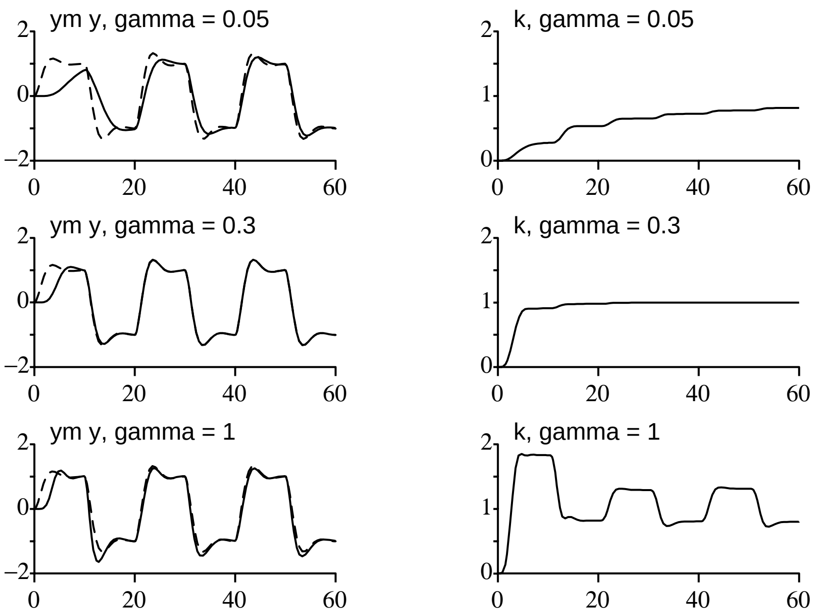

Figure 20. Simulation in Problem 5.12. Left: Process (full) and model (dashed) output. Right: Estimated parameter k for different values of γ.

A simulation of the system is given Fig. 20. This shows the behavior of the system when u_c is a square wave.

SOLUTIONS TO CHAPTER 6

6.1 The process is given by

$$G(s) = \frac{b}{s(s+a)}$$

and the regressor filter should be

$$G_f(s) = \frac{1}{A_f(s)} = \frac{1}{A_m(s)} = \frac{1}{s^2 + 2\zeta\omega s + \omega^2}$$

The controller is given by

$$U(s) = \frac{s_0 s + s_1}{s + r_1} Y(s) + \frac{t_0(s + a_o)}{s + r_1} u_c(s)$$

For the estimation of process parameters we use a continuous-time RLS algorithm. The process is of second order and the controller is of first order. The regressor filter is of second order and both inputs and outputs should be filtered. Hence we need seven states in ξ. The process parameters are contained in θ, and the controller parameters in ϑ. The relation between these are given by

$$\vartheta = \begin{pmatrix} r_1 \\ s_0 \\ s_1 \\ t_0 \end{pmatrix} = \begin{pmatrix} 2\zeta\omega + a_o - \hat{a} \\ (a_o 2\zeta\omega + \omega^2 - \hat{a} r_1)/\hat{b} \\ \omega^2 a_o/\hat{b} \\ \omega^2/\hat{b} \end{pmatrix} = \chi(\theta)$$

To find $A(\vartheta)$, $B(\vartheta)$, $C(\vartheta)$ and $D(\vartheta)$ we start by finding realizations for y, y_f, u_f and the controller. We get

$$\frac{d}{dt}\begin{pmatrix} \dot{y} \\ y \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{y} \\ y \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix} u$$

$$\frac{d}{dt}\begin{pmatrix} \dot{y}_f \\ y_f \end{pmatrix} = \begin{pmatrix} -2\zeta\omega & -\omega^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{y}_f \\ y_f \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} y$$

and

$$\frac{d}{dt}\begin{pmatrix} \dot{u}_f \\ u_f \end{pmatrix} = \begin{pmatrix} -2\zeta\omega & -\omega^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{u}_f \\ u_f \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

and the control law can be rewritten as

$$u = -s_0 y + t_0 u_c + \frac{-s_1 + r_1 s_0}{p + r_1} y + \frac{a_o t_0 - t_0 r_1}{p + r_1} u_c$$

We need one state for the controller and it can be realized as

$$\dot{x} = -r_1 x + (-s_1 + r_1 s_0) y + (a_o t_0 - r_1 t_0) u_c$$

$$u = x - s_0 y + t_0 u_c$$

Combining the states results in

$$\frac{d}{dt}\begin{pmatrix} \dot{y} \\ y \\ \dot{y}_f \\ y_f \\ \dot{u}_f \\ u_f \\ x \end{pmatrix} = \begin{pmatrix} -a & -bs_0 & 0 & 0 & 0 & 0 & b \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2\zeta\omega & -\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -s_0 & 0 & 0 & -2\zeta\omega & -\omega^2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -s_1 + r_1 s_0 & 0 & 0 & 0 & 0 & -r_1 \end{pmatrix}$$

$$\times \begin{pmatrix} \dot{y} \\ y \\ \dot{y}_f \\ y_f \\ \dot{u}_f \\ u_f \\ x \end{pmatrix} + \begin{pmatrix} bt_0 \\ 0 \\ 0 \\ 0 \\ t_0 \\ 0 \\ a_o t_0 - r_1 t_0 \end{pmatrix} u_c$$

which defines the relation

$$\frac{d\xi}{dt} = A(\vartheta)\xi + B(\vartheta)u_c$$

Now we need to express e and φ in the states so that we find the C and D matrices. The estimator tries to find the parameters in

$$y_f = \frac{b}{p(p+a)} u_f$$

which is rewritten as

$$p^2 y_f = \begin{pmatrix} -py_f & u_f \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \varphi^T \theta^0$$

Clearly

$$\varphi = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{y} \\ y \\ \dot{y}_f \\ y_f \\ \dot{u}_f \\ u_f \\ x \end{pmatrix}$$

and

$$e = p^2 y_f - \varphi^T \hat{\theta} = -2\zeta\omega\dot{y}_f - \omega^2 y_f + y + \hat{a}\hat{y}_f - \hat{b}u_f$$

If we use the relations $\hat{a} = -r_1 + 2\zeta\omega + a_o$ and $\hat{b} = \omega^2/t_0$ then e can be written as

$$e = -2\zeta\omega\dot{y}_f - \omega^2 y_f + y + (-r_1 + 2\zeta\omega + a_o)\dot{y}_f - \frac{\omega^2}{t_0} u_f$$

$$= \begin{pmatrix} 0 & 1 & -r_1 + a_o & 0 & 0 & -\frac{\omega^2}{t_0} & 0 \end{pmatrix} \begin{pmatrix} \dot{y} \\ y \\ \dot{y}_f \\ y_f \\ \dot{u}_f \\ u_f \\ x \end{pmatrix}$$

Combining the expressions for φ and e gives

$$\begin{pmatrix} e \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 & 1 & -r_1 + a_o & 0 & 0 & -\frac{\omega^2}{t_0} & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{y} \\ y \\ \dot{y}_f \\ y_f \\ \dot{u}_f \\ u_f \\ x \end{pmatrix} = C(\vartheta)\xi$$

i.e. $D(\vartheta) = 0$. As given in the problem description, the estimator is defined by

$$\frac{d\theta}{dt} = P\varphi e$$

$$\frac{dP}{dt} = \alpha P - P\varphi\varphi^T P$$

where P is a 2×2 matrix and e and φ are given above.

6.3 The averaged equations for the parameter estimates are given by (6.54) on page 303. In this particular case we have

$$G(s) = \frac{ab^2}{(s+a)(s+b)^2}$$

$$G_m(s) = \frac{a}{s+a}$$

To use the averaged equations we need

$$
\begin{aligned}
&\operatorname{avg}\Big((G_m u_c)(G u_c)\Big) = \operatorname{avg}\left(v\left(\frac{b}{(p+b)^2}v\right)\right)\\
&= \frac{1}{2}v^2 \cdot \frac{b^2}{b^2+\omega^2}\cos 2\varphi = \frac{a^2u_0^2b^2}{2(a^2+\omega^2)(b^2+\omega^2)}\left(2\cos^2\varphi - 1\right)\\
&= \frac{a^2b^2u_0^2}{2(a^2+\omega^2)(b^2+\omega^2)}\left(\frac{2b^2}{\omega^2+b^2}-1\right) = \frac{a^2b^2u_0^2(b^2-\omega^2)}{2(a^2+\omega^2)(b^2+\omega^2)^2}
\end{aligned}
$$

where we have introduced

$$
\begin{aligned}
u_c &= u_0 \sin \omega t\\
v &= \frac{a}{p+a}u_c\\
\varphi &= \operatorname{atan}\frac{\omega}{b}
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&\operatorname{avg}(u_c G u_c) = \frac{u_0^2}{2}|G|\cos(2\varphi + \varphi_1)\\
&= \frac{u_0^2}{2}\cdot\frac{ab^2}{\sqrt{a^2+\omega^2}\cdot(b^2+\omega^2)}(\cos 2\varphi\cos\varphi_1 - \sin 2\varphi \sin\varphi_1)\\
&= \frac{u_0^2ab^2(ab^2-\omega^2(a+2b))}{2(a^2+\omega^2)(b^2+\omega^2)^2}
\end{aligned}
$$

where

$$
\varphi = \operatorname{atan}\frac{\omega}{b} \qquad \varphi_1 = \operatorname{atan}\frac{\omega}{a}
$$

It follows from the analysis on page 302–304 that the MIT rule gives a stable system as long as $\omega < b$ while the stability condition for the SPR rule is

$$
\omega < \sqrt{\frac{a}{a+2b}}\,b
$$

with $b = 10a$ we get

$$
\begin{aligned}
\omega_{MIT} &= 10a\\
\omega_{SPR} &= 2.18a
\end{aligned}
$$

6.10 The adaptive system was designed for a process with the transfer function

$$
\hat{G}(s) = \frac{b}{s+a} \tag{1}
$$

The controller has the structure

$$
u = \theta_1 u_c - \theta_2 y \tag{2}
$$

The desired response is

$$G_m(s) = \frac{b_m}{s + a_m} \tag{3}$$

Combining (1) and (2) gives the closed loop transfer function

$$G_{cl} = \frac{b\theta_1}{s + a + b\theta_2}$$

Equating this with $G_m(s)$ given by (3) gives

$$\begin{aligned} b\theta_1 &= b_m \\ a + b\theta_2 &= a_m \end{aligned}$$

If these equations are solved for θ_1 and θ_2 we obtain the controller parameters that give the desired closed loop system. Conversely if the equations are solved for a and b we obtain the parameters of the process model that corresponds to given controller parameters. This gives

$$\begin{aligned} a &= a_m - b_m\theta_2/\theta_1 \\ b &= b_m/\theta_1 \end{aligned}$$

The parameters a and b can thus be interpreted as the parameters of the model the controller believes in. Inserting the expressions for θ_1 and θ_2 from page 318 we get

$$\begin{cases} a(\omega) = \dfrac{229 - 31\omega^2}{259 - \omega^2} \\ b(\omega) = \dfrac{458}{259 - \omega^2} \end{cases} \tag{+}$$

When

$$\omega = \sqrt{\frac{229}{31}} = 2.7179$$

we get $a(\omega) = 0$. The reson for this is that the value of the plant transfer function

$$G(s) = \frac{458}{(s+1)(s^2 + 30s + 229)} \tag{$*$}$$

is $G(2.7179i) = -0.6697i$. The transfer function of the plant is thus purely imaginary. The only way to obtain a purely imaginary value of the transfer function

$$\hat{G} = \frac{b}{s + a} \tag{$**$}$$

is to make $a = 0$. Also notice that $b(2.7179i) = 1.8203$ which gives $\hat{G}(2.7179i) = -0.6697i$. When $\omega = \sqrt{259} = 16.09$ we get infinite values of a and b. Notice that $G(i\sqrt{259}) = -0.0587$ that is real and negative. The only way to make $\hat{G}(i\omega)$ negative and real is to have infinite large values of a and b. It is thus easy to explain the behavior of the algorithm from the system identification point of view. The controller can be interpreted as if it is fitting a model (**) to the process dynamics (*). With a sinusoidal input it is possible to get a perfect fit and the parameters are given by (+).